Robust Explicit MPC Based on Approximate Multiparametric Convex Programming

D. Muñoz de la Peña, Alberto Bemporad, and Carlo Filippi

***Abstract*—Many robust model predictive control (MPC) schemes require the online solution of a computationally demanding convex program. For deterministic MPC schemes, multiparametric programming was successfully applied to move offline most of the computation. In this paper, we adopt a general approximate multiparametric algorithm recently suggested for convex problems and propose to apply it to a classical robust MPC scheme. This approach enables one to implement a robust MPC controller in real time for systems with polytopic uncertainty, ensuring robust constraint satisfaction and robust convergence to a given bounded set.**

***Index Terms*—Model predictive control (MPC), multiparametric programming, robust control, uncertain systems.**

I. Introduction

Model predictive control (MPC) is a control technique that is able to cope in a direct way with multivariable systems, constraints, and uncertainty. At each sampling time, a finite horizon optimal control problem is solved based on a given model of the system. One of the main drawbacks of MPC is the time needed to evaluate the solution of the posed optimization problem. For linear systems, when no uncertainty is taken into account, MPC requires the solution of a quadratic or a linear programming problem. These are well known problems and efficient tools are available for solving them. Also, multiparametric programming has been applied with success to solve offline such optimization problems in order to obtain an explicit description of the control law (see [1]–[4]). Multiparametric programming considers optimization problems where the data depends on one or more parameters. The parameter space is systematically subdivided into characteristic regions where the optimal value and an optimizer are given as explicit functions of the parameters.

One approach used in robust MPC is to minimize the objective function for the worst possible realization of the uncertainty. This strategy is known as min–max and was originally proposed in [5] in the context of robust optimal control. In robust MPC the problem was first tackled in [6]. Several different robust MPC schemes have been proposed in the literature. All of them have in common a high computational burden (see [7]–[12] and the references therein). However, the optimization problem associated with those schemes can often be posed as a convex programming problem.

For linear cost functions, robust MPC controllers have been obtained in explicit form (see [13] and [14]). The piecewise affine nature of the solution for quadratic cost functions for the open-loop formulation with additive uncertainties has been proved in [15]. Also, an efficient offline algorithm for parametric uncertainties was given in [16].

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D. Muñoz de la Peña is with the Departamento de Ingeniería de Sistemas y Automática, University of Seville, 41092 Seville, Spain (e-mail: davidmps@cartuja.us.es).

A. Bemporad is with Dipartimento di Ingegneria dell'Informazione, University of Siena, 53100 Siena, Italy (e-mail: bemporad@dii.unisi.it).

C. Filippi is with the Dipartimento di Metodi Quantitativi, University of Brescia, 25122 Brescia, Italy (e-mail: filippi@eco.unibs.it).

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Recently, approximate multiparametric convex programming solvers have been proposed in [17] and [18]. The latter is based on a general approach that obtains a suboptimal explicit solution for a given convex problem with a guaranteed bound on the error. In this paper we apply the technique of [18] to a classic MPC robust scheme, namely the controller proposed by Kothare *et al.* in [7]. We first obtain an explicit easy-to-implement piecewise affine description of the control law with an arbitrary degree of accuracy, and then prove that, for any chosen degree of accuracy, constraints are handled robustly and the system converges to a bounded set.

II. Problem Formulation

Consider the uncertain linear time-varying (LTV) system with polytopic uncertainty

$$x_{k+1} = A_k x_k + B_k u_k \quad y_k = C x_k, [A_k B_k] \in \Omega \tag{1}$$

where $u_k \in R^{n_u}$ is the control input, $x_k \in R^{n_x}$ is the state vector, $y_k \in R^{n_y}$ is the output, and Ω is the convex hull of given matrices $[A^1 B^1], \dots, [A^L B^L]$.

System (1) is required to satisfy the input and output constraints

$$\begin{aligned} |e_r^T u_k| &\le u_{r,\max}, \qquad k \ge 0, \\ &\qquad\qquad r = 1, 2, \dots, n_u \\ |e_r^T y_k| &\le y_{r,\max}, \qquad k > 0, \\ &\qquad\qquad r = 1, 2, \dots, n_y \end{aligned} \tag{2}$$

where e_r is the rth column of the identity matrix of appropriate dimension.

