Technical Notes and Correspondence

Min–Max Control of Constrained Uncertain Discrete-Time Linear Systems

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***Abstract*—For discrete-time uncertain linear systems with constraints on inputs and states, we develop an approach to determine state feedback controllers based on a min–max control formulation. Robustness is achieved against additive norm-bounded input disturbances and/or polyhedral parametric uncertainties in the state-space matrices. We show that the finite-horizon robust optimal control law is a continuous piecewise affine function of the state vector and can be calculated by solving a sequence of multiparametric linear programs. When the optimal control law is implemented in a receding horizon scheme, only a piecewise affine function needs to be evaluated on line at each time step. The technique computes the robust optimal feedback controller for a rather general class of systems with modest computational effort without needing to resort to gridding of the state–space.**

***Index Terms*—Constraints, multiparametric programming, optimal control, receding horizon control (RHC), robustness.**

I. Introduction

A control system is robust when stability is preserved and the performance specifications are met for a specified range of model variations and a class of noise signals (uncertainty range). Although a rich theory has been developed for the robust control of *linear systems*, very little is known about the robust control of *linear systems with constraints*. This type of problem has been addressed in the context of constrained optimal control, and, in particular, in the context of robust receding horizon control (RHC) and robust model predictive control (MPC); see, e.g., [1] and [2]. A typical robust RHC/MPC strategy consists of solving a min–max problem to optimize robust performance (the minimum over the control input of the maximum over the disturbance) while enforcing input and state constraints for all possible disturbances. Min–max robust RHC was originally proposed by Witsenhausen [3]. In the context of robust MPC, the problem was tackled by Campo and Morari [4], and further developed in [5] for multiple-input–multiple-output finite-impulse response plants. Kothare *et al.* [6] optimize robust performance for polytopic/multimodel and linear fractional uncertainty, Scokaert and Mayne [7] for additive disturbances, and Lee and Yu [8] for linear time-varying and time-invariant state-space models depending on a vector of parameters $\theta \in \Theta$, where Θ is either an ellipsoid or a polyhedron. In all cases, the resulting min–max problems turn out to be computationally demanding, a serious drawback for online receding horizon implementation.

In this note we show how state feedback solutions to min–max robust constrained control problems based on a linear performance index can be computed offline for systems affected by additive norm-bounded exogenous disturbances and/or polyhedral parametric uncertainty. We show that the resulting optimal state feedback control law is piecewise affine so that the online computation involves a simple function evaluation. Earlier results have appeared in [9], and very recently [10] presented various approaches to characterize the solution of the open loop min–max problem with a quadratic objective function. The approach of this note relies on multiparametric solvers, and follows the ideas proposed earlier in [11]–[13] for the optimal control of linear systems and hybrid systems without uncertainty. More details on multiparametric programming can be found in [14], [15] for linear programs, in [16] for nonlinear programs, and in [11], [17], and [18] for quadratic programs.

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II. Problem Statement

Consider the following discrete-time linear uncertain system:

$$x(t+1) = A(w(t))x(t) + B(w(t))u(t) + Ev(t) \tag{1}$$

subject to the constraints

$$Fx(t) + Gu(t) \le f \tag{2}$$

where $x(t) \in \mathbb{R}^n$ and $u(t) \in \mathbb{R}^{n_u}$ are the state and input vector, respectively. Vectors $v(t) \in \mathbb{R}^{n_v}$ and $w(t) \in \mathbb{R}^{n_w}$ are unknown exogenous disturbances and parametric uncertainties, respectively, and we assume that only bounds on $v(t)$ and $w(t)$ are known, namely that $v(t) \in \mathcal{V}$, where $\mathcal{V} \subset \mathbb{R}^{n_v}$ is a given polytope containing the origin, $\mathcal{V} = \{v: Lv \le \ell\}$, and that $w(t) \in \mathcal{W} = \{w: Mw \le m\}$, where $\mathcal{W}$ is a polytope in $\mathbb{R}^{n_w}$. We also assume that $A(w)$, $B(w)$ are affine functions of w, $A(w) = A_0 + \sum_{i=1}^{n_w} A_i w^i$, $B(w) = B_0 + \sum_{i=1}^{n_w} B_i w^i$, a rather general time-domain description of uncertainty, which includes uncertain FIR models [4]. A typical example is a polytopic uncertainty set given as the convex hull of n_w matrices (cf. [6]), namely $\mathcal{W} = \{w: -w^i \le 0, \sum_{i=1}^{n_w} w^i \le 1, -\sum_{i=1}^{n_w} w^i \le -1\}$, $A_0 = 0$, $B_0 = 0$.

The following min–max control problem will be referred to as open-loop constrained robust optimal control (OL-CROC) problem

$$J_N^*(x_0) \triangleq \min_{u_0,\ldots,u_{N-1}} J(x_0, U) \tag{3}$$

$$\text{subj. to } \begin{Bmatrix} Fx_k + Gu_k \le f \\ x_{k+1} = A(w_k)x_k + B(w_k)u_k + Ev_k \\ x_N \in \mathcal{X}^f \\ k = 0,\ldots,N-1 \end{Bmatrix} \quad \forall v_k \in \mathcal{V}, w_k \in \mathcal{W} \quad \forall k = 0,\ldots,N-1 \tag{4}$$

$$J(x_0, U) \triangleq \max_{\substack{v_0,\ldots,v_{N-1}\\ w_0,\ldots,w_{N-1}}} \sum_{k=0}^{N-1} \left(\|Qx_k\|_p + \|Ru_k\|_p\right) + \|Px_N\|_p \tag{5}$$

$$\text{subj. to } \begin{cases} x_{k+1} = A(w_k)x_k + B(w_k)u_k + Ev_k \\ v_k \in \mathcal{V} \\ w_k \in \mathcal{W} \\ k = 0,\ldots,N-1 \end{cases} \tag{6}$$

where x_k denotes the state vector at time k, obtained by starting from the state $x_0 \triangleq x(0)$ and applying to model (1) the input sequence $U \triangleq \{u_0,\ldots,u_{N-1}\}$ and the sequences $V \triangleq \{v_0,\ldots,v_{N-1}\}$, $W \triangleq$

