# Equivalent Piecewise Affine Models of Linear Hybrid Automata

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***Abstract*—In this technical note we examine the relationship between linear hybrid automata (LHA) and piecewise affine (PWA) systems. While a LHA is an autonomous non-deterministic model, a PWA is a deterministic model with inputs. Through the key idea of modeling the uncertainty associated with LHA transitions as input disturbances in a PWA model, by extending continuous-time PWA models to include the dynamics of discrete states and resets we show in a constructive way that a LHA can be equivalently represented as a PWA system, where equivalent means that the two systems generate the same trajectories. Besides filling in a missing theoretical link between the LHA modelling framework and the PWA modelling framework, the result has the practical advantage of enabling the use of several existing control theoretical tools developed for PWA models to a wider class of hybrid systems.**

***Index Terms*—Hybrid systems modeling, linear hybrid automata (LHA), model equivalence, piecewise Affine systems (PWA).**

## I. Introduction

Several mathematical models have been proposed for hybrid systems with different modelling capabilities and different purposes. In computer science, hybrid automata (HA) [1] are probably the most powerful model. System theoretical properties of HA were investigated in [2]. Linear hybrid automata (LHA) [1], [3] and timed automata (TA) [4] are also popular in the computer science community. In systems theory, the proposed hybrid dynamical models include piecewise affine (PWA) systems [5], and other classes of hybrid systems like mixed logical dynamical (MLD) systems, linear complementarity (LC) systems, and min-max plus scaling (MMPS) systems (see [6] and the references therein). Different models have different purposes, in particular contributions in computer science focus on simulation and verification [3], [4], while in control theory are mainly concerned with stability analysis [7], control systems design [8], [9].

Equivalence relations between different classes of hybrid models were investigated in the past. Discrete-time MMPS, LC, MLD and PWA systems were shown in [6], [10] to be equivalent, possibly under suitable conditions. An inclusion relation between HA, LHA and TA was shown in [1]. In this technical note we connect the above two sets of models by showing in a constructive way that a LHA can be represented as a PWA system. This result is useful to apply tools for PWA systems for the analysis and the synthesis of LHA, as shown in [11], [12] as regards mixed integer programming for control and verification. The result also provides the basis for exploiting the synergy of tools for LHA and for PWA, similarly to what is done in [13] for controlling a class of hybrid systems.

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The technical note is organized as follows. In Section II we briefly recall LHA models, in Section III we discuss PWA models and their extension to include discrete dynamics and resets. For a formal and extensive definition of LHA and PWA systems the reader is referred to [1], and [7], [9], respectively. The representation of the continuous and discrete LHA dynamics in the PWA formalism is described in Section IV. A discussion on representing piecewise affine systems by linear hybrid automata, and the conclusions are summarized in Section V.

### A. Notation

We adopt the formalism of [1] for LHA, and the formalism of [7] for PWA systems. The Boolean domain is {0,1}, the sets of reals, positive reals, nonnegative reals are $\mathbb{R}$, $\mathbb{R}_+$, and $\mathbb{R}_{0+}$, respectively. $\mathbb{Z}$ is the set of integers, and given $a, b \in \mathbb{Z}$, $\mathbb{Z}_{[a,b]}$ is the set $\{j \in \mathbb{Z} : a \le j \le b\}$. The relational operators used between vectors denote componentwise relations. The superscript $^T$ indicates the transpose. Given a matrix $H$, $[H]_i$ denotes the $i^{th}$ row, $[H]_i^j$ the scalar at $i$-th row, $j^{th}$ column; given a vector $x \in \mathbb{R}^k$, $[x]_i$ is the $i^{th}$ component of $x$, and the notation $x = c$, where $c \in \mathbb{R}$, denotes $[x]_i = c, \forall i \in \mathbb{Z}_{[1,k]}$. Logical "*and*" is denoted by $\wedge$. Predicates are logical functions of Boolean variables and of Boolean conditions defined by the comparison of real expressions (e.g., $P(X_1, X_2) = ((X_1 < 1) \wedge (X_2 \ge 0)), X_1, X_2 \in \mathbb{R}$). The notation $[P(X_1, \ldots, X_r) | X_1 \leftarrow x_1, \ldots, X_r \leftarrow x_r]$ is the evaluation of $P$ when $X_1, \ldots, X_r$ take values $x_1, \ldots, x_r$, respectively. The short form $[P(X_1, \ldots, X_r) | X \leftarrow x]$, $x \in \mathbb{R}^r$, is also used.