The controller proposed in [7], that will be referred to as "Kothare's controller" from now on, minimizes an upper bound of the worst case infinite time cost function

$$J_\infty(x) = \max_{[A_k B_k] \in \Omega, k \ge 0} \sum_{k=0}^{\infty} x_k^T Q_c x_k + u_k^T R_c u_k$$
$$\text{s.t.}(1), (2) \text{ and} x_0 = x \qquad \forall [A_k B_k] \in \Omega$$

with Q_c and R_c positive definite.

Assume that a state feedback law $u_k = F x_k$ is used, and that there exists a quadratic, strictly convex function $x^T P x$ that satisfies the following constraint for all $[A_k B_k] \in \Omega$:

$$x_{k+1}^T P x_{k+1} - x_k^T P x_k \le -x_k^T Q_c x_k - u_k^T R_c u_k. \tag{3}$$

By summing (3) for all $k \ge 0$ and requiring that $x_k \to 0$ as $k \to \infty$, we obtain the upper bound $J_\infty(x) \le x^T P x$.

Kothare's controller is based on the following result.

Property 1 (cf. [7, Th. 2]): For (1), let γ, Q, Y, Z, and X satisfy the LMI constraints for the state x

$$\begin{aligned} &\begin{bmatrix} 1 & x^T \\ x & Q \end{bmatrix} \ge 0, \; Q > 0 \\ &\begin{bmatrix} Q & QA^{j^T} + Y^T B^{j^T} & QQ_c^{\frac{1}{2}} & Y^T R_c^{\frac{1}{2}} \\ A^j Q + B^j Y & Q & 0 & 0 \\ Q_c^{\frac{1}{2}} Q & 0 & \gamma I & 0 \\ R_c^{\frac{1}{2}} Y & 0 & 0 & \gamma I \end{bmatrix} \ge 0, \\ &j = 1, 2, \dots, L \\ &\begin{bmatrix} X & Y \\ Y^T & Q \end{bmatrix} \ge 0, \text{ with } X_{rr} \le u_{r,\max}^2, \\ &r = 1, 2, \dots, n_u \\ &\begin{bmatrix} Z & C(A^j Q + B^j Y) \\ (A^j Q + B^j Y)^T C^T & Q \end{bmatrix} \ge 0 \\ &\text{with } Z_{rr} \le y_{r,\max}^2, \\ &r = 1, 2, \dots, n_y, \; j = 1, 2, \dots, L \end{aligned} \tag{4}$$

where $M_{rr} = e_r^T M e_r$. Let $F \triangleq YQ^{-1}$ and $P \triangleq \gamma Q^{-1}$. Then $\gamma \geq x^T P x$ and the constraints (2) and (3) are satisfied for the feedback matrix F and the matrix P.

To evaluate the control input for a given state x, Kothare's control algorithm solves the following SDP problem:

$$V^*(x) = \min_{\gamma,Q,Y,X,Z} \gamma \quad \text{s.t. (4) for state } x. \tag{5}$$

From the optimizers $\gamma^*(x), Q^*(x)$, and $Y^*(x)$, the feedback gain $F^*(x) = Y^*(x)Q^*(x)^{-1}$ is obtained. The applied control input is $u = F^*(x)x$, which robustly drives the system to the origin.

From the optimization problem one also gets matrix $P^*(x) = \gamma^*(x)Q^*(x)^{-1}$ which defines an upper bound on the worst case infinite-time cost function for the given feedback law, that is

$$J_\infty(x) \leq x^T P^*(x) x \leq V^*(x).$$

The following property will be used in the sequel.

Property 2 ([7, Lemma 2]): Any quintuple (Y, Q, γ, Z, X) satisfying (4) at time k for x_k also satisfies (4) at time $k+1$ if $u_k = YQ^{-1}x_k$ is applied.

Problem (5) is an SDP problem and efficient tools exist for solving it. However, the computational burden may still be too high in many real applications. An efficient suboptimal offline implementation was presented recently in [16] and is based on the computation of invariant ellipsoids. Here we take a different route and propose to use multiparametric techniques to implement in an efficient way an approximation of this control law. More precisely, we consider the algorithm suggested in [18].