$\{w_0, \ldots, w_{N-1}\}$; $p = 1$ or $p = +\infty$, $\|x\|_\infty$ and $\|x\|_1$ are the standard ∞-norm and one-norm in $\mathbb{R}^n$ (i.e., $\|x\|_\infty = \max_{j=1,\ldots,n} |x^j|$ and $\|x\|_1 = |x^1| + \cdots + |x^n|$, where x^j is the jth component of x), $Q \in \mathbb{R}^{n \times n}$, $R \in \mathbb{R}^{n_u \times n_u}$ are nonsingular matrices, $P \in \mathbb{R}^{m \times n}$, and the constraint $x_N \in \mathcal{X}^f$ forces the final state x_N to belong to the polyhedral set

$$\mathcal{X}^f \triangleq \{x \in \mathbb{R}^n \colon F_N x \leq f_N\}. \tag{7}$$

The choice of $\mathcal{X}^f$ is typically dictated by stability and feasibility requirements when (3)–(6) is implemented in a receding horizon fashion. Receding horizon implementations will be discussed in Section IV. Problem (5)–(6) looks for the worst value $J(x_0, U)$ of the performance index and the corresponding worst sequences V, W as a function of x_0 and U, while problem (3)–(4) minimizes the worst performance subject to the constraint that the input sequence must be feasible *for all* possible disturbance realizations. In other words, worst-case performance is minimized under constraint fulfillment against all possible realizations of V, W.

In the sequel, we denote by $U^* = \{u_0^*, \ldots, u_{N-1}^*\}$ the optimal solution to (3)–(6), where $u_j^* \colon \mathbb{R}^n \to \mathbb{R}^{n_u}$, $j = 0, \ldots, N-1$, and by $\mathcal{X}^0$ the set of initial states x_0 for which (3)–(6) is feasible.

The min–max formulation (3)–(6) is based on an *open-loop* prediction, in contrast to the *closed-loop* prediction schemes of [6]–[8], [19], and [20]. In [19], $u_k = F x_k + \bar{u}_k$, where F is a fixed linear feedback law, and $\bar{u}_k$ are new degrees of freedom optimized on line. In [6] $u_k = F x_k$, and F is optimized online via linear matrix inequalities. In [20], $u_k = F x_k + \bar{u}_k$, where $\bar{u}_k$ and F are optimized on line (for implementation, F is restricted to belong to a finite set of LQR gains). In [7] and [8], the optimization is over general feedback laws.

The benefits of closed-loop prediction with respect to the open-loop prediction can be understood by the following reasoning. In open-loop prediction, it is assumed that the state will not be measured again over the prediction horizon and a single-input sequence is to be found that would guarantee constraint satisfaction for all disturbances; it is well known that this could result in the nonexistence of a feasible input sequence and the infeasibility of (3)–(4). In closed-loop prediction, it is assumed that the state will be measured at each sample instant over the prediction horizon and feedback control laws are typically treated as degrees of freedom to be optimized, rather than input sequences; this approach is less conservative (see, e.g., [7] and [20]).

Without imposing any predefined structure on the closed-loop controller, we define the following closed-loop constrained robust optimal control (CL-CROC) problem [3], [8], [21], [22]:

$$J_j^*(x_j) \triangleq \min_{u_j} J_j(x_j, u_j) \tag{8}$$

$$\text{subj. to } \begin{Bmatrix} F x_j + G u_j \leq f \\ A(w_j) x_j + B(w_j) u_j + E v_j \in \mathcal{X}^{j+1} \end{Bmatrix} \quad \forall v_j \in \mathcal{V}, w_j \in \mathcal{W} \tag{9}$$

$$J_j(x_j, u_j) \triangleq \max_{v_j \in \mathcal{V}, w_j \in \mathcal{W}} \{\|Q x_j\|_p + \|R u_j\|_p + J_{j+1}^*(A(w_j) x_j + B(w_j) u_j + E v_j)\} \tag{10}$$

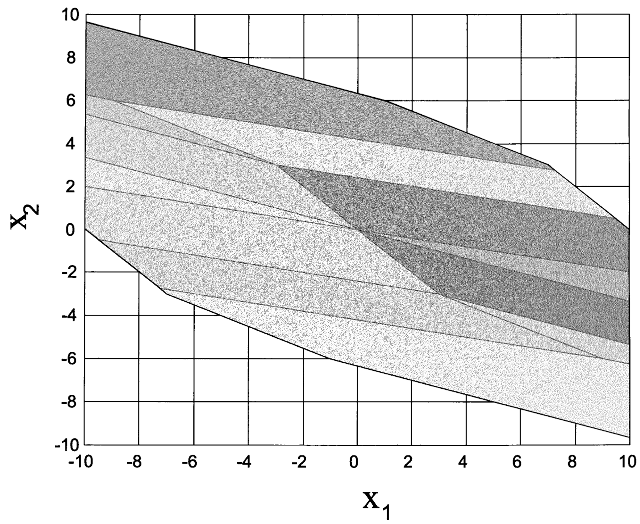
for $j = 0, \ldots, N-1$ and with boundary conditions

