Let (formulation (II.2))

$$\begin{aligned}\hat{Q}(s)v_3 = {} & x_0\left(-\frac{317}{85}+\frac{91}{85}s-\frac{27}{85}s^2+s^3\right)\\ & +x_1\left(-\frac{116}{85}-\frac{46}{85}s+\frac{122}{85}s^2\right)\\ & +x_2\left(-\frac{163}{85}+\frac{157}{85}s-\frac{122}{85}s^2\right)\\ & +x_3\left(-\frac{134}{85}+\frac{197}{85}s-s^2\right)\\ = {} & s^3+3s^2+4s+2\end{aligned}$$

we solve

$$v_3=\left[0,0,1,-\frac{34663}{34686},-\frac{52369}{5781},\frac{143081}{17343}\right]^t.$$

We then normalize $M=[v_1,v_2,v_3]$ to $\left[\begin{smallmatrix}I\\K\end{smallmatrix}\right]$ and obtain

$$K=\begin{bmatrix}-\frac{20216}{17343} & \frac{20216}{17343} & -\frac{34663}{34686}\\ \frac{69860}{5781} & -\frac{75641}{5781} & -\frac{52369}{5781}\\ -\frac{220870}{17343} & \frac{238213}{17343} & \frac{143081}{17343}\end{bmatrix}.$$

So the output feedback $u=Ky$ assigns the closed loop characteristic polynomial to $(s+1)^2(s^2+2s+2)^2$.

REFERENCES

[1] R. W. Brockett and C. I. Byrnes, "Multivariable Nyquist criteria, root loci and pole placement: A geometric viewpoint," *IEEE Trans. Autom. Control*, vol. AC-26, no. 1, pp. 271–284, Feb. 1981.

[2] C. I. Byrnes, "Pole assignment by output feedback," in *Three Decades of Mathematical System Theory*, ser. Lecture Notes in Control and Information Sciences # 135, H. Nijmeijer and J. M. Schumacher, Eds. : Springer–Verlag, 1989, pp. 31–78.

[3] C.-T. Chen, *Linear System Theory and Design*. New York: Holt, Rinehart & Winston, 1984.

[4] E. J. Davison and S. H. Wang, "On pole assignment in linear multivariable systems using output feedback," *IEEE Trans. Autom. Control*, vol. AC-20, no. 4, pp. 516–518, Aug. 1975.

[5] A. Eremenko and A. Gabrielov, "Pole placement static output feedback for generic linear systems," *SIAM J. Control Optimiz.*, vol. 41, pp. 303–312, 2002.

[6] G. D. Forney, "Minimal bases of rational vector spaces, with applications to multivariable linear systems," *SIAM J. Control Optimiz.*, vol. 13, pp. 493–520, 1975.

[7] U. Helmke, J. Rosenthal, and X. Wang, "Output feedback pole assignment for transfer functions with symmetries," *SIAM J. Control Optimiz.*, vol. 45, pp. 1898–1914, 2006.

[8] R. Hermann and C. F. Martin, "Applications of algebraic geometry to system theory part I," *IEEE Trans. Autom. Control*, vol. AC-22, no. 1, pp. 19–25, Jan. 1977.

[9] M. Kim, J. Rosenthal, and X. Wang, "Pole placement and matrix extension problems: A common point of view," *SIAM J. Control Optimiz.*, vol. 42, pp. 2078–2093, 2004.

[10] H. Kimura, "Pole assignment by gain output feedback," *IEEE Trans. Autom. Control*, vol. AC-20, no. 4, pp. 509–516, Aug. 1975.

[11] U. Konigorski, "Pole placement by parametric output feedback," *Syst, & Control Lett,*, vol. 61, pp. 292–297, 2012.

[12] J. Leventides and N. Karcanias, "Global asymptotic linearisation of the pole placement map: A closed form solution for the constant output feedback problem," *Automatica*, vol. 31, pp. 1303–1309, 1995.

[13] C. F. Martin and R. Hermann, "Applications of algebraic geometry to systems theory. II. Feedback and pole placement for linear Hamiltonian systems," *Proc. IEEE*, vol. 65, no. 6, pp. 841–848, Jun. 1977.

[14] C. F. Martin and R. Hermann, "Applications of algebraic geometry to system theory: The McMillan degree and Kronecker indices as topological and holomorphic invariants," *SIAM J. Control Optimiz.*, vol. 16, pp. 743–755, 1978.

[15] J. Rosenthal, "Tuning natural frequencies by output feedback," in *Computation and Control*, ser. Proc. Bozeman Conference, K. Bowers and J. Lund, Eds. Cambridge, MA: Birkhäuser, 1989, pp. 277–282.

[16] J. Rosenthal, J. M. Schumacher, and J. C. Willems, "Generic eigenvalue assignment by memoryless real output feedback," *Syst. & Control Lett.*, vol. 26, pp. 253–260, 1995.

[17] F. Sottile, "Rational curves on Grassmannians: Systems theory, reality transversality," Advances in algebraic geometry motivated by physics (Lowell, MA, 2000) vol. 276, Contemp. Math., pp. 9–42, Amer. Math. Soc.. Providence, RI, 2001.

[18] J. C. Willems and W. H. Hesselink, "Generic properties of the pole placement problem," in *Proc. IFAC*, Helsinki, Finland, 1978, pp. 1725–1729.

[19] X. Wang, "Pole placement by static output feedback," *J. Math. Systems, Estimation, Control*, vol. 2, pp. 205–218, 1992.

[20] X. Wang, "Grassmannian, central projection and output feedback pole assignment of linear systems," *IEEE Trans. Automat. Control*, vol. AC-41, no. 6, pp. 786–794, Jun. 1996.

Stability and Invariance Analysis of Uncertain Discrete-Time Piecewise Affine Systems

Matteo Rubagotti, Sergio Trimboli, and Alberto Bemporad

***Abstract*—This note proposes a method to analyze uniform asymptotic stability and uniform ultimate boundedness of uncertain piecewise affine systems whose dynamics are only defined in a bounded and possibly non-invariant set $\mathcal{X}$ of states. The approach relies on introducing fake dynamics outside $\mathcal{X}$ and on synthesizing a piecewise affine and possibly discontinuous Lyapunov function via linear programming. The existence of such a function proves stability properties of the original system and allows the determination of a region of attraction contained in $\mathcal{X}$. The procedure is particularly useful in practical applications for analyzing the stability of piecewise affine control systems that are only defined over a bounded subset $\mathcal{X}$ of the state space, and to determine whether for a given set of initial conditions the trajectories of the state vector remain within the domain $\mathcal{X}$.**

***Index Terms*—Model predictive control (MPC), piecewise affine (PWA), piecewise quadratic (PWQ).**

I. INTRODUCTION

In the last decade the interest in studying the dynamical properties of piecewise affine (PWA) systems has increased considerably, due to their powerful modeling capabilities. Discrete-time PWA models are a special class of hybrid systems that can represent combinations of finite automata and linear dynamics, are a good approximation of nonlinear

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systems [1], and are equivalent to hybrid systems in mixed logical dynamical form [2], [3].

Stability analysis tools for PWA systems are useful to describe the properties of autonomous hybrid systems, or to check a posteriori the stability of a given closed-loop system [4], [5]. In particular, stability analysis becomes fundamental when a PWA control law is synthesized without a-priori guarantees of closed-loop stability; an example are explicit model predictive control (MPC) laws designed in applications with short prediction horizons and without terminal constraints to limit the number of partitions [6]–[8], or approximated in order to reduce their complexity [9], [10].

The most widely used methods for stability analysis of discrete-time PWA systems are based on piecewise quadratic (PWQ) Lyapunov functions [11] and require the solution of semi-definite programming problems. As highlighted in [12], the search for a PWQ Lyapunov function can be overly conservative, even with the use of the so-called S-procedure (see, e.g., [13]). A valid alternative are PWA Lyapunov functions, computed by linear programming (LP) [12], [14]. Other types of Lyapunov functions can be used for the same purpose, such as piecewise polynomial Lyapunov functions [15].

The approaches for PWA stability analysis recalled above deal with *deterministic* systems. However, in applications it is often important to determine if the state converges to the origin (or to a terminal set) despite *parametric uncertainties* and/or *external disturbances*. Some classical results on stability analysis and control of uncertain linear parameter varying and switched linear systems appeared in [16]–[19] and [20, Chap. 7].