III. Multiparametric Convex Programming

In this section, the multiparametric algorithm suggested in [18] is reviewed. This algorithm obtains, in explicit piecewise affine form, a suboptimal solution of a multiparametric convex optimization problem of the form

$$W^*(\theta) = \min_z \{W(z,\theta) : g_i(z,\theta) \leq 0, \qquad (i = 1, 2, \ldots, p)\} \tag{6}$$

where $z \in R^{n_z}$ are the decision variables, $\theta \in R^{n_\theta}$ are the parameters, and W and g_i are jointly convex functions of the optimization variables and the parameters, so that W^* is a convex function (see [19] and [20]). The multiparametric approach of [18] consists of an algorithm for defining a suboptimal solution $\hat{z}(\theta)$ that is a piecewise affine function of the parameters. The solution is defined for a given full dimensional polyhedron $S = \{\theta \in R^{n_\theta} | A\theta \leq b\}$ of parameters for which (6) is feasible. The suboptimal solution is a piecewise affine function defined over a partition of S made out of n_r critical simplices CS_i

$$\hat{z}(\theta) = \hat{z}^i(\theta) = H_z^i \theta + h_z^i, \ \forall \theta \in \mathrm{CS}_i, \qquad i = 1, 2, \ldots, n_r.$$

The algorithm proposed in [18] is divided in two phases. In the first phase, the polyhedral region S to be characterized is triangulated into a minimal set of simplices. In the second phase, the simplices are subdivided into smaller ones until an upper bound on the maximum error inside each simplex is smaller than a given accuracy threshold ϵ. Because of the recursive nature of the algorithm and of the method for subdividing each simplex, the explicit suboptimizer is a piecewise linear function of the parameters that is organized in a tree structure for evaluation (see [18] for details). Hence, the online computational burden depends only on the maximum tree depth T_d and on the dimension of the parameter vector n_θ. The maximum number of linear inequalities that must be evaluated in order to find the solution is linear in the state dimension and the maximum depth of the tree $(n_\theta T_d)$.

Property 3 ([18]): For all state vectors inside S, the suboptimal solution $\hat{z}(\theta)$, obtained by applying the approximate multiparametric convex programming algorithm of [18] to solve (6) with a fixed $\epsilon > 0$, satisfies

$$g_i(\hat{z}(\theta), \theta) \leq 0, \qquad (i = 1, 2, \ldots, p) \tag{7a}$$
$$W^*(\theta) \leq W(\hat{z}(\theta), \theta) \leq W^*(\theta) + \epsilon. \tag{7b}$$

IV. Multiparametric Approach to Kothare's Controller

Kothare's controller is evaluated at each time step by solving the SDP problem (5) that depends on the current state x. the multiparametric technique reviewed in the previous section can also be applied to SDPs [18]. In particular, let

- the parameter vector θ be defined as the state vector x;
- the optimizer vector consist of the free variables $z = \{\gamma, Q, Y, X, Z\}$ of (5);
- the objective function be linear, $W(z, \theta) = \gamma = c^T z$;
- the constraints $g_i(z, \theta)$ be defined by (4).

The approximate multiparametric convex programming algorithm defines a piecewise affine function for the suboptimizer of Problem (5) with a fixed error bound ϵ. The following piecewise affine functions of interest are obtained

$$\begin{aligned} \gamma(x) &= \gamma^i(x) = H_\gamma^i x + h_\gamma^i && \forall x \in \mathrm{CS}_i \\ Q(x) &= Q^i(x) = H_Q^i x + h_Q^i && \forall x \in \mathrm{CS}_i \\ Y(x) &= Y^i(x) = H_Y^i x + h_Y^i && \forall x \in \mathrm{CS}_i, \\ &&& i = 1, 2, \ldots, n_r \end{aligned} \tag{8}$$

where n_r is the number of critical simplices. Note that matrices X and Z are not used for defining the multiparametric control law, so in the following they will not be taken into account.

Using the previous piecewise affine suboptimizers of the SDP problem, an approximate control law can be efficiently implemented. In the following section, the proposed approach and the main properties are analyzed.

A. Properties of the Proposed Approach

The following lemmas will be used in the proof of the main theorem.

Lemma 1: i) The suboptimizers $\gamma(x), Q(x), Y(x)$ are feasible for (5); ii) $F(x) = Y(x)Q(x)^{-1}$ and $P(x) = \gamma(x)Q(x)^{-1}$ satisfy (2) and (3); iii) the following inequalities hold:

$$V^*(x) \leq x^T P(x) x \leq V^*(x) + \epsilon. \tag{9}$$

Proof: i) Follows from (7a) (Property 3) because the constraints $g_i(z, \theta)$ are defined by (4): ii) follows from Property 1; iii) following (7b) (Property 3), $V^*(x) \leq \gamma(x) \leq V^*(x) + \epsilon$ as $\gamma(x)$ is the upper bound of $V^*(x)$ obtained applying the multiparametric algorithm for a bound on the error ϵ (what in Section III was denoted as $W(\hat{z}(\theta), \theta)$). On the other hand, as the suboptimizers are feasible for (5), it holds $x^T P(x) x \leq \gamma(x)$. ■

