Reference Governor for Constrained Nonlinear Systems

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***Abstract*—This paper addresses the problem of satisfying pointwise-in-time input and/or state hard constraints in nonlinear control systems. The approach is based on conceptual tools of predictive control and consists of adding to a primal compensated nonlinear system a Reference Governor (RG). This is a discrete-time device which on-line handles the reference to be tracked, taking into account the current value of the state in order to satisfy the prescribed constraints. The resulting hybrid system is proved to fulfill the constraints as well as stability and tracking requirements.**

***Index Terms*— Constraint satisfaction problems, nonlinear systems, optimization methods, predictive control, reference input signals.**

I. Introduction

In recent years the field of feedback control of dynamic systems with input and/or state-related constraints has received considerable attention [1], [2]. Most of this research has addressed regulation problems for systems subject to input saturation. More recently, moving horizon optimal control [3]–[5] and model predictive control [6], [7] have proved to be effective tools to deal with tracking problems with input/state constraints. These methods are based on the *receding horizon* philosophy: a sequence of future control actions is chosen according to a prediction of the future evolution of the system and applied to the plant until new measurements are available. Then, a new sequence is evaluated which replaces the previous one. Each sequence is evaluated by means of an optimization procedure, which takes into account two objectives: maximize the tracking performance and protect the system from possible constraint violations. However, when applied to models described by nonlinear differential equations, this requires the on-line solution of high-dimensional nonlinear optimization problems. Unlike other receding horizon approaches which attempt to solve stabilization, tracking, and constraint fulfillment at the same time, we assume that a primal controller has already been designed to stabilize the system and provide nice tracking properties *in the absence of constraints*. The constraint fulfillment task is left to a *reference governor* (RG), a nonlinear device which is added to the primal compensated nonlinear system. Whenever necessary, the RG modifies the reference supplied to the primal control system so as to enforce the fulfillment of the constraints. The RG operates in accordance with the receding horizon strategy, mentioned above, by selecting on-line optimal reference input sequences which, in order to drastically reduce the required computational burden, are parameterized by a scalar quantity.

Studies along similar lines can be found in [8]–[14] for linear control systems. The present paper extends these ideas to nonlinear continuous-time systems and is organized as follows. In Section II we formulate the problem, specify the assumptions on the primal system, and present the RG strategy. Section III is devoted to the derivation of interesting properties of the RG. Computational aspects are considered in Section IV, and a simulative example is reported in Section V.

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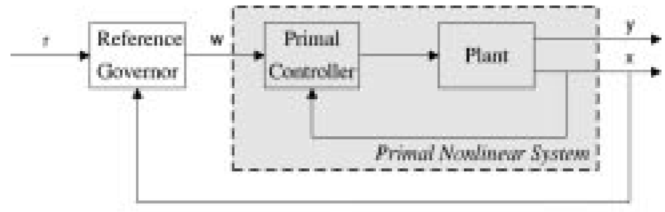

Fig. 1. Control scheme with RG.

II. Problem Formulation and Assumptions

Consider the following nonlinear system:

$$\begin{cases} \dot{x}(t) = \Phi(x(t), w(t)) \\ y(t) = H(x(t), w(t)) \\ c(t) = \begin{bmatrix} x(t) \\ w(t) \end{bmatrix} \end{cases} \tag{1}$$

representing, in general, a (nonlinear) plant under (nonlinear) feedback, where $x(t) \in R^n$ is the state vector, which collects both plant and controller states; $w(t) \in R^p$ is the reference input, which in the absence of constraints would coincide with a desired reference $r(t) \in R^p$; and $y(t) \in R^p$ is the output vector which shall track $r(t)$. Since input and/or state variables of the plant can be expressed as a function of $x(t)$ and $w(t)$, without loss of generality we define $c(t) \in R^{n+p}$ as the vector to be constrained within a given set $\mathcal{C}$.

Assumption 1: $\mathcal{C}$ is compact and has a nonempty interior.

Compactness of $\mathcal{C}$ is nonrestrictive since in practice the desired references and state variables are bounded. Since we are interested in operating on vectors $[x' \;\; w']'$ in $\mathcal{C}$, we restrict the properties required by (1) to the projections of $\mathcal{C}$ on the x-space

$$\mathcal{X} \triangleq \left\{ x \in R^n : \; \exists w \in R^p, \begin{bmatrix} x \\ w \end{bmatrix} \in \mathcal{C} \right\}$$

and the projection on the w-space $\mathcal{W}$, which is defined analogously. It is easy to show that compactness of $\mathcal{C}$ implies that both $\mathcal{X}$ and $\mathcal{W}$ are compact. System (1) is required to fulfill the following assumptions.

Assumption 2: $\forall w \in \mathcal{W}$, there exists a unique equilibrium state $x_w \in \mathcal{X}$.