In analyzing the stability of PWA systems, it is always assumed that the set $\mathcal{X}$ of states in which the PWA dynamics are defined is positively invariant, as the notion of stability has no practical relevance if the state trajectory exits the domain of definition of the dynamics [21]. However, often the PWA system to be analyzed is defined in a set $\mathcal{X}$ that may not be invariant, meaning that, for certain initial conditions, the corresponding state trajectories are not defined for all time steps. A possible approach to tackle the possible non-invariance of $\mathcal{X}$ is to perform a reachability analysis to find the maximum positively invariant subset of $\mathcal{X}$, using a recursive procedure (see [20, Ch. 4–5], [22], [23], and the references therein). Unfortunately this can lead to very involved solutions, due to the exponential complexity of reachability analysis of PWA systems, and in some cases searching the maximum invariant set is even an undecidable problem.

This technical note proposes a stability analysis framework for (possibly discontinuous) discrete-time PWA systems that are affected by additive disturbances bounded within a polytopic set (a similar framework was proposed in [24] to handle the case of parametric uncertainties). A discontinuous PWA Lyapunov function is synthesized via linear programming (LP) either to determine if the state converges to the origin or to find a terminal set where the state is ultimately bounded. As we assume that the PWA dynamics are defined in a closed polytope $\mathcal{X}$ of the state space which *may not be invariant*, by artificially extending the systems dynamics outside $\mathcal{X}$ the proposed method can also determine an invariant subset of $\mathcal{X}$ for the original PWA system. In this case the attractiveness of the origin (or that of the terminal set) is determined with respect to such a region of attraction. A preliminary version of this technical note focusing only on asymptotic stability of deterministic PWA systems has appeared in [25]. A numerical implementation of the overall stability analysis proposed in this work can be found in the MOBY-DIC Toolbox for MATLAB, described in [26].

II. Basic Notation and Definitions

Let $\mathbb{R}$, $\mathbb{Z}$, $\mathbb{Z}_{\geq 0}$, and $\mathbb{Z}_{>0}$ denote the sets of reals, integers, non-negative integers, and positive integers, respectively. Given $a \in \mathbb{R}$, the ceil function $\lceil a \rceil$ is defined as the smallest $b \in \mathbb{Z}$ such that $a \leq b$. Given a vector $v \in \mathbb{R}^n$, let $\|v\|$ denote any vector norm, and let $\|v\|_W \triangleq \|Wv\|_\infty$, where $W \in \mathbb{R}^{n\times n}$ is a full rank diagonal matrix. Given two matrices $A_1 \in \mathbb{R}^{m_1\times n}$, $A_2 \in \mathbb{R}^{m_2\times n}$, $[A_1; A_2]$ denotes the matrix $[A_1' A_2']' \in \mathbb{R}^{(m_1+m_2)\times n}$. Given a discrete-time signal $d : \mathbb{Z}_{\geq 0} \to \mathbb{R}^p$, the sequence of the values of d from time zero to the k-th instant is denoted by $\mathbf{d}_{[k]}$. Given a set $\mathcal{A} \subseteq \mathbb{R}^n$, its interior is denoted by $int(\mathcal{A})$, its closure by $\bar{\mathcal{A}}$, and its convex hull by $conv(\mathcal{A})$. Given a finite number of sets $\mathcal{A}_i$, $i \in \mathcal{I}_a = \{1, \ldots, n_a\}$, we say that $\{\mathcal{A}_i\}$ is a *partition* of $\mathcal{A}$ if $int(\mathcal{A}_i) \neq \emptyset$, $int(\mathcal{A}_i) \cap int(\mathcal{A}_j) = \emptyset$, $\forall i, j \in \mathcal{I}_a$ with $i \neq j$, and $\bigcup_{i=1}^{n_a} \mathcal{A}_i = \mathcal{A}$. If $\{\mathcal{A}_i\}$ is a partition with $\mathcal{A}_i \cap \mathcal{A}_j = \emptyset$ $\forall i, j \in \mathcal{I}_a$ with $i \neq j$, it is denoted as *strict partition*. A *polyhedron* is a set given by the intersection of a finite number of (closed or open) half-spaces. A *polytope* $\mathcal{A}$ is a bounded polyhedron, and the set of the vertices of its closure $\bar{\mathcal{A}}$ is denoted by $vert(\bar{\mathcal{A}})$.

Consider a discrete-time nonlinear system

$$x(k+1) = \varphi\left(x(k), d(k)\right) \tag{1}$$

where $x \in \mathcal{X} \subseteq \mathbb{R}^n$ is the state vector, the input $d \in \mathcal{D} \subset \mathbb{R}^p$ collects both model uncertainties and exogenous disturbances, and both $\mathcal{X}$ and $\mathcal{D}$ are compact sets with $0 \in int(\mathcal{X})$, $0 \in \mathcal{D}$.

Definition 1: For system (1) *the one-step reachable set* from a set $\mathcal{A} \subseteq \mathcal{X}$ is $\mathcal{R}(\mathcal{A}) \triangleq \{y \in \mathbb{R}^n : y = \varphi(x, d), d \in \mathcal{D}, x \in \mathcal{A}\}$.

Definition 2: A set $\mathcal{B} \subseteq \mathcal{X}$ is called *robustly positively invariant (RPI)* with respect to dynamics (1) if, for all $d \in \mathcal{D}$, $\mathcal{R}(\mathcal{B}) \subseteq \mathcal{B}$.

Note that the set $\mathcal{X}$ is not assumed to be RPI with respect to dynamics (1), so that some trajectories may possibly leave $\mathcal{X}$ and be therefore defined only on a finite time interval of time $[0, k_{max}]$.

Definition 3: Consider dynamics (1) and a RPI set $\mathcal{B} \subseteq \mathcal{X}$ with $0 \in int(\mathcal{B})$ and $\varphi(0, d) = 0$ for all $d \in \mathcal{D}$. System (1) is *uniformly asymptotically stable* in $\mathcal{P}(\text{UAS}(\mathcal{P}))$ if there exists a $\mathcal{KL}$-function[1] ϕ such that, for all the initial conditions $x(0) \in \mathcal{P}$ and for all the sequences $\mathbf{d}_{[k]}$ with $d(i) \in \mathcal{D}$, $i = 0, \ldots, k$, $\|x(k)\| \leq \phi(\|x(0)\|, k)$, for all $k \in \mathbb{Z}_{\geq 0}$.

Definition 4: Given two RPI sets $\mathcal{P} \subseteq \mathcal{X}$ and $\mathcal{F} \subseteq \mathcal{P}$, with $0 \in int(\mathcal{F})$, system (1) is *uniformly ultimately bounded* from $\mathcal{P}$ to $\mathcal{F}$ $(\text{UUB}(\mathcal{P}, \mathcal{F}))$ if for all $a > 0$ there exists $T(a) > 0$ such that, for every $x(0) \in \mathcal{P}$ with $\|x(0)\| \leq a$, $x(T) \in \mathcal{F}$ for all the sequences $\mathbf{d}_{[T]}$ with $d(k) \in \mathcal{D}$, $k = 0, \ldots, T$.

III. Stability Analysis Problem

Given a closed polytope $\mathcal{X} \subset \mathbb{R}^n$ that includes the origin in its interior, consider a partition $\{\mathcal{X}_i\}$ of $\mathcal{X}$ that consists of a finite number s of polytopes

$$\mathcal{X}_i \triangleq \left\{x : H_i^1 x \leq h_i^1, H_i^2 x < h_i^2\right\}, \quad i \in \mathcal{I} \triangleq \{1, \ldots, s\} \tag{2}$$

where $H_i^1 \in \mathbb{R}^{q_{i,1}\times n}$, $H_i^2 \in \mathbb{R}^{q_{i,2}\times n}$ are constant matrices with $q_{i,1}$, $q_{i,2} \in \mathbb{Z}_{>0}$, $i \in \mathcal{I}$, and $h_i^1 \in \mathbb{R}^{q_{i,1}}$ and $h_i^2 \in \mathbb{R}^{q_{i,2}}$ are constant vectors, and assume that $\{\mathcal{X}_i\}$ is a *strict* partition. The closure $\bar{\mathcal{X}}_i$ of $\mathcal{X}_i$ is denoted by $\bar{\mathcal{X}}_i = \{x : H_i x \leq h_i\}$, with $H_i = [H_i^1; H_i^2] \in \mathbb{R}^{q_i\times n}$ and $h_i = [h_i^1; h_i^2] \in \mathbb{R}^{q_i}$. The subset of indices $\mathcal{I}_0$ is defined as $\mathcal{I}_0 \triangleq \{i \in \mathcal{I} : 0 \in \bar{\mathcal{X}}_i\}$. Without loss of generality, we assume that $0 \in vert(\bar{\mathcal{X}}_i)$ for all $i \in \mathcal{I}_0$, which is necessary when PWA Lyapunov functions are considered. If this property does not hold for the given partition $\{\mathcal{X}_i\}$, it is always possible to obtain it by further partitioning the regions whose indices belong to $\mathcal{I}_0$.