Lemma 2: Consider a system of the form (1) and the feedback gain given by $F(x) = Y(x)Q(x)^{-1}$, where $\gamma(x), Q(x)$, and $Y(x)$ are taken from a suboptimizer of (5) over a set S with a given bound on the error $\epsilon > 0$. For all states $x_k \in S$, if $u_k = F(x_k)x_k$ the following inequality holds

$$V^*(x_{k+1}) - V^*(x_k) \leq -x_k^T Q_c x_k + \epsilon \qquad \forall [A_k \ B_k] \in \Omega. \tag{10}$$

Proof: For each $x_k \in S$ the multiparametric convex programming algorithm provides a suboptimizer $\gamma(x_k), Q(x_k), Y(x_k)$ of (5) such that (3) and (9) hold for $F(x_k)$ and $P(x_k) = \gamma(x_k)Q(x_k)^{-1}$ (Lemma 1). By Property 2, $\gamma(x_k), Q(x_k), Y(x_k)$ is also a feasible solution of (5) for all possible x_{k+1}, so that $V^*(x_{k+1}) \leq x_{k+1}^T P(x_k) x_{k+1}$. As $R_c > 0$, by replacing $x_k^T P(x_k) x_k$ with

$V^*(x_k)+\epsilon$ and $x_{k+1}^T P(x_k) x_{k+1}$ with $V^*(x_{k+1})$ in (3), we obtain inequality (10). ■

The proposed control law and its convergence and robustness properties are stated in the following theorem.

Theorem 1: Consider the control law

$$u_k = \hat{F}(x_k)x_k$$
$$[\hat{F}(x_k)\hat{P}(x_k)] = \begin{cases} [F(x_k)P(x_k)], & \text{if } x_k \in S \\ [\hat{F}(x_{k-1})\hat{P}(x_{k-1})], & \text{otherwise} \end{cases} \tag{11}$$

where S is a full dimensional polyhedron containing the origin in its interior, $F(x) = Y(x)Q(x)^{-1}$ and $P(x) = \gamma(x)Q(x)^{-1}$ where $\gamma(x), Q(x), Y(x)$ is a suboptimizer of (5) over S with an error bound $\epsilon > 0$ for x_k. Then, if $x_0 \in S$, the controller defined by (11) robustly regulates the system to a bounded set Ω_α of the state–space while satisfying (2) for all possible uncertainties, where $\Omega_\alpha = \{x \in R^{n_x} | V^*(x) \le \alpha\}$, $\alpha = \max_{x\in\Phi_\epsilon}\{V^*(x)+\epsilon - x^T Q_c x\}$, and $\Phi_\epsilon = \{x \in S |\, x^T Q_c x \le \epsilon\}$.

Proof: In order to prove that the closed-loop system is ultimately bounded we will first prove convergence to Φ_ϵ by Lyapunov arguments. Then, we will show that once the state lands in this set, even if it may leave it again, in no case it will go outside the set Ω_α, from which it will return again to Φ_ϵ. In this way, Ω_α is an invariant set for the system.

Let $x_k \notin \Phi_\epsilon$. By Lemma 2, if $x_k \in S$, then $V^*(x_{k+1}) < V^*(x_k)$, for all $[A_k\ B_k] \in \Omega$. If $x_k \in S$ for all $k \ge 0$ then clearly the system converges to Φ_ϵ because $V^*(x)$ acts as a Lyapunov function. Suppose instead there exists k such that $x_k \in S$ and $x_{k+1} \notin S$. For all $h \ge 1$ such that $x_{k+j} \notin S$ $(j = 1,2,\ldots,h)$, by (11) we have $\hat{F}(x_{k+h}) = \hat{F}(x_k)$ and $\hat{P}(x_{k+h}) = \hat{P}(x_k)$. Since $\hat{F}(x_k)$ and $\hat{P}(x_k)$ are defined by the suboptimizer of (5) for $x = x_k$, taking into account Property 2 and (3), the following inequality holds:

$$x_{k+h+1}^T \hat{P}(x_k) x_{k+h+1} < x_{k+h}^T \hat{P}(x_k) x_{k+h} \qquad \forall [A_k\ B_k] \in \Omega.$$

