15th Triennial World Congress, Barcelona, Spain

MAX-PLUS-ALGEBRAIC PROBLEMS AND THE EXTENDED LINEAR COMPLEMENTARITY PROBLEM — ALGORITHMIC ASPECTS

B. De Schutter * **W.P.M.H. Heemels** ** **A. Bemporad** ***

* *Control Systems Engineering, Fac. ITS, Delft University of Technology, P.O. Box 5031, 2600 GA Delft, The Netherlands,* `b.deschutter@its.tudelft.nl`
** *Dept. of Electrical Eng., Eindhoven Univ. of Technology, P.O. Box 513, 5600 MB Eindhoven, The Netherlands,* `w.p.m.h.heemels@tue.nl`
*** *Dip. Ingegneria dell'Informazione, Università di Siena, Via Roma 56, 53100 Siena, Italy,* `bemporad@dii.unisi.it` *& Automatic Control Laboratory, Swiss Federal Institute of Technology, ETH-Z, Physikstrasse 3, 8092 Zurich, Switzerland,* `bemporad@aut.ee.ethz.ch`

Abstract: Many fundamental problems in the max-plus-algebraic system theory for discrete event systems — among which the minimal state space realization problem — can be solved using an Extended Linear Complementarity Problem (ELCP). We present some new, more efficient methods to solve the ELCP. We show that an ELCP with a bounded feasible set can be recast as a standard Linear Complementarity Problem (LCP). Our proof results in three possible numerical solution methods for a given ELCP: regular ELCP algorithms, mixed integer linear programming algorithms, and regular LCP algorithms. We also apply these three methods to a basic max-plus-algebraic benchmark problem.

Keywords: discrete event dynamic systems, optimization problems, complementarity problems, algorithms, mathematical programming

1. INTRODUCTION

In this paper we consider the subclass of discrete event systems that can be described using the max-plus algebra (Baccelli *et al.*, 1992; Cuninghame-Green, 1979), which has maximization and addition as basic operations. Although the description of discrete event systems that belong to this subclass is nonlinear in conventional algebra, the model becomes "linear" when we formulate it in the max-plus algebra.

In our previous work we have shown that many fundamental problems in the max-plus-algebraic system theory (such as computing max-plus-algebraic matrix factorizations, state space transformations, and minimal state space realizations) can be reformulated as a mathematical programming problem: the Extended Linear Complementarity Problem (ELCP). The standard Linear Complementarity Problem (LCP) and most of its "linear" extensions are actually special cases of the ELCP. In this paper we show that the reverse statement holds as well: any ELCP can be recast as an LCP provided that the surplus variables of the ELCP are bounded over the feasible set (A sufficient condition for this is that the feasible set of the ELCP is bounded). In the derivation of our result we obtain another equivalent problem, which will be called the mixed integer linear feasibility problem (MILFP). As a consequence, we have several ways to solve a given ELCP with bounded surplus variables: either as an ELCP using the algorithm of (De Schutter and

De Moor, 1995*a*), as a mixed integer linear programming problem, or as a standard LCP. The computation time of the ELCP algorithm of (De Schutter and De Moor, 1995*a*), which finds *all* solutions of the ELCP, increases rapidly as the number of (in)equalities and variables increases, which may often be prohibitive. However, in many applications for discrete event systems we often only need one solution of the ELCP. In these cases the alternative (mixed integer or LCP) algorithms offer an attractive, more efficient way to solve the ELCP. This also allows us to solve larger-sized max-plus-algebraic problems than before.

This paper is organized as follows. First, we give a brief introduction to the max-plus algebra and to linear complementarity problems. Next, we show that an ELCP with a bounded feasible set or — more general — with bounded surplus variables can be recast as a regular LCP. As an intermediate step we also obtain the MILFP. This leads to three methods to solve an ELCP. Finally, we compare the performance of the three methods for the max-plus-algebraic matrix factorization problem, which is one of the basic problems in the max-plus algebra and in the max-plus-algebraic system theory for discrete event systems.

2. BACKGROUND MATERIAL

2.1 *Max-plus algebra*

One of the frameworks that can be used to model discrete event systems is the max-plus algebra (Baccelli *et al.*, 1992; Cuninghame-Green, 1979). The basic operations of the max-plus algebra are the maximum (represented[1] by $\oplus$) and the addition (represented by $\otimes$):

$$x \oplus y = \max(x,y)$$
$$x \otimes y = x + y$$

with $x,y \in \mathbb{R}_\varepsilon$ where $\mathbb{R}_\varepsilon = \mathbb{R} \cup \{-\infty\}$. The structure $(\mathbb{R} \cup \{-\infty\}, \oplus, \otimes)$ is called the *max-plus algebra*. The operations $\oplus$ and $\otimes$ are extended to matrices as follows. If $\mathbf{A}, \mathbf{B} \in \mathbb{R}_\varepsilon^{m \times n}$ and $\mathbf{C} \in \mathbb{R}_\varepsilon^{n \times p}$ then

$$(\mathbf{A} \oplus \mathbf{B})_{ij} = a_{ij} \oplus b_{ij} = \max(a_{ij}, b_{ij})$$
$$(\mathbf{A} \otimes \mathbf{C})_{ij} = \bigoplus_{k=1}^{n} a_{ik} \otimes c_{kj} = \max_k \{a_{ik} + c_{kj}\} \ .$$