Consider a closed polytope $\mathcal{D} = \{d \in \mathbb{R}^p : \tilde{H} d \leq \tilde{h}\}$, with $\tilde{H} \in \mathbb{R}^{\eta\times p}$, and $\tilde{h} \in \mathbb{R}^\eta$, $p, \eta \in \mathbb{Z}_{>0}$, and $0 \in \mathcal{D}$. The vertices of $\mathcal{D}$ are denoted by $d_1, \ldots, d_\eta \in \mathbb{R}^p$, and, since $\mathcal{D}$ is compact by definition, $\mathcal{D} = conv(d_1, \ldots, d_\eta)$.

[1]See, e.g., Section II.A in [27] for a formal definition of $\mathcal{KL}$-functions.

Consider the autonomous discrete-time uncertain PWA system

$$x(k+1) = A_i x(k) + a_i + E_i d(k) \text{ if } x(k) \in \mathcal{X}_i \tag{3}$$

with $x(k) \in \mathcal{X}$, $d(k) \in \mathcal{D}$, $A_i \in \mathbb{R}^{n\times n}$, $a_i \in \mathbb{R}^n$, $E_i \in \mathbb{R}^{n\times p}$, and $k \in \mathbb{Z}_{\geq 0}$. Note that the dynamics (3) are not required to be continuous on the boundaries of the sets $\mathcal{X}_i$.

A. Problem Statement

Given the uncertain PWA system (3), for which $\mathcal{X}$ is not necessarily a RPI set, prove the properties of stability and convergence to the origin (uniform asymptotic stability or uniform ultimate boundedness) with respect to a RPI set $\mathcal{P} \subseteq \mathcal{X}$. In case of ultimate boundedness, find another RPI set $\mathcal{F}$ including the origin, where the state is driven in finite time.

Nominal stability analysis is considered a special case of the problem when $\mathcal{D} = \{0\}$ or $E_i = 0$ for all $i \in \mathcal{I}$.

IV. Extended System and Reachability Analysis

Consider the one-step reachable set $\mathcal{R}(\mathcal{X})$ from $\mathcal{X}$, $\mathcal{R}(\mathcal{X}) = \{A_i x + a_i + E_i d : d \in \mathcal{D}, x \in \mathcal{X}_i, i \in \mathcal{I}\}$, and define the closed set

$$\mathcal{R}_{\cup}(\mathcal{X}) \triangleq \overline{\mathcal{R}(\mathcal{X})} \cup \mathcal{X}. \tag{4}$$

The closure $\overline{\mathcal{R}(\mathcal{X})}$ of $\mathcal{R}(\mathcal{X})$ can be computed as the union of the one-step reachable sets from all the sets $\bar{\mathcal{X}}_i$, namely $\mathcal{R}(\bar{\mathcal{X}}_i) \triangleq \{A_i x + a_i + E_i d, d \in \mathcal{D}, x \in \bar{\mathcal{X}}_i\}$. By relying on the results in [20, Chap. 6], we can compute the convex sets $\mathcal{R}(\bar{\mathcal{X}}_i)$ as

$$\mathcal{R}(\bar{\mathcal{X}}_i) = conv(A_i v_{i,h} + a_i + E_i d_\mu, \mu = 1, \ldots, \eta, h = 1, \ldots, m_i) \tag{5}$$

where $v_{i,h}$ represents each of the m_i vertices of $\bar{\mathcal{X}}_i$, to get

$$\mathcal{R}_{\cup}(\mathcal{X}) = \bigcup_{i=1}^{s} \left(\mathcal{R}(\bar{\mathcal{X}}_i)\right) \cup \mathcal{X}. \tag{6}$$

As the dynamics (3) are not defined outside $\mathcal{X}$, the proposed strategy consists of defining *fake dynamics* on $\mathcal{R}_{\cup}(\mathcal{X}) \setminus \mathcal{X}$. Let $\mathcal{X}_H \triangleq conv(\mathcal{R}_{\cup}(\mathcal{X}))$, and consider the dynamics

$$x(k+1) = \rho x(k), \text{ if } x(k) \in \mathcal{X}_E \triangleq \mathcal{X}_H \setminus \mathcal{X} \tag{7}$$

where $\rho \in [0, 1)$ is an adjustable parameter of the proposed approach. The region $\mathcal{X}_E$, when $\mathcal{X}_E \neq \emptyset$, can be divided into convex polyhedral regions as in [28, Th. 3]. As a result, new regions $\mathcal{X}_i$, $i = s+1, \ldots, \tilde{s}$, are created, that together with the original $\mathcal{X}_i$ with $i \in \mathcal{I}$ define a strict partition $\{\mathcal{X}_i\}$ of $\mathcal{X}_H$, with $i \in \tilde{\mathcal{I}} \triangleq \{1, \ldots, \tilde{s}\}$. The dynamics of the extended system on $\mathcal{X}_H$ are

$$x(k+1) = \begin{cases} A_i x(k) + a_i + E_i d(k) & \text{if } x(k) \in \mathcal{X}_i,\ i \in \mathcal{I} \\ \rho x(k) & \text{if } x(k) \in \mathcal{X}_i,\ i \in \tilde{\mathcal{I}} \setminus \mathcal{I} \end{cases} \tag{8}$$

For convenience, let $A_i \triangleq \rho I$, $a_i \triangleq 0$, and $E_i \triangleq 0$, for $i \in \tilde{\mathcal{I}} \setminus \mathcal{I}$. Also, notice that, if $\mathcal{X}_E = \emptyset$, then $\tilde{\mathcal{I}} \setminus \mathcal{I} = \emptyset$, and no additional dynamics are introduced.

Lemma 1: The set $\mathcal{X}_H$ is RPI for the extended dynamics (8).

Proof: See Appendix. □

The definitions of $\mathcal{X}_H$ and of the dynamics in $\mathcal{X}_E$ as in (7) are simplistic, yet we will prove their effectiveness. Other choices of $\mathcal{X}_H$ (see, e.g., [25]) and of the dynamics (7) are possible, provided that Lemma 1 still holds.

If $\mathcal{X}$ is a RPI set, one has $\mathcal{R}_{\cup}(\mathcal{X}) = \mathcal{X}$, and therefore $\mathcal{X}_H = \mathcal{X}$, because $\mathcal{X}$ is a convex set by definition. In this case, the definition of fake dynamics is not necessary, and the same approach hereafter described can be applied with $\tilde{s} = s$, and $\mathcal{X}_E = \emptyset$.