$$J_N^*(x_N) = \|P x_N\|_p \tag{11}$$

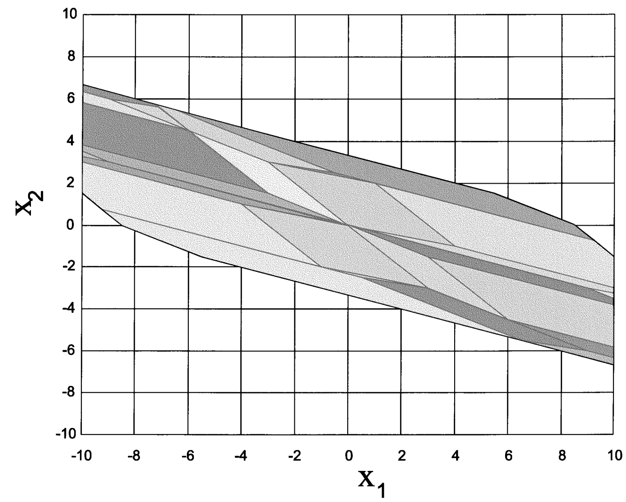
$$\mathcal{X}^N = \mathcal{X}^f \tag{12}$$

where $\mathcal{X}^j$ denotes the set of states x for which (8)–(10) is feasible

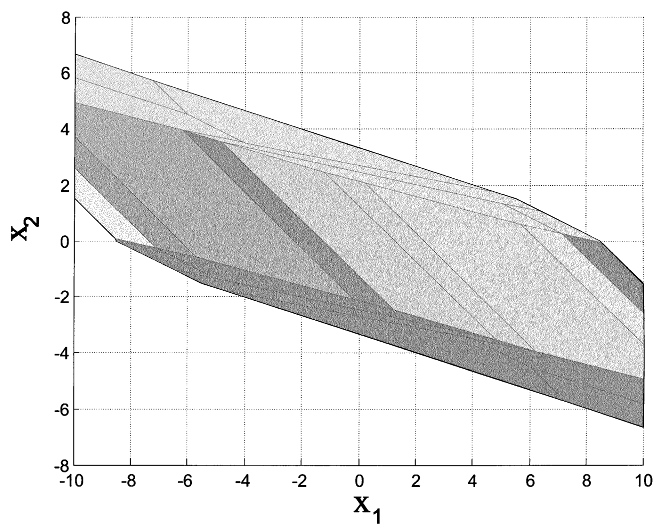
$$\mathcal{X}^j = \{x \in \mathbb{R}^n | \exists u, (F x + G u \leq f, \text{and } A(w) x + B(w) u + E v \in \mathcal{X}^{j+1}, \forall v \in \mathcal{V}, w \in \mathcal{W})\}. \tag{13}$$



(a)



(b)



(c)

Fig. 1. Polyhedral partition of the state–space corresponding to the explicit solution of nominal (a) optimal control, (b) OL-CROC, and (c) CL-CROC.

The reason for including constraints (9) in the minimization problem and not in the maximization problem is that in (10) v_j is free to act regardless of the state constraints. On the other hand, the input u_j has

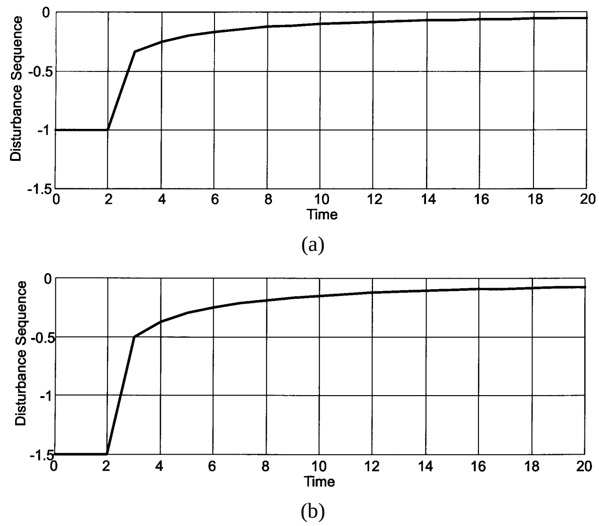


Fig. 2. Disturbances profiles for Example 1.

the duty of keeping the state within the constraints (9) for all possible disturbance realizations.

We will consider different ways of solving OL-CROC and CL-CROC problems in the following sections. First, we will briefly review other algorithms that were proposed in the literature.

For models affected by additive norm-bounded disturbances and parametric uncertainties on the impulse response coefficients, Campo and Morari [4] show how to solve the OL-CROC problem via linear programming. The idea can be summarized as follows. First, the minimization of the objective function (3) is replaced by the minimization of an upper-bound μ on the objective function subject to the constraint that μ is indeed an upper bound for all sequences $V = \{v_0, \ldots, v_{N-1}\} \in \mathcal{V} \times \mathcal{V} \times \cdots \times \mathcal{V}$ (although μ is an upper bound, at the optimum it coincides with the optimal value of the original problem). Then, by exploiting the convexity of the objective function (3) with respect to V, such a continuum of constraints is replaced by a finite number, namely one for each vertex of the set $\mathcal{V} \times \mathcal{V} \times \cdots \times \mathcal{V}$. As a result, for a given value of the initial state $x(0)$, the OL-CROC problem is recast as a linear program (LP).

A solution to the CL-CROC problem was given in [7] using a similar convexity and vertex enumeration argument. The idea there is to augment the number of free inputs by allowing one free sequence U_i for each vertex i of the set $\mathcal{V} \times \mathcal{V} \times \cdots \times \mathcal{V}$, i.e., $N \cdot N_{\mathcal{V}}^N$ free control moves, where $N_{\mathcal{V}}$ is the number of vertices of the set $\mathcal{V}$. By using a causality argument, the number of such free control moves is decreased to $(N_{\mathcal{V}}^N - 1)/(N_{\mathcal{V}} - 1)$. Again, using the minimization of an upper-bound for all the vertices of $\mathcal{V} \times \mathcal{V} \times \cdots \times \mathcal{V}$, the problem is recast as a finite dimensional convex optimization problem, which in the case of ∞-norms or one-norms, can be handled via linear programming as in [4] (see [23] for details). By reducing the number of degrees of freedom in the choice of the optimal input moves, other suboptimal CL-CROC strategies have been proposed, e.g., in [6], [19], and [20].