In general, the behavior of discrete event systems is highly nonlinear. However, time-invariant discrete event systems with only synchronization and no concurrency can be described by a model of the form

$$\mathbf{x}(k+1) = \mathbf{A} \otimes \mathbf{x}(k) \ \oplus \ \mathbf{B} \otimes \mathbf{u}(k) \tag{1}$$
$$\mathbf{y}(k) = \mathbf{C} \otimes \mathbf{x}(k) \tag{2}$$

[1] The reason for choosing these symbols is that many properties from conventional linear algebra can be translated to the max-plus algebra simply by replacing $+$ by $\oplus$ and $\times$ by $\otimes$.

where $\mathbf{A} \in \mathbb{R}_\varepsilon^{n \times n}$, $\mathbf{B} \in \mathbb{R}_\varepsilon^{n \times m}$ and $\mathbf{C} \in \mathbb{R}_\varepsilon^{l \times n}$. In (1)–(2) k is an event counter and $\mathbf{u}(k)$, $\mathbf{x}(k)$ and $\mathbf{y}(k)$ contain respectively the time instants at which the input events, the internal events and the output events occur. Discrete event systems that can be described by the state space model (1)–(2) are called *max-plus-linear*. For more information the interested reader is referred to (Baccelli *et al.*, 1992; Cohen *et al.*, 1985; Cuninghame-Green, 1979).

2.2 *Linear Complementarity Problem (LCP)*

The LCP can be formulated as follows:

Given $\mathbf{M} \in \mathbb{R}^{n \times n}$, $\mathbf{q} \in \mathbb{R}^n$, find $\mathbf{w}, \mathbf{z} \in \mathbb{R}^n$ such that

$$\mathbf{w} = \mathbf{M}\mathbf{z} + \mathbf{q} \tag{3}$$
$$\mathbf{w}, \mathbf{z} \geqslant \mathbf{0} \tag{4}$$
$$\mathbf{w}^T \mathbf{z} = 0 \ . \tag{5}$$

Condition (5) is called the *complementarity condition* of the LCP. For more information on the LCP and its applications we refer the interested reader to (Cottle *et al.*, 1992; Ferris and Pang, 1997*b*; Ferris and Pang, 1997*a*; Leenaerts and van Bokhoven, 1998; Murty, 1988) and the references therein.

2.3 *Extended Linear Complementarity Problem*

In (De Schutter and De Moor, 1995*a*) we have introduced an extension of the LCP, which we have called the Extended Linear Complementarity Problem (ELCP) and which is defined as follows:

Given $\mathbf{A} \in \mathbb{R}^{p \times n}$, $\mathbf{B} \in \mathbb{R}^{q \times n}$, $\mathbf{c} \in \mathbb{R}^p$, $\mathbf{d} \in \mathbb{R}^q$ and $\phi_1, \ldots, \phi_m \subseteq \{1, \ldots, p\}$, find $\mathbf{x} \in \mathbb{R}^n$ such that

$$\mathbf{A}\mathbf{x} \geqslant \mathbf{c} \tag{6}$$
$$\mathbf{B}\mathbf{x} = \mathbf{d} \tag{7}$$
$$\sum_{j=1}^{m} \prod_{i \in \phi_j} (\mathbf{A}\mathbf{x} - \mathbf{c})_i = 0 \ . \tag{8}$$

The surplus variable $\mathrm{s}^+(i,\mathbf{x})$ of the ith inequality of $\mathbf{A}\mathbf{x} \geqslant \mathbf{c}$ is defined as $\mathrm{s}^+(i,\mathbf{x}) = (\mathbf{A}\mathbf{x} - \mathbf{c})_i$. Condition (8) represents the *complementarity condition* of the ELCP and can be interpreted as follows. Since $\mathbf{A}\mathbf{x} \geqslant \mathbf{c}$, all the terms in (8) are nonnegative. Hence, (8) is equivalent to $\prod_{i \in \phi_j} (\mathbf{A}\mathbf{x} - \mathbf{c})_i = 0$ for $j = 1, \ldots, m$. So each set ϕ_j corresponds to a group of inequalities in $\mathbf{A}\mathbf{x} \geqslant \mathbf{c}$, and in each group at least one inequality should hold with equality (i.e., the corresponding surplus variable is equal to 0).