For any pair $(i, j) \in \tilde{\mathcal{I}} \times \tilde{\mathcal{I}}$ define the closed *transition sets*

$$\mathcal{X}_{ij} \triangleq \{x \in \bar{\mathcal{X}}_i : \exists d \in \mathcal{D} : A_i x + a_i + E_i d \in \bar{\mathcal{X}}_j\} \tag{9}$$

of states that can possibly end up in $\bar{\mathcal{X}}_j$ in one step under dynamics i. Note that $\mathcal{X}_H = \bigcup_{i=1}^{\tilde{s}} \bigcup_{j=1}^{\tilde{s}} \mathcal{X}_{ij}$, but $\{\mathcal{X}_{ij}\}$ in general is not a partition of $\mathcal{X}_H$, unless $\mathcal{D} = \{0\}$, or $E_i = 0$, $\forall i \in \tilde{\mathcal{I}}$. To compute the sets $\mathcal{X}_{ij}$ we exploit here controllability analysis [20, Chap. 5] and consider the disturbance vector d as the external input with respect to which the controllability analysis is carried out. First, we compute the set

$$\mathcal{M}_i(\bar{\mathcal{X}}_j) = \left\{ \begin{bmatrix} x \\ d \end{bmatrix} \in \mathbb{R}^{n+p} : \begin{bmatrix} H_j A_i & H_j E_i \\ 0 & \tilde{H} \end{bmatrix} \begin{bmatrix} x \\ d \end{bmatrix} + \begin{bmatrix} H_j a_i \\ 0 \end{bmatrix} \leq \begin{bmatrix} h_j \\ \tilde{h} \end{bmatrix} \right\}$$

which is defined on the extended space (x, d), and we project it onto the first n coordinates (the state space), obtaining the so-called *pre-image set* of $\bar{\mathcal{X}}_j$ with respect to dynamics i

$$\text{Pre}(\bar{\mathcal{X}}_j, i) \triangleq \{x \in \mathbb{R}^n : \exists d \in \mathcal{D} : A_i x + a_i + E_i d \in \bar{\mathcal{X}}_j\}.$$

If we impose $x \in \bar{\mathcal{X}}_i$, we obtain

$$\mathcal{X}_{ij} = \left\{\bar{\mathcal{X}}_i \cap \text{Pre}(\bar{\mathcal{X}}_j, i)\right\}. \tag{10}$$

Let m_{ij} be the number of vertices of $\mathcal{X}_{ij}$, $(i, j) \in (\tilde{\mathcal{I}} \times \tilde{\mathcal{I}})$. In the particular case of no additive disturbances (i.e., $\mathcal{D} = \{0\}$, or $E_i = 0$, $\forall i \in \tilde{\mathcal{I}}$), the transition sets $\mathcal{X}_{ij}$ can be computed without using of the projection operation, as

$$\mathcal{X}_{ij} = \left\{ x \in \mathbb{R}^n : \begin{bmatrix} H_i \\ H_j A_i \end{bmatrix} x + \begin{bmatrix} 0 \\ H_j a_i \end{bmatrix} \leq \begin{bmatrix} h_i \\ h_j \end{bmatrix} \right\}.$$

V. PWA Lyapunov Analysis of the Extended System

Let $V : \mathcal{X}_H \to \mathbb{R}$ be a function such that

$$V(x) \geq \alpha_1 \|x\|_W \tag{11a}$$

$$V(0) = 0 \tag{11b}$$

$$V\left(f(x, d)\right) - \lambda V(x) \leq \delta \tag{11c}$$

$\forall x \in \mathcal{X}_i (i \in \tilde{\mathcal{I}})$ and $\forall d \in \mathcal{D}$, where $f : \mathbb{R}^n \times \mathbb{R}^\eta \to \mathbb{R}^n$ is the PWA state update function defined by (8), $\alpha_1 > 0$, $\lambda \in (0, 1)$, $\delta \geq 0$. $W \in \mathbb{R}^{n\times n}$ can be chosen as any full rank diagonal matrix, but its value is fixed a priori as a design parameter. We call a function $V : \mathbb{R}^n \to \mathbb{R}$ satisfying (11) a *Lyapunov-like* function.

Remark 1: Note that often in the literature an upperbound $\alpha_2 \|x\|_W \geq V(x)$ is imposed, with $\alpha_2 > 0$. Here this is not necessary as V will be defined as a PWA function over the bounded set $\mathcal{X}_H$. As a consequence, it is always possible to find a posteriori $\alpha_2 > 0$ such that $V(x) \leq \alpha_2 \|x\|_W$, $\forall x \in \mathcal{X}_H$, once a Lyapunov-like function V has been determined. Note also that condition (11c) could be replaced by

$$V\left(f(x, d)\right) - V(x) \leq -\alpha_3 \|x\|_W + \delta \tag{12}$$

where $\alpha_3 = (1-\lambda)\alpha_1 > 0$. In fact, by (11a)–(11c), it follows that $V(f(x,d)) - V(x) \leq -(1-\lambda)V(x) + \delta \leq -(1-\lambda)\alpha_1\|x\|_W + \delta$.

To the end of synthesizing a PWA Lyapunov-like function for system (8), define a linear function $V_i : \bar{\mathcal{X}}_i \to \mathbb{R}$ for each region $\mathcal{X}_i$

$$V_i(x) \triangleq F_i x + g_i \tag{13a}$$

for $i \in \tilde{\mathcal{I}}$, where in (13a) $F_i \in \mathbb{R}^{1\times n}$ and $g_i \in \mathbb{R}$ are coefficients to be determined. The overall Lyapunov-like function candidate is

$$V(x) = \max_{i\in\mathcal{N}(x)} V_i(x) \tag{13b}$$

where

$$\mathcal{N}(x) \triangleq \{i \in \tilde{\mathcal{I}} : x \in \bar{\mathcal{X}}_i\}. \tag{13c}$$

Note that simply $V(x) = F_i x + g_i$ for $x \in int(\mathcal{X}_i)$. As for numerical reasons we want to consider closed sets $\bar{\mathcal{X}}_i$ and $V_i(x)$, $V_j(x)$ may not coincide on common boundaries $\bar{\mathcal{X}}_i \cap \bar{\mathcal{X}}_j$ (unless very conservative continuity conditions are imposed), for the states x on the common boundaries we impose (11a) and (11c) on V_i for all $i \in \mathcal{N}(x)$, although only one value (the max) is taken in (13b), as $V(x)$ must be single-valued. In the specific case of (11b), we have $V_i(0) = 0$ for all $i \in \mathcal{I}_0$, because the continuity condition is imposed at the origin. The constraints

$$F_i v_{i,h} + g_i \geq \alpha_1 \|v_{i,h}\|_W \tag{14a}$$

are imposed for all m_i vertices $v_{i,h} \in vert(\bar{\mathcal{X}}_i)$, $i \in \tilde{\mathcal{I}}$, $h = 1,\ldots,m_i$, while

$$F_j(A_i v_{ij,h} + a_i + E_i d_\mu) + g_j - \lambda(F_i v_{ij,h} + g_i) \leq \delta \tag{14b}$$

for all $v_{ij,h} \in vert(\mathcal{X}_{ij})$, with $h = 1,\ldots,m_{ij}$, for all A_i, a_i, E_i with $i \in \tilde{\mathcal{I}}$, and all d_μ with $\mu = 1,\ldots,\eta$. We define $k_{max} \in \mathbb{R}_{>0}$ as the maximum value such that $\mathcal{H} \triangleq \{x : \|x\|_W \leq k_{max}\} \subseteq \mathcal{X}_H$, which can be easily determined by bisection. The further constraints

$$\delta \geq 0 \tag{14c}$$
$$r\delta \leq (1-\lambda)L\alpha_1 \tag{14d}$$
$$g_i = 0, \quad i \in \mathcal{I}_0 \tag{14e}$$
$$\alpha_1 > 0 \tag{14f}$$

are also imposed, where L is the minimum distance between the boundary of the hyper-rectangle $\mathcal{H}$ and the origin, while $r = 1 + \varepsilon$, where $\varepsilon > 0$ is a fixed parameter, chosen as small as possible (e.g., equal to the machine epsilon). Condition (14d) is imposed to ensure that the set where the state is ultimately bounded is included in $\mathcal{X}_H$ (the reader is referred to the proof of Lemma 2, after (24), for details on this aspect), while (14e) ensures the fulfillment of (11b). The vector of variables to be determined is composed by δ, α_1, and the terms F_i and g_i, with $i \in \tilde{\mathcal{I}}$. We define now a procedure to determine a choice for such variables by means of a convex optimization problem.