## II. Linear Hybrid Automata

A linear hybrid automaton (LHA) is a tuple [1], [14]

$$\mathcal{H} = \langle X, \mathbb{V}, \mathbb{E}, flow, inv, init, jump, event, \Sigma \rangle \tag{1}$$

where $X = \{X_1, X_2, \ldots, X_n\}$ is the (ordered) collection of continuous (real-valued) states, the couple $(\mathbb{V}, \mathbb{E})$ defines a graph where $\mathbb{V} = \{v_1, v_2, \ldots, v_l\}$ is the set of vertices (the discrete states of the LHA), each one representing a *control mode*, and $\mathbb{E} \subseteq \mathbb{V} \times \mathbb{V}$ is the set of directed edges, representing the way control modes can switch, that is, the discrete dynamics. Each of the vertex labelling functions $flow$, $inv$ and $init$ assigns a predicate to each control mode: $flow$ defines the allowed continuous state evolutions, $inv$ the *invariant set* in which the continuous states must remain when in that mode,[1] and $init$ the allowed values for initial states. The functions $jump$ and $event$ are edge-labelling functions: $jump$ defines the conditions for changing the control mode, $event$ associates events from the finite set $\Sigma$ to control mode switches.[2] The free variables in $inv$ and $init$ predicates are from $X$, i.e., $inv(v_i) ::= P_i^{inv}(X_1, \ldots, X_N)$, $init(v_i) ::= P_i^{init}(X_1, \ldots, X_N)$, the ones in $flow$ are from the set of the continuous states derivatives $\dot{X}$, i.e., $flow(v_i) ::= P_i^{flow}(\dot{X}_1, \ldots, \dot{X}_N)$. Resets of continuous states are defined by the predicates assigned by the $jump$ function to the vertices. The free variables on such predicates are from $X \cup X'$, where $X'$ is the set of values of state variables after a discrete transition, i.e., $jump(v_i, v_j) ::= P_{i,j}^{jump}(X_1, \ldots, X_N, X_1', \ldots, X_N')$. If the predicate $[jump(v_1, v_2) | X \leftarrow \underline{\chi}, X' \leftarrow \overline{\chi}] = \text{TRUE}$, then the transition from the control mode $v_1$ to the control mode $v_2$, where the continuous state is $\underline{\chi}$ and it is reset to $\overline{\chi}$, is allowed. For linear hybrid automata the $init, inv, flow$ and $jump$ predicates are the conjunction of linear inequalities.[3] Given a generic predicate $P$ acting on a finite

[1]Since these are not invariant sets in the strict control theoretic sense, they are also referred to as *domains* [2].

[2]$event$ and $\Sigma$ are used for verification and composition purposes that are beyond the scope of this technical note. Hence, they will not be further discussed here.

[3]As pointed out in [3, footnote 7], a disjunction of predicates can be implemented by splitting control modes and transitions.

set $Z$ of free variables, for any assignment of the variables in $Z_1 \subset Z$ there may exist more than one assignment of the variables in $Z \setminus Z_1$ such that $P$ holds. Such an ambiguity maps into nondeterministic state evolutions.

The $flow$ function associates a predicate in the form of conjunction of clauses

$$\bigwedge_{h=1}^{r_j} \left[ \underline{p}_h^{(j)} \leq \sum_{k=1}^{N} q_{h,k}^{(j)} \cdot \dot{X}_k \leq \overline{p}_h^{(j)} \right] \tag{2}$$

to each control mode $v_j$, $j \in \mathbb{Z}_{[1,l]}$, where $r_j$ is the number of ranges defining the flow, and $\underline{p}_h^{(j)}$, $q_{h,k}^{(j)}$, $\overline{p}_h^{(j)}$, $k \in \mathbb{Z}_{[1,N]}$, $h \in \mathbb{Z}_{[1,r_j]}$ are the scalars that define the predicate. Given any $v_j \in \mathbb{V}$, the predicate takes value TRUE for the values $\dot{\chi} \in \dot{X}$, such that (2) is satisfied. Thus, the LHA continuous state dynamics (2) is defined by a zero-order differential inclusion. Predicate (2) could also contain strict inequalities, that we skip here for the compactness of notation.