Remark 1. We may without loss of generality assume that the ELCP is written as

$$\mathbf{A}\mathbf{x} \geqslant \mathbf{c} \tag{9}$$
$$\sum_{i=1}^{m} \prod_{j \in \phi_i} (\mathbf{A}\mathbf{x} - \mathbf{c})_j = 0 \ , \tag{10}$$

since we can replace $\mathbf{Bx} = \mathbf{d}$ by $\mathbf{Bx} \geqslant \mathbf{d}$ and obtain equality conditions on these inequalities by adding the index sets $\phi_{m+1} = \{p+1\}, \ldots, \phi_{m+q} = \{p+q\}$.

In general the solution set of the ELCP consists of the union of a subset of the faces of the polyhedron defined by (6)–(7). In (De Schutter and De Moor, 1995*a*) we have developed an algorithm to find a parametric representation of *all* solutions of an ELCP. To the authors' best knowledge no other ELCP algorithms have been described in the literature.

2.4 *Max-plus algebra and the ELCP*

In (De Schutter and De Moor, 1995*b*; De Schutter and De Moor, 1997; De Schutter and De Moor, 1998*b*; De Schutter and van den Boom, 2001) we have shown that many fundamental problems in the max-plus algebra and in max-plus-algebraic system theory for discrete event systems can be solved using the ELCP. More specifically, the following problems can be recast as an ELCP or solved using an ELCP:

- minimal state space realization for max-plus-linear discrete event systems,
- solving systems of max-plus-algebraic polynomial (in)equalities,
- computing max-plus-algebraic matrix factorizations,
- transformation of max-plus-algebraic state space models,
- mixed max-min problems,
- constructing matrices with a given max-plus-algebraic characteristic polynomial,
- determining max-plus-algebraic singular value decompositions and QR decompositions,
- model predictive control for max-plus-linear discrete event systems,
- ...

In addition, in (De Schutter and De Moor, 1998*a*; De Schutter, 2000) we have shown that some analysis and control problems for hybrid systems such as traffic signal controlled intersections, first-order hybrid systems with saturation, and linear complementarity systems can also be solved using ELCPs.

The underlying reason for the link between all these problems and the ELCP is that a system of the form

$$\max_i (a_{i,1}x_1 + \ldots a_{i,n}x_n + b_i) = c_i$$
$$\max_i (e_{i,1}x_1 + \ldots e_{i,n}x_n + d_i) \leqslant f_i$$

can be recast as an ELCP. Basically, the proof of this statement boils down to the fact that $\max(\alpha, \beta) = \gamma$ is equivalent to the system $\alpha \leqslant \gamma$, $\beta \leqslant \gamma$, $(\gamma - \alpha)(\gamma - \beta) = 0$.

3. THE LINK BETWEEN THE LCP AND ELCP

Lemma 2. The LCP is a special case of the ELCP.

PROOF. If we set $\mathbf{x} = [\mathbf{w}^T \; \mathbf{z}^T]^T$, $\mathbf{A} = \mathbf{I}$, $\mathbf{B} = [\mathbf{I} \; -\mathbf{M}]$, $\mathbf{c} = \mathbf{0}$, $\mathbf{d} = \mathbf{q}$ and $\phi_j = \{j, j+n\}$ for $j = 1, \ldots, n$ in the formulation of the ELCP, we get an LCP. $\square$

Theorem 3. If the surplus variables of the inequalities of an ELCP are bounded (from above [2]) over the feasible set of the ELCP, then the ELCP can be rewritten as an LCP.

PROOF. W.l.o.g. we consider the ELCP (9)–(10). The proof consists of two main steps. First, we transform the ELCP into a mixed integer problem to get rid of the complementarity condition [3] of the ELCP at the cost of introducing some additional binary variables. Next, we transform all variables (both binary and real-valued) into nonnegative real ones, which leads to an LCP.

Define a diagonal matrix $\mathbf{D}^{\mathrm{u}} \in \mathbb{R}^{p \times p}$ with $(\mathbf{D}^{\mathrm{u}})_{ii} = d^{\mathrm{u}}_{ii}$ being an upper bound [4] for $\mathrm{s}^+(i, \mathbf{x}) = (\mathbf{Ax} - \mathbf{c})_i$ over the feasible set of the ELCP. Now consider the following system of equations:

$$\boldsymbol{\delta} \in \{0,1\}^p, \; \mathbf{x} \in \mathbb{R}^n \tag{11}$$
$$\mathbf{0} \leqslant (\mathbf{Ax} - \mathbf{c})_i \leqslant d^{\mathrm{u}}_{ii} \delta_i \qquad \text{for } i = 1, \ldots, p, \tag{12}$$
$$\sum_{i \in \phi_j} \delta_i \leqslant \#\phi_j - 1 \qquad \text{for } j = 1, \ldots, m, \tag{13}$$

where $\#\phi_j$ denotes the number of elements of the set ϕ_j. Problem (11)–(13) will be called the equivalent *mixed integer linear feasibility problem* (MILFP).