Lemma 2: Consider the linear-fractional program

$$\text{minimize } \frac{\delta}{\alpha_1} \tag{15a}$$
$$\text{subject to (14)} \tag{15b}$$

associated with the autonomous uncertain PWA dynamics (8) and the candidate Lyapunov-like function (13), and assume it admits a feasible solution. Then:

1) if the optimal δ is equal to zero, the origin is an equilibrium point for system (8), which is $\mathrm{UAS}(\mathcal{X}_H)$;
2) otherwise (the optimal δ is strictly positive), system (8) is $\mathrm{UUB}(\mathcal{X}_H, \mathcal{F})$, with

$$\mathcal{F} = \left\{x \in \mathcal{X}_H : V(x) \leq V_{\mathcal{F}}^+\right\} \tag{16}$$

where

$$V_{\mathcal{F}}^+ \triangleq \max_{x\in\mathcal{R}(\mathcal{Q})} V(x) \tag{17}$$
$$\mathcal{Q} \triangleq \left\{x \in \mathbb{R}^n : \|x\|_W \leq \frac{r\delta}{(1-\lambda)\alpha_1}\right\}. \tag{18}$$

Proof: See Appendix. □

Remark 2: Note that (15) is, strictly speaking, a quasi-convex optimization problem. Even though quasi-convex optimization problems are solved in general by bisection (solving a convex optimization problem at each step), linear-fractional programs like (15) can be transformed into an equivalent LP using standard procedures (see [29, Chap. 4]). The variables of the LP are δ, α_1, F_i and g_i, with $i \in \tilde{\mathcal{I}}$, plus an auxiliary variable introduced by the formulation of the equivalent problem. The total number of variables is $n_v = 3 + \tilde{s}(n+1)$, the total number of constraints is $n_c = 4 + \mathrm{card}(\mathcal{I}_0) + \sum_{i=1}^{\tilde{s}} \left(m_i + \eta \sum_{j=1}^{\tilde{s}} m_{ij}\right)$.

Remark 3: The aim of (15) is to minimize the volume of the hyper-rectangle $\mathcal{Q}$ (in fact, the cost function of (15a) is directly proportional to the volume of $\mathcal{Q}$) for the given choice of the sets $\mathcal{X}_i$, in order to obtain a set $\mathcal{F}$ that is as small as possible (even if not necessarily the smallest one, since there might be another set $\mathcal{F}' \subset \mathcal{F}$ such that system (8) is also $\mathrm{UUB}(\mathcal{X}_H, \mathcal{F}')$). Note that, if (15) is solvable by fixing $\delta = 0$ *a priori*, (15) becomes a test of asymptotic stability of the origin. Note that in this case the constraints in (14) parameterize all the PWA Lyapunov functions that are defined on the same partition $\{\mathcal{X}_i\}$ of $\mathcal{X}_H$. Clearly, a necessary condition for system (8) to be UAS is that $E_i = 0$, $\forall i \in \mathcal{I}_0$, or $\mathcal{D} = \{0\}$, otherwise the origin would not be an equilibrium.

Remark 4: In case (14) is infeasible, besides increasing the value of λ or changing that of ρ, a possibility is to increase the number of regions $\mathcal{X}_i$ of $\mathcal{X}_H$, therefore providing more flexibility in synthesizing the PWA Lyapunov-like function. A possible way is to consider the sets $\mathcal{X}_{ij}$ as the new sets $\bar{\mathcal{X}}_i$ and restart the one-step reachability analysis. Due to space limitations, we refer the reader to [24] for further details.

A practical procedure to calculate $\mathcal{F}$ consists of partitioning $\mathcal{Q}$ into a number of subsets $\mathcal{Q}_i$, $\mathcal{Q}_i = \bar{\mathcal{X}}_i \cap \mathcal{Q}$ with $i \in \tilde{\mathcal{I}}$, and then obtain all the sets $\mathcal{R}(\mathcal{Q}_i)$ as in (5). Then $\mathcal{F}$ can be easily computed as the union of the subsets of the sets $\mathcal{X}_i$ for which $V(x) \leq \hat{V}$, i.e.

$$\mathcal{F} = \bigcup_{i=1}^{\tilde{s}} \mathcal{X}_i^{\mathcal{F}}, \quad \mathcal{X}_i^{\mathcal{F}} \triangleq \left\{x \in \mathcal{X}_i : V(x) \leq \hat{V}\right\}. \tag{19}$$

VI. Invariance Analysis

So far we have analyzed the properties of the *extended* system (8). We want now to derive conditions on the *original* system (3). Consider again system (8) in $\mathcal{X}_H$, assume that a feasible solution to (14) exists, define (recalling that $\mathcal{X}_E = \mathcal{X}_H \setminus \mathcal{X}$)

$$V_E^- \triangleq \begin{cases} \inf_{x\in\mathcal{X}_E} V(x) & \text{if } \mathcal{X}_E \neq \emptyset \\ +\infty & \text{if } \mathcal{X}_E = \emptyset \end{cases} \tag{20}$$

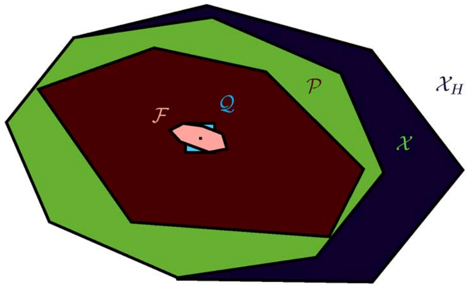

Fig. 1. Graphical representation of the sets $\mathcal{X}_H$, $\mathcal{X}$, $\mathcal{P}$, $\mathcal{Q}$ and $\mathcal{F}$ for an example of second-order system.

and consider the set

$$\mathcal{P} \triangleq \left\{x \in \mathcal{X}_H : V(x) < V_E^-\right\} \subseteq \mathcal{X}. \tag{21}$$

In case $\mathcal{X}$ is a RPI set (and hence $\mathcal{X}_E = \emptyset$), one has $\mathcal{P} \equiv \mathcal{X}$, since $V(x) < \infty$ for all $x \in \mathcal{X}$. Otherwise, the set $\mathcal{P}$ is given by the union of a finite number of polytopes, each of them composed by the points $x \in \mathcal{X}_i, i \in \mathcal{I}$, for which $V(x) < V_E^-$. Being the sublevel set of a discontinuous Lyapunov function, the set $\mathcal{P}$ in this case may not be convex, and not even connected. The following result related to RPI sets can be easily proven.

Lemma 3: Consider system (3), whose dynamics are defined on $\mathcal{X}$, and assume that either the dynamics (7) are defined in $\mathcal{X}_E$, or $\mathcal{X}_E = \emptyset$. Then, if a set $\tilde{\mathcal{X}} \subseteq \mathcal{X}$ is RPI for (8), it is also RPI for (3).

Proof: See Appendix. □

The results for the extended system (8) proved in Lemma 2, and the definition of $\mathcal{P}$ in (21) and Lemma 3 are exploited next to state the main result of this technical note.

Theorem 1: Consider system (3), whose dynamics are defined on $\mathcal{X}$, and assume that either the dynamics (7) are defined in $\mathcal{X}_E$, or $\mathcal{X}_E = \emptyset$. Then:

1) if the optimal δ of (15) is zero, then $\mathcal{P}$ in (21) is a RPI set for (3), and (3) is $\text{UAS}(\mathcal{P})$;
2) otherwise, if $V_E^- \geq V_{\mathcal{F}}^+$, the sets $\mathcal{P}$ and $\mathcal{F}$ are RPI sets with respect to dynamics (3), and system (3) is $\text{UUB}(\mathcal{P}, \mathcal{F})$.

Proof: See Appendix. □

Note that, by construction of $\mathcal{F}$, if $\mathcal{X}_E = \emptyset$, then $\mathcal{F} \subseteq \mathcal{P}$ is always satisfied. Also, note that the result of Theorem 1 refers to the *original* system (3), and the fake dynamics were only employed to determine a suitable Lyapunov-like function. A graphical representation of the sets $\mathcal{X}_H$, $\mathcal{X}$, $\mathcal{P}$, $\mathcal{Q}$ and $\mathcal{F}$ is shown in Fig. 1, where we show that $\mathcal{X}_H \supseteq \mathcal{X} \supseteq \mathcal{P}$. Also, we recall that $\mathcal{Q}, \mathcal{F} \subseteq \mathcal{X}_H$ by definition.

VII. Numerical Complexity

The procedure described in Section IV requires four main numerical computations. First, we need the vertex representation of the polytopes in (2). This can be done by solving a vertex enumeration problem (see the approaches in [30] and [31], for example). The second operation is the computation of convex hulls, which is the dual of the vertex enumeration problem [32]. The complexity of these two operations is exponential in the number of facets of the considered polytopes. The third operation is the projection in (10), for which there exist different methods [33] and whose complexity increases quickly with the number of vertices or facets. The fourth computation is to obtain minimal hyperplane representations of polyhedra to eliminate redundant inequalities [34], which can be done by solving one LP for each inequality defining the given polyhedron. In the stability and invariance analysis procedure described in Sections V and VI, the solution of a single LP is needed to compute $V(x)$, while a number of operations similar to those needed in Section IV is required to compute the sets $\mathcal{P}$ and $\mathcal{F}$.