This means that $x_{k+h}^T \hat{P}(x_k) x_{k+h}$ keeps decreasing while $x_{k+h} \notin S$. As S contains a ball centered in the origin, using Lyapunov arguments it is easy to see that there exists a finite $\bar{h}$ such that $x_{k+\bar{h}} \in S$. Then either $x_{k+\bar{h}} \in \Phi_\epsilon$ or not. In the latter case, in order to prove convergence to Φ_ϵ using Lyapunov arguments, $V^*(x_{k+\bar{h}})$ must be lower than $V^*(x_k)$. Again, taking into account that $\hat{F}(x_k)$ and $\hat{P}(x_k)$ are defined by a suboptimizer of (5) for $x = x_k$, which is also feasible for all x_{k+j} with $j \le \bar{h}$, using Property 2, (9), and (3), the following inequalities can be stated for all $j \le \bar{h}$:

$$\begin{aligned} V^*(x_{k+j}) &\le x_{k+j}^T \hat{P}(x_k) x_{k+j} \le x_k^T \hat{P}(x_k) x_k - x_k^T Q_c x_k \\ &\le V^*(x_k) + \epsilon - x_k^T Q_c x_k. \end{aligned} \tag{12}$$

By taking into account that $x_k \notin \Phi_\epsilon$, it can be seen that $V^*(x_{k+\bar{h}}) < V^*(x_k)$. As $V^*(x)$ is a convex function, $\Phi_\epsilon \subseteq \Omega_\alpha$ because $\alpha \ge \max_{x\in\Phi_\epsilon} V^*(x)$. Hence, it is also proved convergence to Ω_α. Now, we will prove that once in Φ_ϵ, the state will remain inside Ω_α.

As $\Phi_\epsilon \subseteq S$, Lemma 2 holds for all $x_k \in \Phi_\epsilon$ so $V^*(x_{k+1}) \le \alpha$. This means that if $x_k \in \Phi_\epsilon$ then $x_{k+1} \in \Omega_\alpha$. Following the previous ideas, using (9) and (12), it is easy to see that if $x_k \in \Phi_\epsilon$ and $x_{k+1} \notin \Phi_\epsilon$ the system will enter again Φ_ϵ without leaving Ω_α.

Robust satisfaction of the constraints is assured because, by (11) and Property 2, at each time step a feedback gain obtained from a feasible solution of (5) is applied. ■

B. Complexity

The complexity of the controller is measured by both the number of regions and the evaluation time. In general it is not possible to bound *a priori* the number of regions of a multiparametric solution given by the proposed approach (see [18] for a discussion). Because of the recursive triangulation, the complexity of the solution usually grows exponentially with the dimension of the parameter vector. It also depends greatly on the gradient of the optimal cost function $V^*(x)$ that is approximated.

TABLE I
NUMERICAL RESULTS FOR SYSTEMS OF DIFFERENT ORDERS. $S = \{x : \|x\|_\infty \le \mathrm{bnd_x}\}$, T_D IS THE TREE DEPTH, n_r IS THE NUMBER OF REGIONS, T_{LMI} (S) IS THE AVERAGE TIME FOR SOLVING THE LMI (5), T_{mp} (S) THE TIME FOR EVALUATING THE PIECEWISE AFFINE LAW AND $V_{\max}$ IS THE MAXIMUM VALUE OF $V^*(x)$ IN S

n_x	n_u	$\mathrm{bnd_x}$	T_D	n_r	T_{LMI}	T_{mp}	$V_{\max}$
2	1	1	4	44	0.5	0.001	6.2643
2	1	2	6	184	0.5	0.001	25.0720
2	1	5	8	1076	0.5	0.008	156.7513
2	1	10	10	+5000	0.5	0.005	626.4035
3	2	1	5	248	0.6	0.009	4.4669
3	2	2	8	3386	0.6	0.04	17.8547
3	2	5	12	62066	0.7	0.05	111.6369
4	2	1	8	3056	1.0	0.04	5.6029
4	2	2	12	+60000	1.2	0.05	22.4147

Numerical results[1] for three systems (omitted for brevity) are reported in Table I. These systems are unstable and unconstrained. The weighting matrices are equal to the identity matrix of appropriate dimension. The approximated control law has been obtained for different regions of the state–space $S = \{x : \|x\|_\infty \le \mathrm{bnd_x}\}$. Table I shows that the parameter $\mathrm{bnd_x}$ affects greatly the number of regions of the controller. This is due to the fact that, not only the region to be explored is greater, but also that the gradient of $V^*(x)$ increases as x is farther from the origin. For each region S, the maximum value of $V(x)$ is given in entry $V_{\max}$ to show this issue. Despite the high number of regions, it is apparent that the average time T_{LMI} for solving the SDP (5) is sensibly larger than the time T_{mp} for evaluating the piecewise affine function. This is due to the fact that the maximum number of linear inequalities that must be evaluated in order to find the solution is linear in the state dimension and in the maximum depth of the tree $(n_x T_d)$.