Now we show that the MILFP (11)–(13) is equivalent to the ELCP (9)–(10) in the sense that $\mathbf{x}$ is a solution of (9)–(10) if and only if there exists a $\boldsymbol{\delta}$ such that $(\mathbf{x}, \boldsymbol{\delta})$ is a solution of (11)–(13). Equation (9) is implied by (12). Note that (11) and (13) imply that for each j at least one of the δ_i's with $i \in \phi_j$ is equal to 0. If $\delta_{i'} = 0$ then it follows from (12) that $(\mathbf{Ax} - \mathbf{c})_{i'} = 0$. So in each index set ϕ_j there is at least one index $i \in \phi_j$ such that $\mathrm{s}^+(i, \mathbf{x}) = 0$. Hence, the complementarity condition (10) is also implied by (11)–(13). So (11)–(13) imply (9)–(10), and it is easy to verify that the reverse statement also holds. As a consequence, the MILFP is equivalent to the ELCP.

Define $\mathbf{S} \in \mathbb{R}^{m \times p}$ with $s_{ji} = 1$ if $i \in \phi_j$ and $s_{ji} = 0$ otherwise; and $\mathbf{t} \in \mathbb{R}^m$ with $t_j = \#\phi_j - 1$. The MILFP can now be rewritten compactly as

---

[2] We only need boundedness from above since the surplus variables are always nonnegative due to the condition $\mathbf{Ax} \geqslant \mathbf{c}$.

[3] Note that each term of the complementarity condition (10) of the ELCP can have any number of factors in it, whereas in the complementarity condition (5) of the LCP each term $(w_i z_i)$ has exactly two factors.

[4] See Section 4 for efficient methods to compute these upper bounds.

$$\boldsymbol{\delta} \in \{0,1\}^p, \ \mathbf{x} \in \mathbb{R}^n \tag{14}$$

$$\mathbf{0} \leqslant \mathbf{A}\mathbf{x} - \mathbf{c} \leqslant \mathbf{D}^{\mathrm{u}}\boldsymbol{\delta} \tag{15}$$

$$\mathbf{S}\boldsymbol{\delta} \leqslant \mathbf{t} \ . \tag{16}$$

Now we will transform this problem into an LCP.

(1) First, we transform (14) into the LCP framework. All the variables of an LCP should be real-valued, but $\boldsymbol{\delta}$ is a binary vector. However, the condition $\delta_i \in \{0,1\}$ is equivalent to $\delta_i \geqslant 0$, $1-\delta_i \geqslant 0$, $\delta_i(1-\delta_i) = 0$. So if we introduce an auxiliary vector $\mathbf{v}_{\boldsymbol{\delta}} \in \mathbb{R}^p$, then the condition $\boldsymbol{\delta} \in \{0,1\}^p$ is equivalent to

$$\mathbf{v}_{\boldsymbol{\delta}} = \mathbf{1} - \boldsymbol{\delta}, \quad \boldsymbol{\delta}, \mathbf{v}_{\boldsymbol{\delta}} \geqslant 0, \quad \boldsymbol{\delta}^T \mathbf{v}_{\boldsymbol{\delta}} = 0$$

with $\boldsymbol{\delta}, \mathbf{v}_{\boldsymbol{\delta}} \in \mathbb{R}^p$ and where $\mathbf{1}$ is an all-1 vector.

(2) The inequality $\mathbf{0} \leqslant \mathbf{A}\mathbf{x} - \mathbf{c}$ can be adapted to the LCP framework by introducing $\mathbf{v}_{\mathbf{A}} \in \mathbb{R}^p$ with $\mathbf{v}_{\mathbf{A}} = \mathbf{A}\mathbf{x} - \mathbf{c} \geqslant \mathbf{0}$. To obtain a complementarity condition for $\mathbf{v}_{\mathbf{A}}$ we introduce $\mathbf{w}_{\mathbf{A}} \in \mathbb{R}^p$ such that $\mathbf{v}_{\mathbf{A}}, \mathbf{w}_{\mathbf{A}} \geqslant \mathbf{0}$, $\mathbf{v}_A^T \mathbf{w}_A = 0$ (Note that we can always take $\mathbf{w}_{\mathbf{A}} = \mathbf{0}$ to get this condition satisfied). Hence, $\mathbf{0} \leqslant \mathbf{A}\mathbf{x} - \mathbf{c}$ can be rewritten as

$$\mathbf{v}_{\mathbf{A}} = \mathbf{A}\mathbf{x} - \mathbf{c}, \quad \mathbf{v}_{\mathbf{A}}, \mathbf{w}_{\mathbf{A}} \geqslant \mathbf{0}, \quad \mathbf{v}_{\mathbf{A}}^T \mathbf{w}_{\mathbf{A}} = 0$$

with $\mathbf{v}_{\mathbf{A}}, \mathbf{w}_{\mathbf{A}} \in \mathbb{R}^p$. The inequalities $\mathbf{A}\mathbf{x} - \mathbf{c} \leqslant \mathbf{D}^{\mathrm{u}}\boldsymbol{\delta}$ and $\mathbf{S}\boldsymbol{\delta} \leqslant \mathbf{t}$ can be dealt with in a similar way.