VIII. Conclusion

This note has addressed the problem of determining the properties of uniform asymptotic stability and uniform ultimate boundedness of (possibly discontinuous) discrete-time PWA systems subject to additive disturbances. Since the dynamics are defined in the set $\mathcal{X}$ that is a priori not invariant, partially fictitious dynamics are exploited to define a discontinuous PWA Lyapunov-like function. As an outcome of the optimization problem, the RPI set $\mathcal{P} \subseteq \mathcal{X}$ is obtained, together with the terminal set $\mathcal{F}$, or the certificate of asymptotic stability.

Appendix

Proof of Lemma 1: If $x \in \mathcal{X}_H$, then either $x \in \mathcal{X}$ or $x \in \mathcal{X}_E$. If $x \in \mathcal{X}$ then the successor state $A_i x + a_i + E_i d \in \mathcal{R}_\cup(\mathcal{X}) \subseteq \mathcal{X}_H$ by definition of $\mathcal{X}_H$. If $x \in \mathcal{X}_E$, the successor state is $\rho x \in \mathcal{X}_H$, because $\mathcal{X}_H$ is a convex set including the origin. ■

Proof of Lemma 2: To prove the positive definiteness of the Lyapunov-like function, define $\beta_{i,h} \geq 0$, $\sum_{h=1}^{m_i} \beta_{i,h} = 1$, as a set of coefficients which allow one to express $x \in \bar{\mathcal{X}}_i$ as a convex combination of the vertices of $\bar{\mathcal{X}}_i$. Since functions V_i are affine functions defined on convex sets $\bar{\mathcal{X}}_i$, the satisfaction of (14a) for all $v_{i,h} \in vert(\bar{\mathcal{X}}_i)$, with $i \in \tilde{\mathcal{I}}$, $h = 1, \ldots, m_i$, for $x = \sum_{h=1}^{m_i} \beta_{i,h} v_{i,h}$, leads to

$$\begin{aligned}
\alpha_1 \|x\|_W &= \alpha_1 \left\| \sum_{h=1}^{m_i} \beta_{i,h} v_{i,h} \right\|_W \\
&\leq \sum_{h=1}^{m_i} \beta_{i,h} \alpha_1 \|v_{i,h}\|_W \leq \sum_{h=1}^{m_i} \beta_{i,h} (F_i v_{i,h} + g_i) \\
&= F_i \left(\sum_{h=1}^{m_i} \beta_{i,h} v_{i,h} \right) + g_i \sum_{h=1}^{m_i} \beta_{i,h} \\
&= F_i x + g_i.
\end{aligned} \tag{22}$$

For this reason, for $x \in int(\mathcal{X}_i)$, since $V_i(x) = F_i x + g_i$, (11a) holds. Moreover, on the boundaries of $\bar{\mathcal{X}}_i$, according to (13b), one has $\alpha_1 \|x\|_W \leq F_i x + g_i$ for all $i \in \mathcal{N}(x)$, and therefore $\alpha_1 \|x\|_W \leq \max_{i \in \mathcal{N}(x)} \{F_i x + g_i\} = V(x)$. This implies that (11a) holds for all $x \in \mathcal{X}_H$, since $\mathcal{X}_H = \bigcup_{i \in \tilde{\mathcal{I}}} \mathcal{X}_i$.

As for the decay of the Lyapunov-like function, for $x \in \mathcal{X}_{ij}$, define $\beta_{ij,h} \geq 0$, $\sum_{h=1}^{m_{ij}} \beta_{ij,h} = 1$, as a set of coefficients such that $x \in \mathcal{X}_{ij}$ can be expressed as a convex combination of the vertices of $\mathcal{X}_{ij}$. Therefore,

$$V\left(f(x,d)\right) = F_j \left[A_i \left(\sum_{h=1}^{m_{ij}} \beta_{ij,h} v_{ij,h} \right) + a_i + E_i \left(\sum_{\mu=1}^{\eta} c_\mu d_\mu \right) \right] + g_j \tag{23}$$

where $c_\mu \geq 0$, $\sum_{\mu=1}^{\eta} c_\mu = 1$, are coefficients used to express the fact that any point $d \in \mathcal{D}$ can be written as a convex combination of the vertices of $\mathcal{D}$. Recalling that (14b) holds for all the vertices of $\mathcal{X}_{ij}$, and that $\sum_{h=1}^{m_{ij}} \beta_{ij,h} = \sum_{\mu=1}^{\eta} c_h = 1$, from (23) we get

$$\begin{aligned} &V(f(x,d)) \\ &= F_j\left[\sum_{h=1}^{m_{ij}} \beta_{ij,h} A_i v_{ij,h} + a_i + E_i\left(\sum_{\mu=1}^{\eta} c_\mu d_\mu\right)\right] + g_j \\ &= F_j \sum_{h=1}^{m_{ij}} \beta_{ij,h}\left(A_i v_{ij,h} + a_i + \sum_{\mu=1}^{\eta} c_\mu E_i d_\mu\right) + g_j \\ &= \sum_{h=1}^{m_{ij}} \beta_{ij,h} \sum_{\mu=1}^{\eta} c_\mu \left(F_j(A_i v_{ij,h} + a_i + E_i d_\mu)\right) + g_j \\ &\le \sum_{h=1}^{m_{ij}} \beta_{ij,h} \sum_{\mu=1}^{\eta} c_\mu \left(\delta + \lambda(F_i v_{ij,h} + g_i) - g_j\right) + g_j \\ &= \delta + \lambda \sum_{h=1}^{m_{ij}} \beta_{ij,h}(F_i v_{ij,h} + g_i) - \sum_{h=1}^{m_{ij}} \beta_{ij,h} g_j + g_j \\ &= \delta + \lambda\left(F_i \sum_{h=1}^{m_{ij}} \beta_{ij,h} v_{ij,h} + \sum_{h=1}^{m_{ij}} \beta_{ij,h} g_i\right) \\ &= \delta + \lambda(F_i x + g_i) = \delta + \lambda V(x), \end{aligned}$$

which proves that (11c) holds for all $x \in int(\mathcal{X}_{ij})$. Also, on the boundaries of $\mathcal{X}_{ij}$, the decreasing condition (11c) is imposed for all $(i,j) \in \mathcal{N}(x) \times \mathcal{N}(f(x,d))$, and therefore

$$\max_{j\in\mathcal{N}(f(x,d))} (F_j(f(x,d)) + g_j) \le \delta + \lambda \max_{i\in\mathcal{N}(x)} (F_i x + g_i).$$

Since by definition of the sets $\mathcal{X}_{ij}$ we have that $\mathcal{X}_H = \bigcup_{i=1}^{\tilde{s}} \bigcup_{j=1}^{\tilde{s}} \mathcal{X}_{ij}$, (11c) holds for all $x \in \mathcal{X}_H$, and then (11) holds for all $x \in \mathcal{X}_H$.

Up to this point, the proven results hold for any $\delta \ge 0$. Considering the case $\delta = 0$, the decay condition (11c) becomes $V(f(x,d)) - \lambda V(x) \le 0$, which also implies $f(0,d) = 0$, $\forall d \in \mathcal{D}$. From $V(f(x,d)) - \lambda V(x) \le 0$ we can easily obtain $V(x(k)) \le \lambda^k V(x(0))$, which leads to $\alpha_1\|x(k)\|_W \le V(x(k)) \le \lambda^k V(x(0)) \le \lambda^k \alpha_2 \|x(0)\|_W$. Therefore,

$$\|x(k)\|_W \le \frac{\alpha_2}{\alpha_1}\lambda^k \|x(0)\|_W,$$

which implies that system (8) is $\mathrm{UAS}(\mathcal{X}_H)$, according to Definition 3, with $\phi(\|x(0)\|, k) = \alpha_2\lambda^k/\alpha_1$.