We denote by

$$X(\cdot) : R^p \mapsto R^n \tag{2}$$

the function implicitly defined by $\Phi(X(\cdot), \cdot) = 0$ and define $x_w \triangleq X(w)$, $c_w \triangleq [x'_w \;\; w']'$. Notice that in general $w \in \mathcal{W} \not\Rightarrow c_w \in \mathcal{C}$.

Assumption 3: The mapping $\Phi(x, w) : \mathcal{X} \times \mathcal{W} \mapsto R^n$ is continuous in (x, w).

Consider now an arbitrarily small scalar $\delta > 0$, and define the following set:

$$\hat{\mathcal{W}}_\delta \triangleq \{ w \in \mathcal{W} : B(c_w, \delta) \subseteq \mathcal{C} \} \tag{3}$$

where $B(c_w, \delta)$ denotes the closed ball $\{c \in R^{n+p} : \; \|c - c_w\| \leq \delta\}$. We restrict the set of reference inputs w which can be supplied by assuming the information as depicted in Fig. 2.

Assumption 4 (Reference Input Conditioning): The class of reference inputs is restricted to a convex, nonempty, and compact set $\mathcal{W}_\delta \subseteq \hat{\mathcal{W}}_\delta \subset \mathcal{W}$.

Assumption 4 is needed to prevent that the border of $\mathcal{C}$ is approached in steady state and is required later to prove Theorem 2. The constraint $c \in \mathcal{C}$ and the reference input conditioning can be

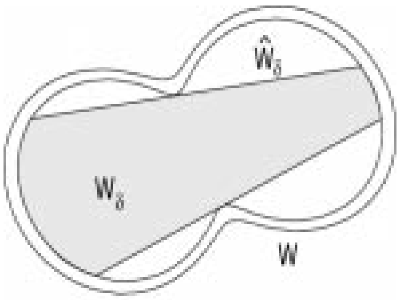

Fig. 2. Sets $\mathcal{W}, \hat{\mathcal{W}}_\delta$, and $\mathcal{W}_\delta$.

summarized as the unique constraint

$$c \in \mathcal{C}_\delta \triangleq \mathcal{C} \bigcap (\mathcal{X} \times \mathcal{W}_\delta) \tag{4}$$

where $\mathcal{C}_\delta$ is compact. We fix $\delta > 0$ such that $\mathcal{C}_\delta$ is nonempty. In order to derive the properties proved in Section III, (1) is supposed to satisfy some extra assumptions.

Assumption 5: For all piecewise constant reference input signals $w(t) \in \mathcal{W}_\delta$, $t \in [0, +\infty)$, and for all initial states $x(0) \in \mathcal{X}$, there exists a unique solution $x(t, x(0), w(t))$ of (1) defined $\forall t \in [0, +\infty)$.

In the following we shall denote by $x(t, x(0), w)$ the solution corresponding to a constant reference $w(t) \equiv w$, $\forall t \in [0, +\infty)$.

Assumption 6 (Converging Input Converging State Stability): Let $w(t) \to w \in \mathcal{W}_\delta$ and each component of vector $w(t)$ be monotonically convergent. Then, $\forall x(0) \in \mathcal{X}$, $\lim_{t\to\infty} x(t, x(0), w(t)) = x_w$.

In particular, Assumption 6 ensures that x_w is an asymptotically stable solution of $\dot{x}(t) = \Phi(x(t), w)$.

Assumption 7 (Uniform-in-$\mathcal{W}_\delta$ Stability): Let $w(t) \equiv w \in \mathcal{W}_\delta$. Then, $\forall \lambda > 0$ there exists $\alpha(\lambda) > 0$ such that $\|x(0) - x_w\| \le \alpha(\lambda) \Rightarrow \|x(t, x(0), w) - x_w\| \le \lambda$, $\forall t \ge 0$, $\forall w \in \mathcal{W}_\delta$.

The aim of this paper is to design an RG, a discrete-time device which, based on the current state $x(t)$ and desired reference $r(t)$, generates the reference input $w(t)$ so as to satisfy the constraint (4) and minimize the tracking error. As depicted in Fig. 1, the RG can be seen as a reference filter which modifies the desired reference r whenever this, if directly supplied to (1), causes constraint violation. Since the filtering action requires a finite computational time τ, the RG operates in discrete-time in that it is applied every *RG period* T, $T \ge \tau$. The reference input $w(t)$ is generated on-line in a predictive manner: during the time interval $(t - \tau, t]$ a *virtual* reference input signal $\{w(t + \sigma)\}$, $\sigma \in (0, +\infty)$ is selected in such a way that the corresponding predicted evolution $c(t + \sigma, x(t), w(t + \sigma))$ lies within $\mathcal{C}_\delta$, $\forall \sigma > 0$. Then, according to a *receding horizon* strategy, the virtual signal is applied during the following interval $(t, t + T]$; at time $t + T$ a new selection is performed. For the sake of notational simplicity, we shall consider hereafter $\tau = 0$. However, a significant delay $\tau > 0$ can be considered in the following developments by suitable changes.