C. Modified Error Bound

This section presents an alternative approach, based on modifying the multiparametric algorithm to enforce that the error inside each simplex is small enough to guarantee that the optimal cost function decreases at each time step.

Proposition 1: Consider controller (11) based on an approximate solution (8) of the multiparametric convex program (5) on S, where S is a full dimensional polyhedron containing the origin in its interior, such that the error inside each simplex CS_i is less than $\epsilon_{QSi} = \min_{x\in \mathrm{CS}_i} x^T Q_c x$. If $x_0 \in S$ then (11) robustly stabilizes system (1).

Proof: For any state vector inside $\mathrm{CS}_i \subseteq S$ Lemma 2 holds, and therefore if $x_k^T Q_c x_k > \epsilon_{QSi}$, then $V^*(x_{k+1}) - V^*(x_k) < 0, \forall [A_k\ B_k] \in \Omega$. Following the same ideas as in the proof of Theorem 1, it is easy to see that if the state leaves the set S, the controller assures that it will enter again with a lower value of $V^*(x)$. Following Lyapunov arguments, as S contains a ball centered in the origin, it is immediate to prove that the closed-loop system is regulated to the origin. Robust satisfaction of the constraints is assured because, by (11) and Property 2, at each time step a feedback gain obtained from a feasible solution of (5) is applied. ■

The approximate multiparametric convex programming algorithm can be modified to enforce the error bound ϵ_{QSi} by modifying the stopping criterion of the second phase of the algorithm: A given simplex is then subdivided if the upper bound on the error is greater than or

[1]Numerical results obtained in a AMD Athlom(tm) XP 2800+ using MATLAB.

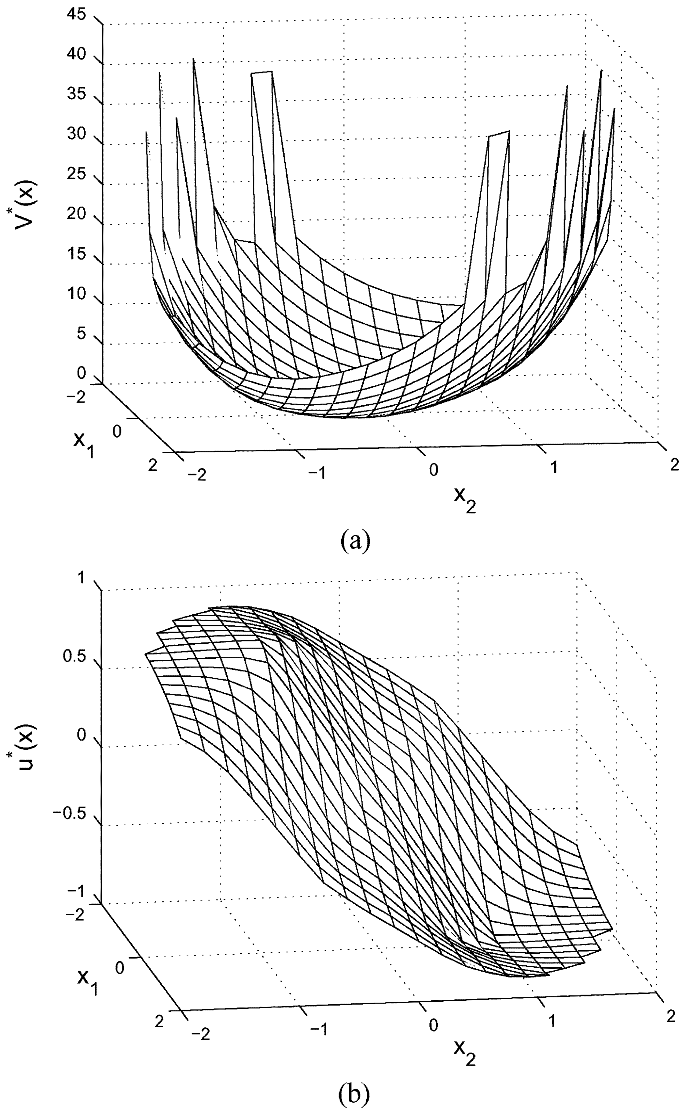

Fig. 1. (a) Optimal cost function $V^*(x)$. (b) Control input $u^*(x) = F^*(x)x$.

equal to ϵ_{QSi}, which can be evaluated solving a quadratic programming problem. The state space partition obtained is more complex around the origin (where $\epsilon_{QSi} \simeq 0$). In fact, to obtain a finite partition, an additional subdivision criterion must be added to deal with the simplex that contains the origin. In this work, a minimum volume criterion is adopted.