Consider now the case $\delta > 0$. By (11a) and (11c) it follows that $V(f(x,d)) - V(x) \le -(1-\lambda)V(x) + \delta \le -(1-\lambda)\alpha_1\|x\|_W + \delta = -\alpha_3\|x\|_W + \delta$, where $\alpha_3 = (1-\lambda)\alpha_1$. By definition of $\mathcal{Q}$ in (18), for all $x \in \mathcal{X}_H \setminus \mathcal{Q}$ we obtain $-\alpha_3\|x\|_W + r\delta < 0$, leading to

$$V(f(x,d)) - V(x) < -\varepsilon\delta, \quad x \in \mathcal{X}_H \setminus \mathcal{Q}. \tag{24}$$

By definition of $\mathcal{F}$, the fulfillment of the constraint (14d) implies $\mathcal{Q} \subseteq \mathcal{H} \subseteq \mathcal{X}_H$ and, by the invariance of $\mathcal{X}_H$ (Lemma 1), $\mathcal{R}(\mathcal{Q}) \subseteq \mathcal{X}_H$. This implies that the max in (17) exists (since $\mathcal{Q}$, and therefore $\mathcal{R}(\mathcal{Q})$, are closed sets) and therefore $\mathcal{F}$ in (16) is well defined.

To prove invariance of $\mathcal{F}$, we distinguish two cases:

- if $x \in \mathcal{F} \cap \mathcal{Q}$, then $f(x,d) \in \mathcal{R}(\mathcal{Q}) \subseteq \mathcal{F}$;
- if $x \in \mathcal{F} \setminus \mathcal{Q}$, then (24) holds, and therefore $V(f(x,d)) < V(x) \le V_{\mathcal{F}}^+$ (this latter being defined in (17)), meaning that $f(x,d) \in \mathcal{F}$ by definition of $\mathcal{F}$.

Therefore, $\mathcal{F}$ is a RPI set for (8).

To prove that system (8) is $\mathrm{UUB}(\mathcal{X}_H, \mathcal{F})$ according to Definition 4, we must show that for all $a > 0$ there exists $T = T(a)$ such that, for all $x(0) \in \mathcal{X}_H$ with $\|x(0)\|_W \le a$, $x(T) \in \mathcal{F}$ for any admissible sequence of the disturbance term d. We define $\mathcal{Q}_a \triangleq \{x \in \mathbb{R}^n : \|x\|_W \le a\}$, and distinguish two cases:

- If $0 < a \le r\delta/\alpha_3$, condition $x(0) \in \mathcal{Q}_a$ implies $x(0) \in \mathcal{Q}$. Therefore, simply taking $T(a) = 1$, we have that for every $x(0) \in \mathcal{Q}_a$, $x(T) \in \mathcal{F}$ (by definition of $\mathcal{F}$).
- If $a > r\delta/\alpha_3$, let $V_H^- \triangleq \inf_{x\in\mathcal{X}_H\setminus\mathcal{Q}} V(x) > 0$. Then, from (24) it follows that

$$\forall x(0) \in \mathcal{X}_H \setminus \mathcal{Q}, \quad \exists \bar{k} = \bar{k}(a) \in \mathbb{Z}_{>0} : V\left(x(\bar{k})\right) < V_H^- \tag{25}$$

meaning that $x(\bar{k}) \in \mathcal{Q}$, and therefore $x(\bar{k}+1) \in \mathcal{F}$. In order to explicitely find $\bar{k}$ as a function of a, let $V_a^+ \triangleq \sup_{x\in\mathcal{Q}_a\setminus\mathcal{Q}} V(x)$. From (24), one has

$$\bar{k}(a) \triangleq \left\lceil \frac{V_a^+ - V_H^-}{r\delta} \right\rceil.$$

In conclusion, if $a > r\delta/\alpha_3$, there exists $T(a) = \bar{k}(a) + 1$ such that, for every $x(0) \in \mathcal{Q}_a$, $x(T) \in \mathcal{F}$.

■

Proof of Lemma 3: Since $\tilde{\mathcal{X}} \subseteq \mathcal{X}$, from (8) we can easily see that, given $x(k) \in \tilde{\mathcal{X}}$, we have that $x(k+1) = A_i x(k) + a_i + E_i d(k)$, with $x(k) \in \mathcal{X}_i$ and $i \in \mathcal{I}$. Notice that dynamics (3) and (8) coincide for $x(k) \in \tilde{\mathcal{X}}$. This means that we applied dynamics (3), and the same will hold for all subsequent time instants, since $x(k+1) \in \tilde{\mathcal{X}}$ by assumption ($\tilde{\mathcal{X}}$ is a RPI set). We conclude that $\tilde{\mathcal{X}}$ is a RPI set for dynamics (3). ■

Proof of Theorem 1: The proof consists of showing that the PWA Lyapunov-like function $V_{\mathcal{P}}(x) : \mathcal{P} \to \mathbb{R}$,

$$V_{\mathcal{P}}(x) \equiv V(x), \quad \forall x \in \mathcal{P} \tag{26}$$

where $V(x)$ is found as in Lemma 2 for the extended system (8) in $\mathcal{X}_H$, is a Lyapunov-like function for (3) over the set $\mathcal{P}$. First of all, we prove that $\mathcal{P}$ is a RPI set for (8) in $\mathcal{X}_H$. In case $\delta = 0$, for all $x \in \mathcal{P}$, $V(f(x,d)) \le \lambda V(x)$, and then $f(x,d) \in \mathcal{P}$ by definition of $\mathcal{P}$. In case $\delta > 0$, first of all notice that, by definition of $\mathcal{P}$ and $\mathcal{F}$ as sublevel sets, it follows that the assumption $V_E^- \ge V_{\mathcal{F}}^+$ implies $\mathcal{F} \subseteq \mathcal{P}$. Then, we distinguish two cases:

- if $x \in \mathcal{F}$, then $f(x,d) \in \mathcal{F}$ (which is a RPI set for (8)), and then $f(x,d) \in \mathcal{P}$ since $\mathcal{F} \subseteq \mathcal{P}$;
- if $x \in \mathcal{P} \setminus \mathcal{F}$, we distinguish again two sub-cases:
  - —if $x \in \mathcal{Q}$, then $f(x,d) \in \mathcal{F} \subseteq \mathcal{P}$, since $\mathcal{R}(\mathcal{Q}) \subseteq \mathcal{F}$ by construction (see (16)–(18);
  - —if $x \notin \mathcal{Q}$, then, from (24), $V(f(x,d)) < V(x)$, leading to $f(x,d) \in \mathcal{P}$.

Since $\mathcal{P} \subseteq \mathcal{X}$, by Lemma 3 we obtain that $\mathcal{P}$ is a RPI set for dynamics (3). This fact leads to the possibility of defining a Lyapunov-like function for system (3) in $\mathcal{P}$. Considering that $\mathcal{P} \subseteq \mathcal{X}_H$, if (11) are satisfied with $\delta = 0$ for all $x \in \mathcal{X}_H$ (and then for all $x \in \mathcal{P}$), we conclude that $V_{\mathcal{P}}(x)$ is a Lyapunov function for system (3) in $\mathcal{P}$, and that system (3) is $\mathrm{UAS}(\mathcal{P})$.

In case $\delta > 0$, if (11) hold for all $x \in \mathcal{X}_H$ (and then for all $x \in \mathcal{P}$), the invariance of $\mathcal{F}$ with respect to dynamics (3) is guaranteed by Lemmas 2 and 3. We have already proved that $\mathcal{P}$ is RPI for (3), and we know that $\mathcal{F} \subseteq \mathcal{P}$ by construction. We also know from Lemma 2 that the extended system (8) is $\mathrm{UUB}(\mathcal{X}_H, \mathcal{F})$, which means that, for all $a > 0$, there exists $T(a) > 0$ such that, for every $x(0) \in \mathcal{X}_H$ with $\|x(0)\| \le a$, $x(T) \in \mathcal{F}$ for all the sequences $\mathbf{d}_{[T]}$ with $d(k) \in \mathcal{D}$, $k = 0, \dots, T$. However, since dynamics (3) and (8) coincide in $\mathcal{P}$, one can use Definition 4 again considering dynamics (3). We know that for all $x(0) \in \mathcal{P}$, for all $a > 0$ there exists $T(a) > 0$ such that, for every $x(0) \in \mathcal{P}$ with $\|x(0)\| \le a$, $x(T) \in \mathcal{F}$ for all the sequences $\mathbf{d}_{[T]}$ with $d(k) \in \mathcal{D}$, $k = 0, \dots, T$. In conclusion, system (3) is $\mathrm{UUB}(\mathcal{P}, \mathcal{F})$.