For reasons that will be clearer soon, we restrict our attention to the class of virtual constant reference input signals, introduced by [13], which are parameterized by the scalar β and defined by

$$\begin{cases} w(kT + \sigma, \beta) \equiv r(kT) + \beta[w((k-1)T) - r(kT)] \\ \qquad\qquad \triangleq w_\beta, \forall \sigma > 0, \; k \in N, \\ \qquad w(-T) = w_0 \end{cases} \tag{5}$$

where $N = \{0, 1, \cdots\}$. At each time kT a parameter $\beta(kT) \in R$, and the corresponding constant reference input $w_k \triangleq w_{\beta(kT)}$, are selected in accordance with the optimization criterion

$$\beta(kT) = \begin{cases} \arg\min_{\beta \ge 0} \beta^2 \\ \text{subj. to } c(kT + \sigma, x(kT), w(kT + \sigma, \beta)) \in \mathcal{C}_\delta, \\ \forall \sigma \in (0, +\infty). \end{cases} \tag{6}$$

and

$$w(t) \equiv w_k, \qquad \forall t \in (kT, (k+1)T].$$

Notice that by minimizing β^2 one attempts to minimize $\|w - r\|^2$ and therefore $\|y - r\|^2$. A parameter β, or a constant reference w, satisfying the constraints in (6) will be referred to as *admissible.*

Assumption 8 (Feasible Initial Condition): The initial state $x(0)$ is such that there exists at least one admissible virtual constant reference input $w_0 \in \mathcal{W}_\delta$.

For instance, Assumption 8 is satisfied for an equilibrium state $x(0) = x_{w_0}$ corresponding to $w_0 \in \mathcal{W}_\delta$.

III. Main Results

Lemma 1: Suppose that Assumptions 1–3 hold. Then, the function $X : \mathcal{W} \mapsto \mathcal{X}$ defined in (2) is continuous.

Proof: Consider a generic $w^* \in \mathcal{W}$. By contradiction, suppose $\exists \epsilon \ge 0$ such that $\forall i \in N$ there exists a reference input $w_i \in \mathcal{W}$, $\|w_i - w^*\| \le \frac{1}{i}$, and $\|x_{w_i} - x_{w^*}\| > \epsilon$, $x_{w_i} = X(w_i)$, $x_{w^*} = X(w^*)$. By Assumption 2, the sequence $\{x_{w_i}\} \subseteq \mathcal{X}$. By Assumption 1, $\mathcal{X}$ is compact, and hence there exists a subsequence $\{x_{w_j}\}$ converging to a point $\bar{x} \in \mathcal{X}$, with $\|\bar{x} - x_{w^*}\| > \epsilon$, or equivalently $\bar{x} \ne x_{w^*}$. Because of the continuity of Φ, $0 = \lim_{j\to\infty} \Phi(x_{w_j}, w_j) = \Phi(\bar{x}, w^*)$, which contradicts Assumption 2. □

The next proposition shows that, for constant desired reference trajectories, the RG yields a converging reference input.

Proposition 1: Suppose that $r(t) \equiv r$, $\forall t \ge 0$, and Assumptions 3 and 4 hold. Then there exists $\lim_{t\to\infty} w(t) \triangleq w_\infty \in \mathcal{W}_\delta$. In addition, each component of vector $w(t)$ is monotonically convergent.

Proof: If $w_0 = r$, then $\beta(kT) = 0$ is admissible, $\forall k \in N$. Therefore, $w(t) = r$, $\forall t > 0$, and $w_\infty = r$ (the RG behaves as an all-pass filter). Suppose $w_0 \ne r$. Since $\beta(kT) \ge 0$, $w_k = r + \frac{d_k}{\|w_0 - r\|}[w_0 - r]$, where $d_k \triangleq \|w_k - r\|$. By construction, at time $(k+1)T$, $\beta = 1$ is admissible, and hence $\beta((k+1)T) \le 1$. Then, $0 \le d_{k+1}^2 = \beta^2((k+1)T) d_k^2 \le d_k^2$, $\forall k \in N$, and hence there exists $d_\infty = \lim_{k\to\infty} d_k$. Consequently, $\lim_{t\to\infty} w(t) = w_\infty \triangleq r + \frac{d_\infty}{\|w_0 - r\|}[w_0 - r]$. By compactness of $\mathcal{W}_\delta$, $w_\infty \in \mathcal{W}_\delta$ follows. □

Next Lemma 2 and Proposition 2 show that w_∞ is the admissible reference input which is closest to r along the line segment $\rho w_0 + (1 - \rho) r$, $\rho \in [0, 1]$.

Lemma 2: Suppose that Assumptions 1–5 and 7 hold. Consider two reference inputs w_a, $w_b \in \mathcal{W}_\delta$, $w_a \ne w_b$. Let $x(kT) = x_{w_a} + \Delta x \in \mathcal{X}$, and let η such that $B(c_{w_a}, \eta) \subseteq \mathcal{C}$. Then there exists a $\bar{\gamma} > 0$, dependent on w_a and η, such that reference input $w_a + \gamma(w_b - w_a)$ is admissible for all $\|\Delta x\| \le \frac{1}{2}\alpha(\eta/2)$ and for all $0 \le \gamma \le \bar{\gamma}$.