(3) All variables in an LCP are nonnegative whereas this condition is not present in the MILFP or the ELCP. Therefore, we split $\mathbf{x}$ in its positive part $\mathbf{x}^+ = \max(\mathbf{x}, \mathbf{0})$ and its negative part $\mathbf{x}^- = \max(-\mathbf{x}, \mathbf{0})$. So $\mathbf{x} = \mathbf{x}^+ - \mathbf{x}^-$ and $\mathbf{x}^+, \mathbf{x}^- \geqslant \mathbf{0}$, $(\mathbf{x}^+)^T \mathbf{x}^- = 0$. To obtain independent complementarity conditions for $\mathbf{x}^+$ and $\mathbf{x}^-$ we introduce $\mathbf{v}^+, \mathbf{v}^- \in \mathbb{R}^n$ with $\mathbf{v}^+ = \mathbf{x}^+$ and $\mathbf{v}^- = \mathbf{x}^-$ such that $\mathbf{v}^-, \mathbf{x}^+ \geqslant \mathbf{0}$, $(\mathbf{v}^-)^T \mathbf{x}^+ = 0$ and $\mathbf{v}^+, \mathbf{x}^- \geqslant \mathbf{0}$, $(\mathbf{v}^+)^T \mathbf{x}^- = 0$.

Combining the three above steps results in the following equivalent LCP:

$$\underbrace{\begin{bmatrix} \mathbf{v}_{\boldsymbol{\delta}} \\ \mathbf{v}^- \\ \mathbf{v}^+ \\ \mathbf{v}_{\mathbf{A}} \\ \mathbf{v}_{\mathbf{D}^{\mathrm{u}}} \\ \mathbf{v}_{\mathbf{S}} \end{bmatrix}}_{\mathbf{w}} = \underbrace{\begin{bmatrix} -\mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{A} & -\mathbf{A} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{D}^{\mathrm{u}} & -\mathbf{A} & \mathbf{A} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ -\mathbf{S} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}}_{\mathbf{M}} \underbrace{\begin{bmatrix} \boldsymbol{\delta} \\ \mathbf{x}^+ \\ \mathbf{x}^- \\ \mathbf{w}_{\mathbf{A}} \\ \mathbf{w}_{\mathbf{D}^{\mathrm{u}}} \\ \mathbf{w}_{\mathbf{S}} \end{bmatrix}}_{\mathbf{z}} + \underbrace{\begin{bmatrix} \mathbf{1} \\ \mathbf{0} \\ \mathbf{0} \\ -\mathbf{c} \\ \mathbf{c} \\ \mathbf{t} \end{bmatrix}}_{\mathbf{q}} \tag{17}$$

$$\mathbf{w}, \mathbf{z} \geqslant \mathbf{0}, \quad \mathbf{w}^T \mathbf{z} = 0 \tag{18}$$

with $\mathbf{w}, \mathbf{z} \in \mathbb{R}^{3p+2n+m}$.
Furthermore, it is easy to verify that a solution of the LCP (17)–(18) results in a solution of the MILFP if we set $\mathbf{x} = \mathbf{x}^+ - \mathbf{x}^-$. $\square$

The introduction of the MILFP in this proof is inspired by the paper (Heemels *et al.*, 2001).

Even if we have an ELCP for which the surplus variables are not bounded over the feasible set, then in many practical applications for discrete event systems it often occurs that we are only interested in solutions that lie in a given (bounded) region. In that case we can use the following theorem which can be proved in a similar way as Theorem 3.

Theorem 4. Consider the ELCP (6)–(8) and let X be an arbitrary bounded subset of $\mathbb{R}^n$. Then every solution $\mathbf{x} \in X$ of the ELCP can be obtained as a solution of an equivalent MILFP or LCP (where the upper bounds d_{ii}^{u} are now defined as upper bounds of the surplus variables over the set X).