V. Numerical Examples

In this section, we exemplify the ideas developed above on the following simple LTV second-order uncertain system:

$$A^1 = \begin{bmatrix} 0.9 & 0.9 \\ 0 & 0.9 \end{bmatrix} \quad A^2 = \begin{bmatrix} 0.9 & 0.5 \\ 0 & 0.5 \end{bmatrix}$$
$$B^{1,2} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{13}$$

with $\|x\|_\infty \leq 2, \|u\|_\infty \leq 1, Q_c = I$, and $R_c = 1$.

For this system, Fig. 1(a) and (b), respectively, show the optimal upper bound $V^*(x)$ defined by Kothare's controller and the corresponding optimal control law $u^*(x)$. Note that the value of $V^*(x)$ goes up to 30. Table II shows the number of regions in the state partition for different values of ϵ. Fig. 2(a) shows the state partition obtained for the absolute error bound $\epsilon = 0.1$. The state partition is more complex near the boundary of the feasible region. This is due to the fact that towards the boundaries the optimal cost function to be approximated has a larger gradient. Fig. 2(b) shows the state partition of a suboptimizer which assures a bound on the error on each simplex lower than ϵ_{QSi} as in Section IV-C. It can be noticed how the partition is rather complex around the origin (where $V^*(x) \simeq 0$) but less towards the boundary.

TABLE II
Number of Regions n_r of the State Partition for Different Values of the Error Bound ϵ for System (13)