The set of initial states $(X_0 \times \mathbb{V}_0) \subseteq (\mathbb{R}^n \times \mathbb{V})$ contains couples $(\chi_0, v_i)$ such that $[init(v_i)|X \leftarrow \chi_0] = \mathrm{TRUE}$. Let us indicate the value of the continuous state and of the control mode of the LHA at any time $t$ by $\chi(t)$ and $v(t)$, respectively, a simple illustration of the LHA evolution follows (see [1] for an extensive discussion). From an initial state $(\chi_0, v_i)$ at time $\underline{t}_0$ such that $[init(v_i)|X \leftarrow \chi_0] = \mathrm{TRUE}$, the continuous state evolves for $t \in T_0 = [\underline{t}_0, \bar{t}_0]$ in a way such that for all $t \in T_0$, $[flow(v_i)|\dot{X} \leftarrow \dot{\chi}(t)] \wedge [inv(v_i)|X \leftarrow \chi(t)] = \mathrm{TRUE}$. Let the instant $\bar{t}_0 = \underline{t}_1$ be the control mode switch instant so that there exists $e = (v_i, v_j) \in \mathbb{E} : [jump(e)|X \leftarrow \chi(\bar{t}_0), X' \leftarrow \chi(\underline{t}_1)] = \mathrm{TRUE}$ and $[inv(v_j)|X \leftarrow \chi(\underline{t}_1)] = \mathrm{TRUE}$. Then, the evolution proceeds from $\chi(\underline{t}_1)$ through a continuous flow for $t \in T_1 = [\underline{t}_1, \bar{t}_1]$, when a new switch occurs. The evolution of the LHA can be defined by a sequence of epochs $T_i = [\underline{t}_i, \bar{t}_i]$, $\underline{t}_i = \bar{t}_{i-1}$, where continuous evolution takes place, interleaved by instants at which the control mode changes and the continuous state is reset. Such a sequence of epochs $\mathcal{T} = [T_0, T_1, \ldots]$ is defined as the *hybrid time-trajectory* [2]. If $\{\chi(t), v_i(t)\}$ is such that both $flow$ and $jump$ take value FALSE for all possible values of the respective free variables, the system is in *deadlock* [4].

## III. Continuous-Time Piecewise Affine Systems

Continuous-time piecewise affine (PWA) systems [5], [7] are defined by

$$\dot{x}_c(t) = A_{i(t)}x_c(t) + B_{i(t)}u_c(t) + f_{i(t)} \tag{3a}$$
$$i(t) : H_{i(t)}x_c(t) + J_{i(t)}u_c(t) \leq K_{i(t)} \tag{3b}$$
$$\tilde{H}_{i(t)}x_c(t) + \tilde{J}_{i(t)}u_c(t) < \tilde{K}_{i(t)} \tag{3c}$$

where $x_c(t) \in \mathbb{R}^{n_c}$ is the state vector at time $t$, and $u_c(t) \in \mathbb{R}^{m_c}$ is the input vector. The index $i(t) \in \mathcal{I} \triangleq \{1, \ldots, s\}$ labels the *active mode* of the system, which is uniquely determined by the condition $[x_c(t)^T \; u_c(t)^T]^T \in \mathcal{P}_{i(t)}$, where the polyhedral region $\mathcal{P}_{i(t)} \subseteq \mathbb{R}^{n_c+m_c}$ is defined by inequalities (3b), (3c). A PWA system is well-posed if for all $i, j \in \mathcal{I}$, $i \neq j$, $\mathcal{P}_i \cap \mathcal{P}_j = \emptyset$. A PWA system is globally defined on $\mathbb{R}^{n_c \times m_c}$ if $\bigcup_{i \in \mathcal{I}} \mathcal{P}_i \equiv \mathbb{R}^{n_c+m_c}$. A condition similar to LHA deadlock occurs if at time $t$, $x_c(t) \in \mathbb{R}^{n_c}$ is such that $\forall u_c(t) \in \mathbb{R}^{m_c}$, $[x_c(t)^T \; u_c(t)^T]^T \notin \bigcup_{i \in \mathcal{I}} \mathcal{P}_i$.