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REFERENCES

[1] E. D. Sontag, "Nonlinear regulation: The piecewise linear approach," *IEEE Trans. Autom. Control*, vol. 26, no. 2, pp. 346–358, Feb. 1981.

[2] A. Bemporad and M. Morari, "Control of systems integrating logic, dynamics, and constraints," *Automatica*, vol. 35, pp. 407–428, 1999.

[3] A. Bemporad, "Efficient conversion of mixed logical dynamical systems into an equivalent piecewise affine form," *IEEE Trans. Autom. Control*, vol. 49, no. 5, pp. 832–838, May 2004.

[4] F. J. Christophersen, M. Baotic, and M. Morari, "Stability analysis of hybrid systems with a linear performance index," in *Proc. Conf. Decision Control*, Atlantis, Paradise Island, Bahamas, Dec. 2004, pp. 4589–4594.

[5] A. Bemporad, W. Heemels, and B. De Schutter, "On hybrid systems and closed-loop MPC systems," *IEEE Trans. Autom. Control*, vol. 47, no. 5, pp. 863–869, May 2002.

[6] S. Di Cairano, A. Bemporad, I. Kolmanovsky, and D. Hrovat, "Model predictive control of magnetically actuated mass spring dampers for automotive applications," *Int. J. Control*, vol. 80, no. 11, pp. 1701–1716, 2007.

[7] F. Borrelli, A. Bemporad, M. Fodor, and D. Hrovat, "An MPC/hybrid system approach to traction control," *IEEE Trans. Control Syst. Technol.*, vol. 14, no. 3, pp. 541–552, May 2006.

[8] N. Giorgetti, A. Bemporad, H. Tseng, and D. Hrovat, "Hybrid model predictive control application towards optimal semi-active suspension," *Int. J. Control*, vol. 79, no. 5, pp. 521–533, 2006.

[9] A. Alessio and A. Bemporad, , L. Magni, F. Allgower, and D. M. Raimondo, Eds., "A survey on explicit model predictive control," in *Nonlinear Model Predictive Control: Towards New Challenging Applications*, ser. Lecture Notes in Control and Information Sciences. Berlin, Germany: Springer-Verlag, 2009, vol. 384, pp. 345–369.

[10] A. Bemporad, A. Oliveri, T. Poggi, and M. Storace, "Ultra-fast stabilizing model predictive control via canonical piecewise affine approximations," *IEEE Trans. Autom. Control*, vol. 56, no. 12, pp. 2883–2897, Dec. 2011.

[11] G. Ferrari Trecate, F. A. Cuzzola, D. Mignone, and M. Morari, "Analysis of discrete-time piecewise affine and hybrid systems," *Automatica*, vol. 38, no. 12, pp. 2139–2146, 2002.

[12] P. Grieder, M. Kvasnica, M. Baotić, and M. Morari, "Stabilizing low complexity feedback control of constrained piecewise affine systems," *Automatica*, vol. 41, no. 10, pp. 1683–1694, 2005.

[13] S. Boyd, L. El Ghaoui, E. Feron, and V. Balakrishnan*, Linear Matrix Inequalities in System and Control Theory*. Philadelphia, PA: SIAM, 1994.

[14] M. Lazar and A. Jokić, "On infinity norms as Lyapunov functions for piecewise affine systems," in *Proc. Hybrid Syst.: Computat. Control*, Stockholm, Sweden, Apr. 2010, pp. 417–432.

[15] S. Prajna and A. Papachristodoulou, "Analysis of switched and hybrid systems—Beyond piecewise quadratic methods," in *Proc. Amer. Control Conf.*, Denver, CO, Jun. 2003, pp. 2779–2784.

[16] A. Polanski, "Lyapunov function construction by linear programming," *IEEE Trans. Autom. Control*, vol. 42, no. 7, pp. 1013–1016, Jul. 1997.

[17] E. C. Kerrigan and D. Q. Mayne, "Optimal control of constrained, piecewise affine systems with bounded disturbances," in *Proc. Conf. Decision Control*, Las Vegas, NV, Dec. 2002, pp. 1552–1557.

[18] G. Zhai, H. Lin, and P. J. Antsaklis, "Quadratic stabilizability of switched linear systems with polytopic uncertainties," *Int. J. Control*, vol. 76, no. 7, pp. 747–753, 2003.

[19] H. Lin and P. Antsaklis, "Synthesis of uniformly ultimate boundedness switching laws for discrete-time uncertain switched linear systems," in *Proc. Conf. Decision Control*, Maui, HI, Dec. 2003, pp. 4806–4811.

[20] F. Blanchini and S. Miani*, Set-Theoretic Methods in Control*. Berlin, Germany: Springer-Verlag, 2008.

[21] P. Biswas, P. Grieder, J. Löfberg, and M. Morari, "A survey on stability analysis of discrete-time piecewise affine systems," in *Proc. IFAC World Congress*, Prague, Czech Republic, Jun. 2005, pp. 331–343.

[22] S. V. Raković, P. Grieder, M. Kvasnica, D. Q. Mayne, and M. Morari, "Computation of invariant sets for piecewise affine discrete time systems subject to bounded disturbances," in *Proc. Conf. Decision Control*, Atlantis, Paradise Island, Bahamas, Dec. 2004, pp. 1418–1423.

[23] S. Raković, E. Kerrigan, D. Mayne, and J. Lygeros, "Reachability analysis of discrete-time systems with disturbances," *IEEE Trans. Autom. Control*, vol. 51, no. 4, pp. 546–561, Apr. 2006.

[24] S. Trimboli, M. Rubagotti, and A. Bemporad, "Stability and invariance analysis of uncertain PWA systems based on linear programming," in *Proc. Conf. Decision Control*, Orlando, FL, Dec. 2011, pp. 7398–7403.

[25] M. Rubagotti, S. Trimboli, D. Bernardini, and A. Bemporad, "Stability and invariance analysis of approximate explicit MPC based on PWA Lyapunov functions," in *Proc. IFAC World Congress*, Milan, Italy, Aug. 2011, pp. 5712–5717.

[26] A. Oliveri, D. Barcelli, A. Bemporad, B. A. G. Genuit, W. P. M. H. Heemels, T. Poggi, M. Rubagotti, and M. Storace, "MOBY-DIC: A Matlab toolbox for the circuit design of explicit MPC," in *Proc. IFAC Conf. Nonlin. Model Predictive Control*, Noordwijkerhout, The Netherlands, Aug. 2012, pp. 218–225.

[27] M. Lazar, W. P. M. H. Heemels, and A. R. Teel, "Lyapunov functions, stability and input-to-state stability subtleties for discrete-time discontinuous systems," *IEEE Trans. Autom. Control*, vol. 54, no. 10, pp. 2421–2425, Oct. 2009.

[28] A. Bemporad, M. Morari, V. Dua, and E. N. Pistikopoulos, "The explicit linear quadratic regulator for constrained systems," *Automatica*, vol. 38, no. 1, pp. 3–20, 2002.

[29] S. P. Boyd and L. Vandenberghe*, Convex Optimization*. Cambridge, U.K.: Cambridge Univ. Press, 2004.

[30] D. Avis, "LRS: A revised implementation of the reverse search vertex enumeration algorithm," in *Polytopes, Combinatorics and Computation*, ser. DMV-Seminar 29, G. Kalai and G. Ziegler, Eds. Boston, MA: Birkhauser Verlag, 2000, pp. 177–198.

[31] K. Fukuda and A. Prodon, "Double description method revisited," in *Combinatorics and Computer Science*, ser. Lecture Notes in Computer Science. New York: Springer, 1996, pp. 91–111.

[32] K. Fukuda, Institute for Operations Research ETH-Zentrum 1997, cdd/cdd+ Reference Manual.

[33] C. N. Jones, E. C. Kerrigan, and J. M. Maciejowski, Equality Set Projection: A New Algorithm for the Projection of Polytopes in Halfspace Representation Univ. of Cambridge, Cambridge, U.K., Tech. Rep., Jun. 2004.

[34] R. Suard, J. Lofberg, P. Grieder, M. Kvasnica, and M. Morari, "Efficient computation of controller partitions in multi-parametric programming," in *Proc. Conf. Decision Control*, Atlantis, Bahamas, Dec. 2004, pp. 3643–3648.