Proof: Let $\alpha = \alpha(\eta/2)$ in accordance with Assumption 7. By continuity of the mapping $X(w)$ in w_a there exists a $\lambda = \lambda(w_a, \eta)$, $0 < \lambda < \|w_b - w_a\|$ such that, $\forall w \in \mathcal{W}_\delta$, $\|w_a - w\| \le \lambda \Rightarrow \|x_{w_a} - x_w\| \le \frac{\alpha}{2}$. Let $\bar{\gamma} \triangleq \frac{\lambda}{\|w_b - w_a\|}$ and γ such that $0 < \gamma \le \bar{\gamma}$; by Assumption 4, the reference input $w_\gamma \triangleq w_a + \gamma(w_b - w_a)$ lies within $\mathcal{W}_\delta$. By taking $\|\Delta x\| \le \frac{\alpha}{2}$, $\|x(t) - x_{w_\gamma}\| \le \|x_{w_a} - x_{w_\gamma}\| + \|\Delta x\| \le \alpha$, and by Assumption 7 $\|c(kT + \sigma, x(kT), w_\gamma) - c_{w_\gamma}\| = \|x(kT + \sigma, x(kT), w_\gamma) - x_{w_\gamma}\| \le \eta$, $\forall \sigma > 0$. Therefore, each reference w_γ is admissible at time kT. □

Proposition 2: Suppose that $r(t) \equiv r$, $\forall t \ge 0$, and Assumptions 1–8 hold. Then $\lim_{t\to\infty} w(t) = w_r \in \mathcal{W}_\delta$ with

$$w_r = \arg\min_{\rho \in [0,1]} \begin{cases} \|w - r\| \\ \text{subject to } w = r + \rho[w_0 - r] \in \mathcal{W}_\delta \end{cases} \tag{7}$$

where $w_0 \in \mathcal{W}_\delta$ is an admissible reference input at time $t = 0$.

Proof: By Proposition 1 there exists $\lim_{t\to\infty} w(t) = w_\infty \in \mathcal{W}_\delta$, and the convergence is component-by-component monotonic. Suppose by contradiction $w_\infty \neq w_r$. By Assumption 6, there exists a time t_0 such that $\|x(t_0, x(0), w(t_0)) - x_{w_\infty}\| \leq \alpha\left(\frac{\delta}{2}\right)$. Hence, by Lemma 2, there exists a constant $\bar{\gamma} > 0$ such that $w_\gamma \triangleq w_\infty + \gamma(w_r - w_\infty)$ is admissible at time t_0, $\forall\gamma$ such that $0 < \gamma \leq \bar{\gamma}$. Then, $\|w(t) - r\| \leq \|w_\gamma - r\|$. Since r, $w(t)$, w_γ, w_∞ are collinear, it follows that $\|w(t) - w_\infty\| = \|w(t) - w_\gamma\| + \|w_\gamma - w_\infty\| \geq \gamma\|w_r - w_\infty\| > 0$, $\forall t \geq t_0$, which contradicts the hypothesis $\lim_{t\to\infty} w(t) = w_\infty$. □

Lemma 3: Under the hypotheses of Proposition 2, there exists a stopping time t_s such that $w(t) = w_r$ for all $t \geq t_s$.

Proof: Since by Proposition 2 $\lim_{k\to\infty} w_k = w_r$, by Assumptions 6 and 7 there exists an index M such that $\|x(MT, x(0), w(MT)) - x_{w_r}\| < \alpha(\delta)$ which implies $\|c(M+\sigma, x(MT), w_r) - c_{w_r}\| = \|x(MT+\sigma, x(MT), w_r) - x_{w_r}\| \leq \delta$, $\forall\sigma \in R_+$ or, equivalently, that w_r is admissible at time $t_s \triangleq MT$. □

Next Theorem 1 summarizes the previous results.

Theorem 1: Suppose $r(t) \equiv r$, $\forall t \geq 0$, and Assumptions 1–8 hold. Then, after a finite time t_s the RG generates a constant reference input $w(t) \equiv w_r$, where w_r is given by (7). Consequently, (1) is asymptotically driven from $x(0)$ to x_{w_r} with no constraint violation.

Notice that when $r \in \mathcal{W}_\delta$, the RG has no effect on the asymptotic behavior of $y(t)$, which instead depends on the original tracking properties of the primal system (1).

A. Finite Constraint Horizon

The optimization criterion (6) requires that the constraint $c(kT + \sigma, x(kT), w_\beta) \in \mathcal{C}_\delta$ is checked for all $\sigma > 0$. In this section, we show that it suffices to verify this condition over a finite prediction horizon $(0, T_\infty]$.