III. State Feedback Solution to CROC Problems

In Section II, we have reviewed different approaches to compute numerically the optimal input sequence solving the CROC problems for a given value of the initial state x_0. Here we want to find a *state feedback* solution to CROC problems, namely a function $u_k^*: \mathbb{R}^n \to \mathbb{R}^{n_u}$ (and an explicit representation of it) mapping the state x_k to its corresponding optimal input $u_k^*, \forall k = 0, \ldots, N-1$.

For a very general parameterization of the uncertainty description, in [8] the authors propose to solve CL-CROC in state feedback form via dynamic programming by discretizing the state–space. Therefore, the technique is limited to simple low-dimensional prediction models. In this note we aim at finding the exact solution to CROC problems via multiparametric programming [11], [14], [15], [24], and in addition, for the CL-CROC problem, by using dynamic programming.

For the problems defined previously, the task of determining the sequence of optimal control actions can be expressed as a mathematical program with the initial state as a fixed parameter. To determine the optimal state feedback law we consider the initial state as a parameter which can vary over a specified domain. The resulting problem is referred to as a multiparametric mathematical program. In the following, we will first define and analyze various multiparametric mathematical programs. Then we will show how they can be used to solve the different robust control problems. Finally, we will demonstrate the effectiveness of these tools on some numerical examples from the literature.

A. Preliminaries on Multiparametric Programming

Consider the multiparametric program

$$\begin{aligned} J^*(x) &= \min_z g'z \\ \text{subj. to } & Cz \le c + Sx \end{aligned} \tag{14}$$

where $z \in \mathbb{R}^{n_z}$ is the optimization vector, $x \in \mathbb{R}^n$ is the vector of parameters, and $g \in \mathbb{R}^{n_z}, C \in \mathbb{R}^{n_c \times n_z}, c \in \mathbb{R}^{n_c}, S \in \mathbb{R}^{n_c \times n}$ are constant matrices. We refer to (14) as a (right-hand side) *multi-parametric linear program* (mp-LP) [14], [15].

For a given polyhedral set $X \subseteq \mathbb{R}^n$ of parameters, solving (14) amounts to determining the set $X_f \subseteq X$ of parameters for which (14) is feasible, the value function $J^*: X_f \to \mathbb{R}$, and the optimizer function[1] $z^*: X_f \to \mathbb{R}^{n_z}$.

Theorem 1: Consider the mp-LP (14). Then, the set X_f is a convex polyhedral set, the optimizer $z^*: \mathbb{R}^n \to \mathbb{R}^{n_z}$ is a continuous[2] and piecewise affine function[3] of x, and the optimizer function $J^*: X_f \to \mathbb{R}$ is a convex and continuous piecewise affine function of x.

Proof: See [14]. □

The following lemma deals with the special case of a multiparametric program where the cost function is a convex function of z and x.

Lemma 1: Let $J : \mathbb{R}^{n_z} \times \mathbb{R}^n \to \mathbb{R}$ be a convex piecewise affine function of (z, x). Then, the multiparametric optimization problem

$$\begin{aligned} J^*(x) &\triangleq \min_z \ J(z,x) \\ \text{subj. to } & Cz \le c + Sx. \end{aligned} \tag{15}$$

is an mp-LP.

Proof: As J is a convex piecewise affine function, it follows that $J(z,x) = \max_{i=1,\ldots,s}\{L_i z + H_i x + K_i\}$ [25]. Then, it is easy to show that (15) is equivalent to the following mp-LP: $\min_{z,\varepsilon} \varepsilon$ subject to $Cz \le c + Sx, L_i z + H_i x + K_i \le \varepsilon, i = 1, \ldots, s$. □

Lemma 2: Let $f: \mathbb{R}^{n_z} \times \mathbb{R}^n \times \mathbb{R}^{n_d} \to \mathbb{R}$ and $g: \mathbb{R}^{n_z} \times \mathbb{R}^n \times \mathbb{R}^{n_d} \to \mathbb{R}^{n_g}$ be functions of (z, x, d) convex in d for each (z, x)[4]. Assume

[1] In case of multiple solutions, we define $z^*(x)$ as one of the optimizers [15].

[2] In case the optimizer is not unique, a continuous optimizer function $z^*(x)$ can always be chosen; see [15, Remark 4] for details.

[3] We recall that, given a polyhedral set $X \subseteq \mathbb{R}^{n_1}$, a continuous function $h: X \to \mathbb{R}^{n_2}$ is *piecewise affine (PWA)* if there exists a partition of X into convex polyhedra $X_1, \ldots, X_N$, and $h(x) = H_i x + k_i (H_i \in \mathbb{R}^{n_1 \times n_2}, k_i \in \mathbb{R}^{n_2}), \forall x \in X_i, i = 1, \ldots, N$.

[4] We define a vector-valued function to be convex if all its single-valued components are convex functions.

that the variable d belongs to the polyhedron $\mathcal{D}$ with vertices $\{\bar{d}_i\}_{i=1}^{N_{\mathcal{D}}}$. Then, the min–max multiparametric problem

$$\begin{aligned} J^*(x) &= \min_z \max_{d\in\mathcal{D}} f(z,x,d) \\ \text{subj. to } g(z,x,d) &\le 0 \quad \forall d\in\mathcal{D} \end{aligned} \tag{16}$$

is equivalent to the multiparametric optimization problem

$$\begin{aligned} J^*(x) &= \min_{\mu,z} \ \mu \\ \text{subj. to } \mu &\ge f(z,x,\bar{d}_i), \qquad i=1,\dots,N_{\mathcal{D}} \\ g(z,x,\bar{d}_i) &\le 0, \qquad i=1,\dots,N_{\mathcal{D}}. \end{aligned} \tag{17}$$