4. THREE ELCP ALGORITHMS

The proof of Theorem 3 results in three possible methods to solve a given ELCP:

- as an ELCP using the ELCP algorithm of (De Schutter and De Moor, 1995*a*);
- as an MILFP using mixed integer linear programming algorithms: If we are only interested in obtaining one solution of an ELCP, we can transform the corresponding MILFP into a mixed integer linear programming problem by adding a dummy linear objective function. This problem can then be solved using, e.g., a branch-and-bound method (Bemporad and Mignone, 2000; Fletcher and Leyffer, 1998; Taha, 1987) or a branch-and-cut method (Cordier *et al.*, 1999).
- as an LCP using standard LCP algorithms [5] such as the PATH algorithm of (Ferris and Munson, 2000), or via a reformulation as a quadratic programming (QP) problem [6].

The removal of $\mathbf{B}\mathbf{x} = \mathbf{d}$ in the proof of Theorem 3 is only done to simplify the proof. If we want to solve an ELCP by reformulating it as an MILFP, then it is numerically better to keep the equalities. In that case the full ELCP (6)–(8) would result in the MILFP (11)–(13) but with the additional constraint $\mathbf{B}\mathbf{x} = \mathbf{d}$.

A sufficient condition for the surplus variables of the inequalities of the ELCP to be bounded is that the feasible set of the ELCP is bounded [7]. Upper bounds

---

[5] The matrix $\mathbf{M}$ of the LCP (17)–(18) is not positive definite. Moreover, this LCP is in general not strictly feasible (i.e., in general the set $\{\mathbf{z} \in \mathbb{R}^n \mid \mathbf{M}\mathbf{z} + \mathbf{q} > \mathbf{0}, \mathbf{z} > \mathbf{0}\}$ is empty). This may prevent us from using, e.g., Lemke's method or strictly feasible interior point methods to solve the LCP.

[6] If we consider the QP problem $\min_{\mathbf{z}} \mathbf{z}^T(\mathbf{M}\mathbf{z} + \mathbf{q})$ subject to $\mathbf{z} \geq \mathbf{0}$ and $\mathbf{M}\mathbf{z} + \mathbf{q} \geq \mathbf{0}$, then it is easy to verify that the optimal solution $\mathbf{z}_{\mathrm{opt}}$ of the QP problem yields a solution of the LCP (3)–(5) if the value of the objective function in $\mathbf{z}_{\mathrm{opt}}$ is equal to 0. Note that we can significantly reduce the size of the QP problem by setting $\mathbf{w}_{\mathbf{A}} = 0$, $\mathbf{w}_{\mathbf{D}^{\mathrm{u}}} = 0$, and $\mathbf{w}_{\mathbf{S}} = 0$ (cf. item (2) of the last part of the proof of Theorem 3).

[7] However, boundedness of the feasible set is *not* a necessary condition for boundedness of the surplus variables. Consider, e.g., the ELCP $\mathbf{x} \in \mathbb{R}^2$, $-1 \leqslant x_1 \leqslant 1$, $(x_1+1)(1-x_1) = 0$, which has an unbounded feasible set $\{\mathbf{x} \in \mathbb{R}^2 \mid x_1 = -1 \text{ or } x_1 = 1\}$, but bounded surplus variables $s^+(1,\mathbf{x}), s^+(2,\mathbf{x}) \in [0,2]$ over the feasible set.

for the surplus variables over the feasible set can be determined as follows:

- The upper bounds d^{u}_{jj} can be computed efficiently using a linear programming (LP) problem:

$$d^{\mathrm{u}}_{ii} = \max_{\mathbf{Ax}-\mathbf{c}\geqslant\mathbf{0}} (\mathbf{Ax}-\mathbf{c})_i \quad \text{for } i=1,\ldots,p\,.$$

  If any of these LP problems yields an unbounded objective function, then the ELCP does not have a bounded feasible set and then the condition of Theorem 3 does not hold.
- If we know upper bounds $\mathbf{x}_{\mathrm{upp}}$ and lower bounds $\mathbf{x}_{\mathrm{low}}$ for the components of $\mathbf{x}$, e.g., as a consequence of physical or other constraints or because of additional information that is available, then we can even more efficiently compute upper bounds as

$$d^{\mathrm{u}}_{ii} = (\mathbf{A}^{+}\mathbf{x}_{\mathrm{upp}} - \mathbf{A}^{-}\mathbf{x}_{\mathrm{low}} - \mathbf{c})_i \quad \text{for } i=1,\ldots,p,$$

  with $\mathbf{A}^{+}$ and $\mathbf{A}^{-}$ defined by $(\mathbf{A}^{+})_{ij} = \max(a_{ij},0)$ and $(\mathbf{A}^{-})_{ij} = \max(-a_{ij},0)$ respectively.

5. PERFORMANCE OF THE THREE ELCP ALGORITHMS

In order to assess the performance of the three methods to solve an ELCP algorithm, we will use them to solve a benchmark problem, namely max-plus-algebraic matrix factorization. This problem can be considered as one of the fundamental problems in max-plus algebra and also in system theory for max-linear discrete event systems since it is one of the basic components for solving the state space transformation problem and the (partial) minimal realization problem.