Definition 1 (Constraint Horizon): The constraint horizon T_∞ is defined as the shortest prediction horizon such that $c(t+\sigma, x(t), w) \in \mathcal{C}_\delta$, $\forall\sigma > 0 \Leftrightarrow c(t+\sigma, x(t), w) \in \mathcal{C}_\delta$, $\forall 0 < \sigma \leq T_\infty$, $\forall x(t) \in \mathcal{X}$, $\forall w \in \mathcal{W}_\delta$.

In order to prove that such a T_∞ exists, we recall the following result [15, pp. 58–60] for time-invariant systems.

Result 1 (Variation of Solutions w.r.t. Initial Conditions and Parameters): Consider generic $x^*(0) \in \mathcal{X}$ and $w^* \in \mathcal{W}_\delta$. Let $\eta > 0$, $\eta \leq \delta$, and D_η the set of all c satisfying $x \in \mathcal{X}$, $w \in B(w^*, \eta) \subseteq \mathcal{W}$. Suppose we have Φ continuous and bounded on D_η. Then, there exists a $\gamma > 0$ such that for all $x(0)$, w satisfying $\|x(0) - x^*(0)\| < \gamma$, $\|w - w^*\| < \gamma$ the solution $x(t, x(0), w)$ exists over any bounded interval $[0, T^*]$, and as $(x(0), w) \to (x^*(0), w^*)$, $x(t, x(0), w) \to x(t, x^*(0), w^*)$ uniformly over $[0, T^*]$.

Note that Assumption 3 and compactness of $\mathcal{X}$ and $\mathcal{W}$ imply that $\Phi(x, w)$ is bounded on $\mathcal{X} \times \mathcal{W}$.

When $w(t) \equiv w$, Theorem 2 proves that, for a fixed scalar $\lambda > 0$, the state $x(t)$ converges to the ball $B(x_w, \lambda)$ in a finite time T which is not dependent of the initial state $x(0) \in \mathcal{X}$ and reference input $w \in \mathcal{W}_\delta$.

Theorem 2: Let Assumptions 1, 3, and 5–7 be satisfied. Then for all $\lambda > 0$ there exists a finite time $T(\lambda)$ such that $\forall c(0) = [x'(0) \quad w']' \in \mathcal{C}_\delta$

$$\|x(t, x(0), w) - x_w\| \leq \lambda, \qquad \forall t \geq T(\lambda). \tag{8}$$

Proof: By Assumption 6 it is immediate to show that (8) is verified for some $T(\lambda, x(0), w)$. Suppose by contradiction that $\sup_{c(0)\in\mathcal{C}_\delta} T(\lambda, w, x(0)) = +\infty$. Then, there exists a sequence $\{c_i(0)\}_{i=0}^{\infty}$ such that $\lim_{i\to\infty} T(\lambda, x_i(0), w_i) = +\infty$. By compactness of $\mathcal{C}_\delta$, there exists a subsequence $\{c_j(0)\}_{j=0}^{\infty}$ converging to a point $c^*(0) \in \mathcal{C}_\delta$. By Assumption 7, there exists an $\alpha = \alpha(\lambda)$, independent of w, such that $\|x(t_0) - x_w\| \leq \alpha \Leftrightarrow \|x(t, x(t_0), w) - x_w\| \leq \lambda$, $\forall t \geq t_0$. Let $w \in \mathcal{W}_\delta$ such that $\|x_w - x_{w^*}\| < \alpha/3$, $T^* \triangleq T(\alpha/3, x^*(0), w^*)$ and define $x(t) \triangleq x(t, x(0), w)$, $x^*(t) \triangleq x(t, x^*(0), w^*)$. By Result 1, setting $\eta \triangleq \delta/2$, there exists a $\gamma = \gamma(T^*, \alpha/3)$ such that $\|x(0) - x^*(0)\| < \gamma$, $\|w - w^*\| < \gamma \Rightarrow \|x(t) - x^*(t)\| < \frac{\alpha}{3}$, $\forall t \in [0, T^*]$. Then, $\|x(T^*) - x_w\| \leq \|x(T^*) - x^*(T)\| + \|x^*(T) - x_{w^*}\| + \|x_{w^*} - x_w\| \leq \frac{\alpha}{3} + \frac{\alpha}{3} + \frac{\alpha}{3} \leq \alpha$, which implies $\|x(t) - x_w\| \leq \lambda$ for all $t \geq T^*$. Hence, $T(\lambda, x(0), w) \leq T^*$. In conclusion, there exists an index j_0 such that, $\forall j \geq j_0$, $\|x_j(0) - x^*(0)\| \leq \gamma$, $\|w_j - w^*\| \leq \gamma$, $\|x_{w_j} - x_{w^*}\| \leq \frac{\alpha}{3}$, and $T(\lambda, x_j(0), w_j) \leq T^*$. This contradicts the assumption $\lim_{j\to\infty} T(\lambda, x_j(0), w_j) = +\infty$. □

By (3) and Assumption 4, Theorem 2 proves that T_∞ exists and satisfies the inequality $T_\infty \leq T(\delta)$.