Proof: Easily follows by the fact that the maximum of a convex function over a convex set is attained at an extreme point of the set, cf. also [7]. □

Corollary 1: If f is also convex and piecewise affine in (z,x), i.e., $f(z,x,d)=\max_{i=1,\dots,s}\{L_i(d)z+H_i(d)x+K_i(d)\}$ and g is linear in (z,x) for all $d\in\mathcal{D}, g(z,x,d)=K_g(d)+L_g(d)x+H_g(d)z$ (with $K_g(\cdot),L_g(\cdot),H_g(\cdot),L_i(\cdot),H_i(\cdot),K_i(\cdot), i=1,\dots,s$, convex functions), then the min–max multiparametric problem (16) is equivalent to the mp-LP problem

$$\begin{aligned} J^*(x) &= \min_{\mu,z} \mu \\ \text{subj. to } \mu &\ge K_j(\bar{d}_i)+L_j(\bar{d}_i)z+H_j(\bar{d}_i)x \\ &\forall i=1,\dots,N_{\mathcal{D}}, \forall j=1,\dots,s \\ &L_g(\bar{d}_i)x+H_g(\bar{d}_i)z\le -K_g(\bar{d}_i) \\ &\forall i=1,\dots,N_{\mathcal{D}}. \end{aligned} \tag{18}$$

Remark 1: In case $g(z,x,d)=g_1(z,x)+g_2(d)$, the second constraint in (17) can be replaced by $g_1(z,x)\le -\bar{g}$, where $\bar{g}\triangleq[\bar{g}^1,\dots,\bar{g}^{n_g}]'$ is a vector whose ith component is

$$\bar{g}^i = \max_{d\in\mathcal{D}} \ g_2^i(d) \tag{19}$$

and $g_2^i(d)$ denotes the ith component of $g_2(d)$. Similarly, if $f(z,x,d)=f_1(z,x)+f_2(d)$, the first constraint in (17) can be replaced by $\mu\ge f_1(z,x)+\bar{f}$, where

$$\bar{f}^i = \max_{d\in\mathcal{D}} \ f_2^i(d). \tag{20}$$

Clearly, this has the advantage of reducing the number of constraints in the multiparametric program (17) from $N_{\mathcal{D}}n_g$ to n_g for the second constraint and from $N_{\mathcal{D}}s$ to s for the first constraint. Note that (19)–(20) does not require $f_2(\cdot),g_2(\cdot),\mathcal{D}$ to be convex. □

In the following sections, we propose an approach based on multiparametric linear programming to obtain solutions to CROC problems in state feedback form.

B. CL-CROC

Theorem 2: By solving N mp-LPs, the solution of CL-CROC is obtained in state feedback piecewise affine form

$$\begin{aligned} u_k^*(x_k) &= F_i^k x_k + g_i^k, \quad \text{if} \\ x_k\in\mathcal{X}_i^k &\triangleq \left\{x: T_i^k x\le S_i^k\right\}, \qquad i=1,\dots,s_k \end{aligned} \tag{21}$$

for all $x_k\in\mathcal{X}^k$, where $\mathcal{X}^k=\cup_{i=1}^{s_k}\mathcal{X}_i^k$ is the set of states x_k for which (8)–(10) is feasible with $j=k$.

Proof: Consider the first step $j=N-1$ of dynamic programming applied to the CL-CROC problem (8)–(10)

$$\begin{aligned} J_{N-1}^*(x_{N-1}) &\triangleq \min_{u_{N-1}} J_{N-1}(x_{N-1},u_{N-1}) \end{aligned} \tag{22a}$$

$$\text{subj. to } \begin{cases} Fx_{N-1}+Gu_{N-1}\le f \\ A(w_{N-1})x_{N-1}+B(w_{N-1})u_{N-1}+ \\ Ev_{N-1}\in\mathcal{X}^f \\ \forall v_{N-1}\in\mathcal{V}, w_{N-1}\in\mathcal{W} \end{cases} \tag{22b}$$

$$\begin{aligned} J_{N-1}&(x_{N-1},u_{N-1}) \\ &\triangleq \max_{v_{N-1}\in\mathcal{V},w_{N-1}\in\mathcal{W}}\{\|Qx_{N-1}\|_p \\ &\quad+\|Ru_{N-1}\|_p+\|P(A(w_{N-1})x_{N-1} \\ &\quad+B(w_{N-1})u_{N-1}+Ev_{N-1})\|_p\}. \end{aligned} \tag{22c}$$

The cost function in the maximization problem (22c) is piecewise affine and convex with respect to the optimization vector v_{N-1},w_{N-1} and the parameters u_{N-1},x_{N-1}. Moreover, the constraints in the minimization problem (22b) are linear in (u_{N-1},x_{N-1}) for all vectors v_{N-1},w_{N-1}. Therefore, by Lemma 2 and Corollary 1, $J_{N-1}^*(x_{N-1}),u_{N-1}^*(x_{N-1})$ and $\mathcal{X}^{N-1}$ are computable via the mp-LP[5]:

$$J_{N-1}^*(x_{N-1}) \triangleq \min_{\mu,u_{N-1}} \mu \tag{23a}$$

$$\begin{aligned} \text{subj. to } \mu &\ge \|Qx_{N-1}\|_p+\|Ru_{N-1}\|_p \\ &\quad+\|P(A(\bar{w}_h)x_{N-1}+B(\bar{w}_h)u_{N-1}+E\bar{v}_i)\|_p \end{aligned} \tag{23b}$$

$$Fx_{N-1}+Gu_{N-1}\le f \tag{23c}$$

$$A(\bar{w}_h)x_{N-1}+B(\bar{w}_h)u_{N-1}+E\bar{v}_i\in\mathcal{X}^N \tag{23d}$$

$$\forall i=1,\dots,N_{\mathcal{V}} \qquad \forall h=1,\dots,N_{\mathcal{W}}$$

where $\{\bar{v}_i\}_{i=1}^{N_{\mathcal{V}}}$ and $\{\bar{w}_h\}_{h=1}^{N_{\mathcal{W}}}$ are the vertices of the disturbance sets $\mathcal{V}$ and $\mathcal{W}$, respectively. By Theorem 1, J_{N-1}^* is a convex and piecewise affine function of x_{N-1}, the corresponding optimizer u_{N-1}^* is piecewise affine and continuous, and the feasible set $\mathcal{X}^{N-1}$ is a convex polyhedron. Therefore, the convexity and linearity arguments still hold for $j=N-2,\dots,0$ and the procedure can be iterated backward in time j, proving the theorem. □