More specifically, we have performed the following experiment: For a given integer ℓ and a given integer bound b, we have constructed two random integer matrices $\mathbf{A} \in \mathbb{R}_{\varepsilon}^{\ell\times 2}$ and $\mathbf{B} \in \mathbb{R}_{\varepsilon}^{2\times\ell}$ with the entries uniformly distributed over the integer set $\{-b,-b+1,\ldots,b\}$. Next, we have defined $\mathbf{C} = \mathbf{A}\otimes\mathbf{B}$ and constructed the ELCP that solves the following problem:

> Given $\mathbf{C}$, find $\mathbf{P} \in [-b,b]^{\ell\times 2}$ and $\mathbf{Q} \in [-b,b]^{2\times\ell}$ such that $\mathbf{P}\otimes\mathbf{Q} = \mathbf{C}$.

This ELCP has $n = 4\ell$ variables, $p = 2\ell^2+8\ell$ inequalities, and $m = \ell^2$ index sets.

We have performed 1000 experiments for ℓ a random integer in the range $1,2,\ldots,10$ and with $b = 10$. Each ELCP has been solved using the different methods of Section 4 with all the algorithms implemented in C. Since only one solution of the ELCP was required, we have only performed the first part of the ELCP algorithm of (De Schutter and De Moor, 1995*a*), i.e., the computation of the extreme points of the solution set, and skipped the second part of the algorithm in which it is determined which combinations of extreme points also yield a solution. Moreover, due to excessive CPU times we have only applied the ELCP algorithm of (De Schutter and De Moor, 1995*a*) if $\ell \leqslant 7$. The MILFP has been solved using a branch-and-bound method, and the LCP using a quadratic programming approach (cf. Footnote 6). Since the matrix $\mathbf{M}$ is not positive definite, we have a non-convex QP problem. So not every run of the QP solver yields a (global) solution. However, in each experiment we have repeated the QP algorithm with different initial solutions until we obtained a sufficiently accurate solution of the LCP, i.e., a solution for which $\mathbf{z}^T(\mathbf{Mz}+\mathbf{q}) \leq \varepsilon$ with $\varepsilon = 10^{-6}$.

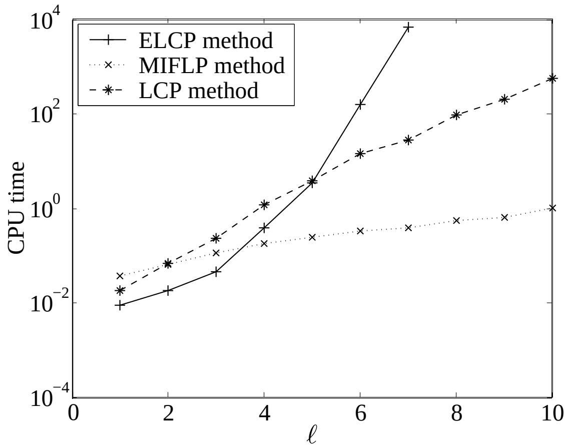

Fig. 1. Average CPU time used by the three algorithms for the benchmark problem (CPU time on a 1 GHz PC).

In Figure 1 we have plotted the average CPU time of the three methods to solve the ELCPs. Note that the scale on the y-axis of the plot is logarithmic. Clearly, the MILFP and the LCP approach outperform the ELCP algorithm of (De Schutter and De Moor, 1995*a*). This also implies that we can now tackle instances of the problems mentioned in Section 2.4 of a significantly larger size than we could before (i.e., using the algorithm of (De Schutter and De Moor, 1995*a*)). Note that even more improvement can be obtained by selecting mixed-integer and/or LCP algorithms that are more suited for the equivalent MILFPs and LCPs than the algorithms we have used, and by reducing the equivalent MILFPs and LCPs before solving them. This will be a topic for future research.

6. CONCLUSIONS AND FUTURE RESEARCH

The ELCP can be used to solve many problems for max-plus-linear discrete event systems (such as the minimal state space realization problem). We have shown that an ELCP with bounded surplus variables over the feasible set (or with a bounded feasible set) can be rewritten as an LCP. In our constructive equivalence proof we have introduced a mixed integer linear feasibility problem that is also equivalent to the ELCP. As a consequence, we now have three different ways to solve an ELCP: using the ELCP algorithm of (De Schutter and De Moor, 1995*a*), using a mixed integer (linear programming) algorithm, or using one of the many LCP algorithms. The main advantage of the latter two approaches is that they can be used for many applications for max-plus-algebraic discrete event systems in which we only need one solution of the ELCP since in that case they will be much more efficient than the ELCP algorithm of (De Schutter and De Moor, 1995*a*), which computes all solutions and, as a consequence, requires much more computation time. We have also compared the performance of the three methods to solve an ELCP for a typical max-plus-algebraic benchmark problem (max-plus-algebraic matrix factorization). For this problem the MILFP-based and LCP-based approaches were significantly faster than the approach that uses the ELCP algorithm of (De Schutter and De Moor, 1995*a*). We expect that this also holds for the ELCPs that arise from other max-plus-algebraic problems.