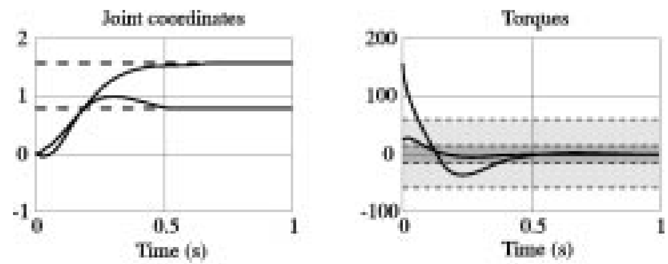

Fig. 3. Response without RG.

IV. Computations

In order to implement the RG described in the previous sections, the optimization (6) is solved by using a bisection algorithm over the interval $[0, 1]$. Testing the admissibility of a given β requires the numerical integration of (1) from initial state $x(kT)$. The fulfillment of the constraints $c(kT+\sigma, x(kT), w_\beta) \in \mathcal{C}$ is checked at integration steps. Let N denote the number of parameters β which can be evaluated during one RG period T. For a given T, N is determined by both the desired integration accuracy and the constraint horizon T_∞. Since admissibility of $\beta = 0$ is always tried first, the optimal $\beta(kT)$ is evaluated with a worst case precision of $2^{-(N-1)}$. Because $\mathcal{C}$ is generic and the plant is nonlinear, no convexity properties of the set of admissible β can be invoked. Then, the adopted bisection algorithm only provides local minima. However, this does not affect the convergence results proved in Section III. In fact, if at time t after N evaluations no admissible $\beta < 1$ is found, $\beta(t) = 1$ is selected, which is admissible by construction. Consequently, Proposition 1 still holds. By Lemma 2, an admissible interval $[1 - \bar{\gamma}, 1]$ can be found after a finite time. For N large enough, the bisection method can therefore find admissible $\beta < 1$, and hence the proof of Proposition 2 holds. Since $\beta = 0$ is always tested, Lemma 3 and Theorem 1 hold as well. It is clear that if global minimization procedures were adopted in selecting $\beta(t)$, better tracking properties might be achieved at the expense of an increased computational effort.

V. An Example

The performance of the RG presented in the previous sections has been tested by computer simulations on a two-link robot moving on a horizontal plane.

A. Nonlinear Model

Each joint is equipped with the following: a motor for providing input torque and encoders and tachometers for measuring the joint positions θ_1, θ_2 and velocities $\dot{\theta}_1$, $\dot{\theta}_2$. By using Lagrangian equations, and by setting

$$x = \begin{bmatrix}\theta_1\\ \dot{\theta}_1\\ \theta_2\\ \dot{\theta}_2\end{bmatrix}, \qquad y = \begin{bmatrix}\theta_1\\ \theta_2\end{bmatrix}, \qquad \mathcal{T} = \begin{bmatrix}\mathcal{T}_1\\ \mathcal{T}_2\end{bmatrix}, \qquad w = \begin{bmatrix}\theta_{1d}\\ \theta_{2d}\end{bmatrix}$$

where θ_{1d}, θ_{2d} denote the desired values for joint positions and $\mathcal{T}_1, \mathcal{T}_2$ the motor torques, the dynamic model of the robot can be expressed as

$$H(x)\begin{bmatrix}\dot{x_2}\\ \dot{x_4}\end{bmatrix} + C(x)\begin{bmatrix}x_2\\ x_4\end{bmatrix} = \mathcal{T} \tag{9}$$

where

$$\begin{aligned} H(x) &= \begin{bmatrix}h_{11} & h_{12}\\ h_{12} & h_{22}\end{bmatrix}\\ h_{11} &= m_1 l_{c_1}^2 + I_1 + m_2\left[l_1^2 + l_{c_2}^2 + 2l_1 l_{c_2}\cos(x_3)\right] + I_2\\ h_{12} &= m_2 l_1 l_{c_2}\cos(x_3) + m_2 l_{c_2}^2 + I_2\\ h_{22} &= m_2 l_{c_2}^2 + I_2\\ C(x) &= m_2 l_1 l_{c_2}\sin(x_3)\begin{bmatrix}-x_4 & -x_2 - x_4\\ x_2 & 0\end{bmatrix}. \end{aligned}$$