Remark 2: Let n_a and n_b be the number of inequalities in (23b) and (23d), respectively, for any i and h. In case of additive disturbances only $(w(t)\equiv 0)$ the total number of constraints in (23b) and (23d) for all i and h can be reduced from $(n_a+n_b)N_{\mathcal{V}}N_{\mathcal{W}}$ to n_a+n_b as shown in Remark 1. □

The following corollary is an immediate consequence of the continuity properties of the mp-LP recalled in Theorem 1, and of Theorem 2:

Corollary 2: The piecewise affine solution $u_k^*: \mathbb{R}^n\to\mathbb{R}^{n_u}$ to the CL-CROC problem is a continuous function of $x_k, \forall k=0,\dots,N-1$.

C. OL-CROC

Theorem 3: The solution $U^*: \mathcal{X}^0\to\mathbb{R}^{Nn_u}$ to OL-CROC with parametric uncertainties in the B matrix only $(A(w)\equiv A)$, is a piecewise affine function of $x_0\in\mathcal{X}^0$, where $\mathcal{X}^0$ is the set of initial states for which a solution to (3)–(6) exists. It can be found by solving an mp-LP.

Proof: Since $x_k=A^k x_0+\sum_{k=0}^{k-1}A^i[B(w)u_{k-1-i}+Ev_{k-1-i}]$ is a linear function of the disturbances W,V for any given U and x_0, the cost function in the maximization problem (5) is convex and piecewise affine with respect to the optimization vectors V,W and the parameters U,x_0. The constraints in (4) are linear in U and x_0, for any V and W. Therefore, by Lemma 2 and Corollary 1, problem (3)–(6) can be solved by solving an mp-LP through the enumeration of all the vertices of the sets $\mathcal{V}\times\mathcal{V}\times\cdots\times\mathcal{V}$ and $\mathcal{W}\times\mathcal{W}\times\cdots\times\mathcal{W}$. □

We remark that Theorem 3 covers a rather broad class of uncertainty descriptions, including uncertainty on the coefficients of the impulse and step response [4]. In case of OL-CROC with additive disturbances

[5]In case $p=\infty$ (23a), (23b) can be rewritten as: $\min_{\mu_1,\mu_2,\mu_3,u_{N-1}}\mu_1+\mu_2+\mu_3$, subject to $\mu_1\ge\pm P^i x_N, \forall i=1,2,\dots,m, \mu_2\ge\pm Q^i x_{N-1}, \forall i=1,2,\dots,n, \mu_3\ge\pm R^i u_{N-1}, \forall i=1,2,\dots,n_u$, where ^{i} denotes the ith row. The case $p=1$ can be treated similarly.

only ($w(t) \equiv 0$) the number of constraints in (4) can be reduced as explained in Remark 1.

The following is a corollary of the continuity properties of mp-LP recalled in Theorem 1 and of Theorem 3:

Corollary 3: The piecewise affine solution $U^*: \mathcal{X}^0 \to \mathbb{R}^{Nn_u}$ to the OL-CROC problem with additive disturbances and uncertainty in the B matrix only ($A(w) \equiv A$) is a continuous function of x_0.

IV. Robust RHC

A robust RHC for (1) which enforces the constraints (2) at each time t in spite of additive and parametric uncertainties can be obtained immediately by setting

$$u(t) = u_0^*(x(t)) \tag{24}$$

where $u_0^*: \mathbb{R}^n \to \mathbb{R}^{n_u}$ is the piecewise affine solution to the OL-CROC or CL-CROC problems developed in the previous sections. In this way, we obtain a state feedback strategy defined at all time steps $t = 0, 1, \ldots$, from the associated finite time CROC problem.

While the stability of the closed-loop system (1)–(24) cannot be guaranteed (indeed, no robust RHC schemes with a stability guarantee are available in the literature in the case of general parametric uncertainties) we demonstrate through examples that our feedback solution performs satisfactorily.

For a discussion on stability of robust RHC we refer the reader to previously published results, e.g., [1], [2], [23], and [26]. Also, some stability issues are discussed in [27], which extends the ideas of this note to the class of piecewise-affine systems.

When the optimal control law is implemented in a moving horizon scheme, the online computation consists of a simple function evaluation. However, when the number of constraints involved in the optimization problem increases, the number of regions associated with the piecewise affine control map may increase exponentially. In [28] and [29], efficient algorithms for the online evaluation of the explicit optimal control law were presented, where efficiency is in terms of storage and computational complexity.

V. Examples

In [9], we compared the state feedback solutions to nominal RHC [12], open-loop robust RHC, and closed-loop robust RHC for the example considered in [7], using infinity norms instead of quadratic norms in the objective function. For closed-loop robust RHC, the offline computation time in Matlab 5.3 on a Pentium III 800 was about 1.3 s by using Theorem 2 (mp-LP). Below we consider another example.