ϵ	0.9	0.8	0.7	0.6	0.5	0.4	0.3	0.2	0.1
n_r	70	86	124	148	170	176	248	379	766

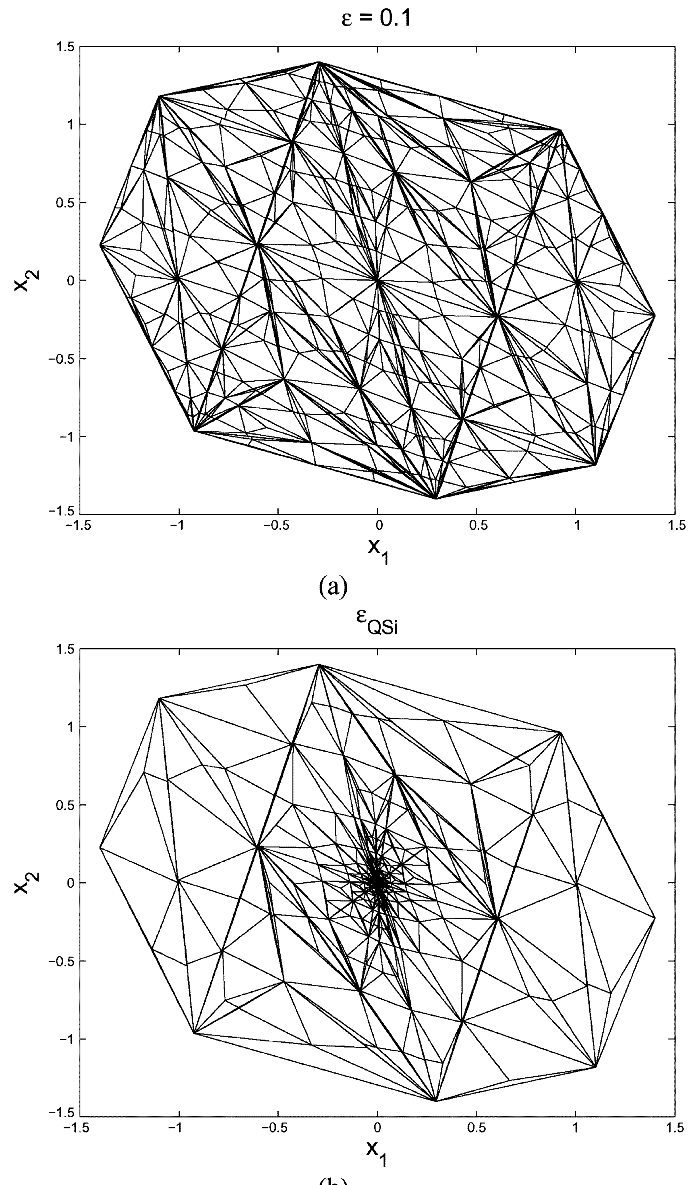

Fig. 2. State–space partition corresponding to (a) the approximate solution with absolute error bound $\epsilon = 1$ and to (b) the stabilizing criterion described in Section IV-C.

VI. Conclusion

Multiparametric quadratic and linear programming theory has been applied with success for implementing deterministic MPC controllers. In this note, we have proposed to apply the approximate multiparametric convex programming solver of [18] to the robust MPC control scheme proposed in [7]. An explicit description of the control law is obtained for ease of implementation of robust MPC. The control law assures robust constraint handling and robust convergence to a given bounded set. Also, an alternative approach has been given in order to assure convergence to the origin.

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Comments on "On the Global Stability of Delayed Neural Networks"

Chuandong Li and Xiaofeng Liao

***Abstract*—In this note, we show that the theorem and its corollary given in the above paper are not correctly stated. In addition, a revised version is proposed in the light of the original idea.**

***Index Terms*—Asymptotical stability, equilibrium point, neural networks, time delay.**

I. Introduction

Consider the delayed neural network given in [1]

$$\dot{u}_i(t) = -c_i u_i(t) + \sum_{j=1}^{n} a_{ij} f_j\left(u_j(t)\right) + \sum_{j=1}^{n} b_{ij} g_j\left(u_j(t-\tau_j)\right) + I_i, \qquad i = 1,2,\ldots,n. \quad (1)$$

Assume that $u^* = (u_1^*, \ldots u_n^*)^T$ is an equilibrium point of system (1), the transformation $x(t) = u(t) - u^*$ puts (1) into the following form:

$$\dot{x}_i(t) = -c_i x_i(t) + \sum_{j=1}^{n} a_{ij} \phi_j\left(x_j(t)\right) + \sum_{j=1}^{n} b_{ij} \psi_j\left(x_j(t-\tau_j)\right), \qquad i = 1,2,\ldots,n. \quad (2)$$

The main theorem presented in [1] is restated as follows.

Theorem 1: The equilibrium u^* of (1) is globally asymptotically stable if there exist constants $p_k > 0 (k = 1,2,\ldots,L_1)$, $q_k > 0 (k = 1,2,\ldots,L_2)$, $\gamma_j > 0$, $\alpha_{ij}, \alpha_{ij}^*, \beta_{ij}, \beta_{ij}^*, \xi_{ij}, \xi_{ij}^*, \eta_{ij}, \eta_{ij}^* \in R$, $i,j = 1,2,\ldots,n$ such that

$$\sum_{j=1}^{n} \left(\sum_{k=1}^{L_1} p_k |a_{ij}|^{\frac{r\alpha_{ij}}{p_k}} m_j^{r\xi_{ij}} + \frac{\gamma_j}{\gamma_i} m_i^{r\xi_{ji}^*} |a_{ji}|^{r\alpha_{ji}^*} + \sum_{k=1}^{L_2} q_k |b_{ij}|^{\frac{r\beta_{ij}}{q_k}} n_j^{r\eta_{ij}} + \frac{\gamma_j}{\gamma_i} n_i^{r\eta_{ji}^*} |b_{ji}|^{r\beta_{ji}^*} \right) < rc_i \quad (3)$$

holds for each $i = 1,2,\ldots,n$, in which $L_1\alpha_{ij} + \alpha_{ij}^* = 1$, $L_1\xi_{ij} + \xi_{ij}^* = 1$, $L_2\beta_{ij} + \beta_{ij}^* = 1$, $L_2\eta_{ij} + \eta_{ij}^* = 1$ for all $i,j = 1,2,\ldots,n$ and $r - 1 = \sum_{k=1}^{L_1} p_k = \sum_{k=1}^{L_2} q_k$.

Two errors appear in the proof of Theorem 1 when the authors attempt to obtain (7) by using [1, eq. (5)]. Concisely, the following

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The authors are with the Department of Computer Science and Engineering, Chongqing University, Chongqing 400044, China (e-mail: licd@cqu.edu.cn).

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