Individual joint PD controllers

$$\mathcal{T} = -\begin{bmatrix}k_{p1}(x_1 - w_1) + k_{d1}x_2\\ k_{p2}(x_3 - w_2) + k_{d2}x_4\end{bmatrix} \tag{10}$$

provide reference tracking. As a general rule, to design controllers to be used in connection with an RG, in order to maximize the properties of tracking one should try to select a primal controller which provides a fast closed-loop response (1). Usually this corresponds to large violations of the constraints, which therefore can be enforced by inserting an RG. In order to show that system (9) and (10) fulfills the required assumptions, consider the following function:

$$\begin{aligned} V(x) &= \frac{1}{2}\begin{bmatrix}x_2\\ x_4\end{bmatrix}' H(x)\begin{bmatrix}x_2\\ x_4\end{bmatrix}\\ &\quad + \frac{1}{2}\begin{bmatrix}w_1 - x_1\\ w_2 - x_3\end{bmatrix}' K_p\begin{bmatrix}w_1 - x_1\\ w_2 - x_3\end{bmatrix}\\ K_p &= \begin{bmatrix}k_{p1} & 0\\ 0 & k_{p2}\end{bmatrix} > 0 \end{aligned}$$

which is a Lyapunov function for (9) and (10) [16]. Since its derivative along the trajectories of the system is

$$\begin{aligned} \dot{V}(x) &= -\begin{bmatrix}x_2\\ x_4\end{bmatrix}' K_d\begin{bmatrix}x_2\\ x_4\end{bmatrix} \le 0\\ K_d &= \begin{bmatrix}k_{d1} & 0\\ 0 & k_{d2}\end{bmatrix} > 0 \end{aligned}$$

and $V(x) = 0$, iff $x = [w_1\ 0\ w_2\ 0]'$ Assumption 2 is satisfied. Moreover, in practice the reference input $w(t)$ is expressed by a finite numerical precision; therefore, if $w(t)$ monotonically tends toward w, after a finite time $w(t) \equiv w$, and hence Assumption 6 is verified as well. The fulfillment of Assumption 7 is proved as follows. By contradiction, suppose that there exists a $\lambda > 0$ such that $\forall\alpha > 0$, there exists w and t_w with $\|x(0) - x_w\| \le \alpha$ and $\|x(t_w, x(0), w) - x_w\| > \lambda$. Since $\gamma_1 I \le H(x) \le \gamma_2 I$ for some positive γ_1, γ_2, by denoting by $\lambda_m(K_p)$ and $\lambda_M(K_p)$, respectively, the minimum and maximum eigenvalue of K_p, and by setting $\gamma_3 = \min\{\lambda_m(K_p), \gamma_1\}$, $\gamma_4 = \max\{\lambda_M(K_p), \gamma_3\}$, it follows that $\|x(t_w, x(0), w) - x_w\| \le \frac{2}{\gamma_3}V(x(t_w)) \le \frac{2}{\gamma_3}V(x(0)) \le \frac{\gamma_4}{\gamma_3}\alpha$ for any arbitrary positive α, a contradiction.

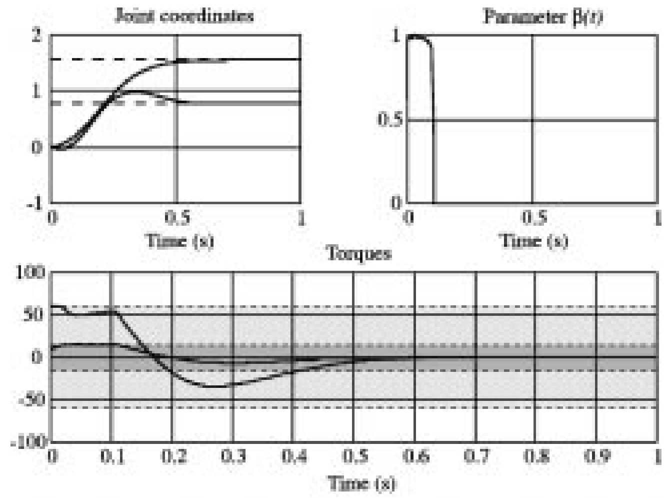

Fig. 4. Response with RG ($T = 0.001$ s).

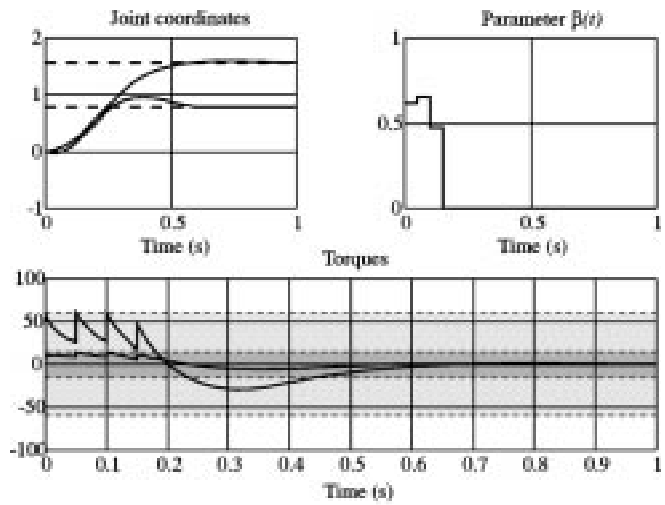

Fig. 5. Response with RG ($T = 0.05$ s).