Example 1: Consider the problem of robustly regulating to the origin the system

$$x(t+1) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} v(t).$$

We consider the performance measure $\|Px_N\|_\infty + \sum_{k=0}^{N-1}(\|Qx_k\|_\infty + |Ru_k|)$ where

$$N = 4 \quad P = Q = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \quad R = 1.8$$

and $U = \{u_0, \ldots, u_3\}$, subject to the input constraints $-3 \le u_k \le 3, k = 0, \ldots, 3$, and the state constraints $-10 \le x_k \le 10, k = 0, \ldots, 3$. The two-dimensional disturbance v is restricted to the set $\mathcal{V} = \{v : \|v\|_\infty \le 1.5\}$.

We compare the control law (24) for the nominal case, OL-CROC, and CL-CROC. In all cases, the closed-loop system is simulated from the initial state $x(0) = [-8, 0]$ with two different disturbances profiles shown in Fig. 2.

1) Nominal Case: We ignore the disturbance $v(t)$, and solve the resulting multiparametric linear program by using the approach of [12]. The piecewise affine state feedback control law is computed in 23 s, and the corresponding polyhedral partition (defined over 12 regions) is depicted in Fig. 1(a) (for lack of space, we do not report here the different affine gains for each region). Figs. 3(a)–(b) report the corresponding evolutions of the state vector. Note that the second disturbance profile leads to infeasibility at step 3.

2) OL-CROC: The min–max problem is formulated as in (3)–(6) and solved offline in 582 s. The resulting polyhedral partition (defined over 24 regions) is depicted in Fig. 1(b). In Fig. 3(c)–(d) the closed-loop system responses are shown.

3) CL-CROC: The min–max problem is formulated as in (8)–(10) and solved in 53 s using the approach of Theorem 2. The resulting polyhedral partition (defined over 21 regions) is depicted in Fig. 1(c). In Fig. 3(e)–(f), the closed-loop system responses can be seen.

Remark 3: As shown in [23], the approach of [7] to solve CL-CROC, requires the solution of one mp-LP where the number of constraints is proportional to the number $N_\mathcal{V}^N$ of extreme points of the set $\mathcal{V} \times \mathcal{V} \times \cdots \times \mathcal{V} \subset \mathbb{R}^{Nn_v}$ of disturbance sequences, and the number of optimization variables, as observed earlier, is proportional to $(N_\mathcal{V}^N - 1)/(N_\mathcal{V} - 1)$, where $N_\mathcal{V}$ is the number of vertices of $\mathcal{V}$. Let $n_{J_i^*}$ and $n_{\mathcal{X}_i}$ be the number of the affine gains of the cost-to-go function J_i^* and the number of constraints defining $\mathcal{X}^i$, respectively. The dynamic programming approach of Theorem 2 requires N mp-LPs where at step i the number of optimization variables is $n_u + 1$ and the number of constraints is equal to a quantity proportional to $(n_{J_i^*} + n_{\mathcal{X}_i})$. Simulation experiments have shown that $n_{J_i^*}$ and $n_{\mathcal{X}_i}$ do not increase exponentially during the recursion $i = N-1, \ldots, 0$ (although, in the worst case, they could). For instance in Example 1, we have at step 0 $n_{J_0^*} = 34$ and $n_{\mathcal{X}_0} = 4$ while $N_\mathcal{V} = 4$ and $N_\mathcal{V}^N = 256$. As the complexity of an mp-LP depends mostly (in general combinatorially) on the number of constraints, one can expect that the approach presented here is numerically more efficient than the approach of [7] [23]. On the other hand, it is also true that the latter approach could benefit from the elimination of redundant inequalities before solving the mp-LP (how many inequalities is quite difficult to quantify *a priori*). □

We remark that the offline computational time of CL-CROC is about ten times smaller than the one of OL-CROC, where the vertex enumeration would lead to a problem with 12 288 constraints, reduced to 52 by applying Remark 1, and further reduced to 38 after removing redundant inequalities in the extended space of variables and parameters. We finally remark that by enlarging the disturbance v to the set $\tilde{\mathcal{V}} = \{v : \|v\|_\infty \le 2\}$ the OL-RRHC problem becomes infeasible for all the initial states, while the CL-RRHC problem is still feasible for a certain set of initial states. □

Example 2: We consider here the problem of robustly regulating to the origin the active suspension system [30]

$$x(t+1) = \begin{bmatrix} 0.809 & 0.009 & 0 & 0 \\ -36.93 & 0.80 & 0 & 0 \\ 0.191 & -0.009 & 1 & 0.01 \\ 0 & 0 & 0 & 1 \end{bmatrix} x(t) + \begin{bmatrix} 0.0005 \\ 0.0935 \\ -0.005 \\ -0.0100 \end{bmatrix} u(t) + \begin{bmatrix} -0.009 \\ 0.191 \\ -0.0006 \\ 0 \end{bmatrix} v(t)$$