An important topic for further research is a more thorough evaluation and comparison of the performance of several different mixed integer and LCP algorithms for the special cases of the ELCP that arise in specific applications involving discrete event systems. The LCP or MILFP reformulation of a given ELCP that results from our equivalence proof is not necessarily the most efficient one (i.e., with a minimal number of variables or equations). If we aim at using the equivalent LCP or MILFP to solve the original ELCP in a computationally very efficient way, then it might be useful to look for techniques to reduce the LCP or MILFP by removing redundant variables or inequalities before actually solving the LCP or MILFP. This will also be a topic for future research.

7. REFERENCES

Baccelli, F., G. Cohen, G.J. Olsder and J.P. Quadrat (1992). *Synchronization and Linearity*. John Wiley & Sons. New York.

Bemporad, A. and D. Mignone (2000). miqp.m: A Matlab function for solving Mixed Integer Quadratic Programs. Technical Report AUT00-22. Institut für Automatik, Swiss Federal Institute of Technology (ETHZ). Zürich, Switzerland.

Cohen, G., D. Dubois, J.P. Quadrat and M. Viot (1985). A linear-system-theoretic view of discrete-event processes and its use for performance evaluation in manufacturing. *IEEE Transactions on Automatic Control* **30**(3), 210–220.

Cordier, C., H. Marchand, R. Laundy and L.A. Wolsey (1999). *bc-opt*: a branch-and-cut code for mixed integer programs. *Mathematical Programming, Series A* **86**(2), 335–353.

Cottle, R.W., J.S. Pang and R.E. Stone (1992). *The Linear Complementarity Problem*. Academic Press. Boston.

Cuninghame-Green, R.A. (1979). *Minimax Algebra*. Vol. 166 of *Lecture Notes in Economics and Mathematical Systems*. Springer-Verlag. Berlin, Germany.

De Schutter, B. (2000). Optimal control of a class of linear hybrid systems with saturation. *SIAM Journal on Control and Optimization* **39**(3), 835–851.

De Schutter, B. and B. De Moor (1995*a*). The extended linear complementarity problem. *Mathematical Programming* **71**(3), 289–325.

De Schutter, B. and B. De Moor (1995*b*). Minimal realization in the max algebra is an extended linear complementarity problem. *Systems & Control Letters* **25**(2), 103–111.

De Schutter, B. and B. De Moor (1997). The extended linear complementarity problem and its applications in the max-plus algebra. In: *Complementarity and Variational Problems: State of the Art* (M.C. Ferris and J.S. Pang, Eds.). pp. 22–39. SIAM. Philadelphia, Pennsylvania.

De Schutter, B. and B. De Moor (1998*a*). The Linear Dynamic Complementarity Problem is a special case of the Extended Linear Complementarity Problem. *Systems & Control Letters* **34**(1–2), 63–75.

De Schutter, B. and B. De Moor (1998*b*). Optimal traffic light control for a single intersection. *European Journal of Control* **4**(3), 260–276.

De Schutter, B. and T. van den Boom (2001). Model predictive control for max-plus-linear discrete event systems. *Automatica* **37**(7), 1049–1056.

Ferris, M.C. and J.S. Pang (1997*a*). Engineering and economic applications of complementarity problems. *SIAM Review* **39**(4), 669–713.

Ferris, M.C. and Pang, J.S., Eds.) (1997*b*). *Complementarity and Variational Problems: State of the Art*. Philadelphia, Pennsylvania: SIAM. (Proceedings of the International Conference on Complementarity Problems, Baltimore, Maryland, November 1995).

Ferris, M.C. and T.S. Munson (2000). Complementarity problems in GAMS and the PATH solver. *Journal of Economic Dynamics and Control* **24**(2), 165–188.

Fletcher, R. and S. Leyffer (1998). Numerical experience with lower bounds for MIQP branch-and-bound. *SIAM Journal on Optimization* **8**(2), 604–616.

Heemels, W.P.M.H., B. De Schutter and A. Bemporad (2001). Equivalence of hybrid dynamical models. *Automatica* **37**(7), 1085–1091.

Leenaerts, D.M.W. and W.M.G. van Bokhoven (1998). *Piecewise Linear Modeling and Analysis*. Kluwer Academic Publishers. Boston.

Murty, K.G. (1988). *Linear Complementarity, Linear and Nonlinear Programming*. Helderman Verlag. Berlin, Germany.

Taha, H.A. (1987). *Operations Research: An Introduction*. 4th ed.. Macmillan Publishing Company. New York.