B. Simulations

Simulations have been carried out with the system parameters reported in [17]. On-line optimization has been performed by using the bisection method mentioned in Section IV, and a standard fourth-order Runge–Kutta method with adaptive stepsize control has been adopted for numerical integration. Fig. 3 shows the closed-loop system behavior for a constant desired reference $r_1(t) \equiv \frac{\pi}{2}$, $r_2(t) \equiv \frac{\pi}{4}$, $t \in R_+$, in the absence of the RG. In order to bound the input torques within the range

$$|\mathcal{T}_1| \le 60\ \text{Nm}, \qquad |\mathcal{T}_2| \le 15\ \text{Nm} \tag{11}$$

which has been represented by shadowed areas in Fig. 3, the RG is applied. The initial conditions $\theta_1(0) = \theta_2(0) = 0$, $\dot{\theta}_1(0) = \dot{\theta}_2(0) = 0$, and $w_0 = [0\ \ 0]'$ satisfy Assumption 8. An RG period $T = 0.001$ s, a constraint horizon $T_\infty = 0.4$ s, $N = 10$ admissibility evaluations per period, and $\delta \approx 0$ are selected as parameters of the RG. The set $\mathcal{C}$ is determined by (11) and by further limiting the state and reference input in such a manner that only constraints (11) become active. The resulting trajectories are depicted in Fig. 4. In Fig. 5, the RG period is increased to $T = .05$ s, which causes a transient chatter on the input torques. The further constraint

$$|\theta_1 - \theta_2| \le 0.2\ \text{rad}$$

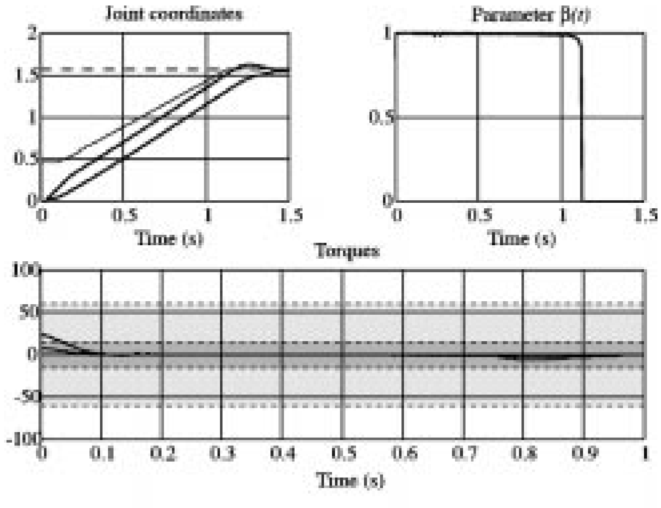

Fig. 6. Response with RG, torque constraints, and the constraint $|\theta_1 - \theta_2| \leq 0.2$ rad. The generated reference input is depicted (thin line) together with the joint trajectories (thick lines).

is taken into account by the RG, and the related simulated trajectories are depicted in Fig 6 with $r_1(t) = r_2(t) \equiv \frac{\pi}{4}$, $T = 0.001$ s. The slight chatter on the β and torque trajectories is caused by the approximations involved in the optimization procedure described in Section IV. The results described above were obtained on a 486 DX2/66 personal computer, using Matlab 4.2 and Simulink 1.3 with embedded C code. The CPU time required by the RG to select a single $\beta(t)$ ranged between 7 and 18 ms.

VI. Conclusion

For a broad class of nonlinear continuous-time systems and input/state hard constraints, this paper has addressed the RG problem, viz. the one of filtering the desired reference trajectory in such a way that a nonlinear primal compensated control system can operate in a stable way with satisfactory tracking performance and no constraint violation. The resulting computational burden turns out to be moderate and the related operations executable with current computing hardware. Alternatively, in some applications, the trajectory generated by the RG can be computed off-line and stored for subsequent task executions. Future developments of this research will be addressed toward numerical criteria for the determination of the constraint horizon and to an independent parameterization of the components of the reference.

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Geometric/Asymptotic Properties of Adaptive Nonlinear Systems with Partial Excitation

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***Abstract*—In this paper we continue the study of geometric/asymptotic properties of adaptive nonlinear systems. The long-standing question of whether the parameter estimates converge to *stabilizing* values—stabilizing if used in a *nonadaptive* controller—is addressed in the general set-point regulation case. The key quantifier of excitation in an adaptive system is the rank r of the regressor matrix at the resulting equilibrium. Our earlier paper showed that when either $r = 0$ or $r = p$ (where p is the number of uncertain parameters), the set of initial conditions leading to destabilizing estimates is of *measure zero*. Intuition suggests the same for the intermediate case $0 < r < p$ studied in this paper. We present a surprising result: the set of initial conditions leading to destabilizing estimates can have *positive measure*. We present results for the backstepping design with tuning functions; the same results can be established for other Lyapunov-based adaptive designs.**

***Index Terms*—Adaptive nonlinear control, invariant manifold, partial excitation.**

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