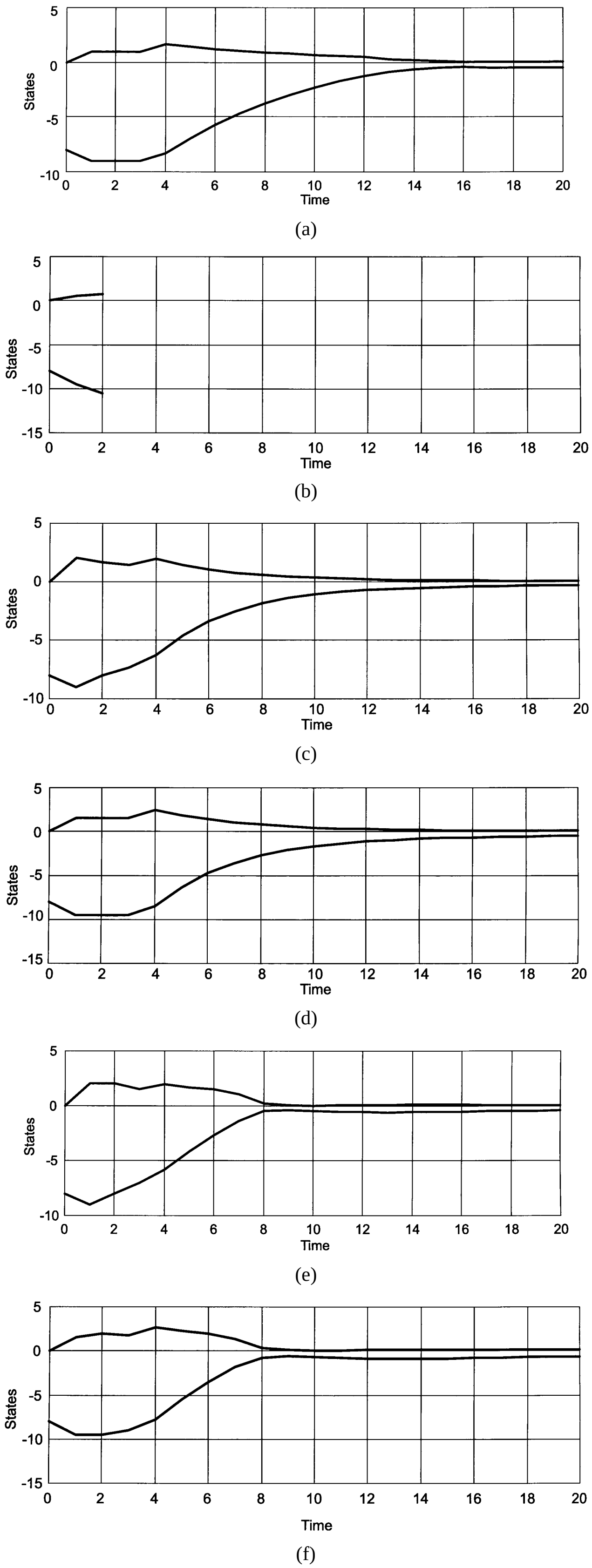


Fig. 3. Closed-loop simulations for the two disturbances shown in Fig. 2: nominal case (a, b), OL-CROC (c, d), and CL-CROC (e, f).

where the input disturbance $v(t)$ represents the vertical ground velocity of the road profile and $u(t)$ the vertical acceleration.

We solved the CL-CROC (8)–(10) with $N = 4, P = Q = \operatorname{diag}\{5000, 0.1, 400, 0.1\}, \mathcal{X}^f = \mathbb{R}^4$, and $R = 1.8$, with input constraints $-5 \le u \le 5$, and the state constraints

$$\begin{bmatrix} -0.02 \\ -\infty \\ -0.05 \\ -\infty \end{bmatrix} \le x \le \begin{bmatrix} 0.02 \\ +\infty \\ 0.05 \\ +\infty \end{bmatrix}.$$

The disturbance v is restricted to the set $-0.4 \le v \le 0.4$. The problem was solved in less then 5 min for the subset

$$X = \left\{ x \in \mathbb{R}^4 | \begin{bmatrix} -0.02 \\ -1 \\ -0.05 \\ -0.5 \end{bmatrix} \le x \le \begin{bmatrix} 0.02 \\ 1 \\ 0.50 \\ 0.5 \end{bmatrix} \right\}$$

of states, and the resulting piecewise-affine robust optimal control law is defined over 390 polyhedral regions. □

VI. CONCLUSION

This note has shown how to find state feedback solutions to constrained robust optimal control problems based on min–max optimization, for both open-loop and closed-loop formulations. The resulting robust optimal control law is piecewise affine. Such a characterization is especially useful in those applications of robust receding horizon control where online min–max constrained optimization may be computationally prohibitive. In fact, our technique allows the design of robust optimal feedback controllers with modest computational effort for a rather general class of systems.

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Upper Bounds for Approximation of Continuous-Time Dynamics Using Delayed Outputs and Feedforward Neural Networks

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***Abstract*—The problem of approximation of unknown dynamics of a continuous-time observable nonlinear system is considered using a feedforward neural network, operating over delayed sampled outputs of the system. Error bounds are derived that explicitly depend upon the sampling time interval and network architecture. The main result of this note broadens the class of nonlinear dynamical systems for which adaptive output feedback control and state estimation problems are solvable.**

***Index Terms*—Adaptive estimation, adaptive output feedback, approximation, continuous-time dynamics, feedforward neural networks.**

I. Introduction

We consider approximation of *continuous-time* dynamics of an observable nonlinear system given *delayed sampled values* of the system output. Although the problem has obvious application to system identification, our primary motivation originates within the context of adaptive *output* feedback control of nonlinear continuous-time systems with both parametric and dynamic uncertainties. A reasonable assumption in identification and control problems is *observability* of the system, which for discrete-time systems, given by difference equations, enables state estimation, system identification and output feedback control [1]. In [1], it is shown that given an arbitrary strongly observable nonlinear discrete-time system

$$x(k+1) = f[x(k), u(k)] \quad y(k) = h[x(k)] \tag{1}$$

where $x(k) \in X \subset \mathbb{R}^n$ is the internal state of the system, $u(k) \in U \subset \mathbb{R}$ is the input to the system, $y(k) \in Y \subset \mathbb{R}$ is the output,[1] there exists an equivalent input-output representation, i.e., there exists a function $g(\cdot)$ and a number l, such that future outputs can be determined based on a number of past observations of the inputs and outputs

$$\begin{aligned} y(k+1) = g[&y(k), y(k-1), \ldots, y(k-l+1) \\ &u(k), u(k-1), \ldots, u(k-l+1)]. \end{aligned} \tag{2}$$

Based on this property, adaptive state estimation, system identification and adaptive output feedback control for a general class of discrete-time systems are addressed and solved in [1] using neural networks. An equivalence, such as the one between (1) and (2), has not been demonstrated for continuous-time systems. Therefore, adaptive output feedback control of unknown continuous-time systems has been formulated and solved for a limited class of systems [2].

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[1]For simplicity, we consider here only the single-input–single-output (SISO) case.