Given an initial state $x_0$, an initial instant $\underline{t}_0$ and an input function $u_c : [\underline{t}_0, t_f] \to \mathbb{R}^{m_c}$, the PWA system (3) evolves as follows. Let $i_0$ be the active mode at $t = \underline{t}_0$, that is $i_0 \in \mathcal{I}$ such that (3b), (3c) are satisfied for $i(t) = i_0$, $x_c(t) = x_0$, $u_c(t) = u_c(\underline{t}_0)$. We call $\bar{t}_0 = \sup_{t \in [\underline{t}_0, t_f]} \{t : i(t) = i, \forall t \in [\underline{t}_0, \bar{t}_0)\} = \underline{t}_1$ the mode switching instant. The mode at $\underline{t}_1$ is $i_1 \in \mathcal{I}$ such that $[x(\underline{t}_1)^T \; u(\underline{t}_1)^T]^T \in \mathcal{P}_{i_1}$. For general PWA systems, the vector field might be discontinuous, hence extended solutions concepts (such as Filippov's or Utkin's [15]) might be needed to define the trajectory. However, this does not happen with the class of PWA systems considered in this technical note, as it will become clear later.

In order to capture the features of LHA, we include possible resets and discrete dynamics in continuous-time PWA systems, similarly to what done in [10] for discrete-time PWA systems. Because of the impulsive behavior during a state reset, the system trajectory cannot be defined as the solution of a classical differential equation on the whole time axis. Instead, as for LHA, we define the PWA trajectory as a sequence of intervals of right-continuous evolution interleaved by impulsive resets. These are triggered by the state and input vectors entering certain regions and are thus modelled by additional *reset modes* $\mathcal{I}_r = \{s+1, \ldots, s_r\}$. In the presence of a reset, the system evolves as follows. Let the system be evolving in mode $i \in \mathcal{I}$, $\bar{t}$ be a switching instant, and assume a reset mode $h \in \mathcal{I}_r$ is activated by $x(\bar{t})$ and $u(\bar{t})$. Then the state vector is reset to a value $\hat{x}$, which in turn forces the mode to change to an evolution mode $k \in \mathcal{I}$. To define the evolution at non-differentiable points of the trajectory we introduce the "+" operator that indicates "immediately after" ($z^+(t) = z(t^+)$ is the value of $z$ immediately after time $t$), similarly to [9] where resets and discrete dynamics have also been proposed.

By using resets, we can include discrete dynamics, defined for instance by a finite state machine. The variables in (3) become $x(t) = \begin{bmatrix} x_c(t) \\ x_b(t) \end{bmatrix} \in \mathbb{R}^{n_c} \times \{0,1\}^{n_b}$, $u(t) = \begin{bmatrix} u_c(t) \\ u_b(t) \end{bmatrix} \in \mathbb{R}^{m_c} \times \{0,1\}^{m_b}$. The discrete state-update function is modelled as a mode-dependent constant vector with Boolean components [10], which is impulsively updated during reset modes. In details, $\dot{x}_b(t) = 0$, for $i(t) \in \mathcal{I}$, $x_b(t^+) = T_{i(t)}^b$ for $i(t) \in \mathcal{I}_r$, can be simplified into $x_b(t^+) = T_{i(t)}^b$ for $i(t) \in \mathcal{I}_r \cup \mathcal{I}$, where $T_{i(t)}^b = x_b(t)$ for $i(t) \in \mathcal{I}$. The discrete state can be also used to model dependency of the current mode from previously activated modes.

In summary, the overall PWA dynamics can be expressed as[4]

$$\dot{x}_c(t) = A_{i(t)}x_c(t) + B_{i(t)}u_c(t) + f_{i(t)}, \quad i(t) \in \mathcal{I} \tag{4a}$$
$$x_c^+(t) = S_{i(t)}x_c(t) + R_{i(t)}u_c(t) + T_{i(t)}, \quad i(t) \in \mathcal{I}_r \tag{4b}$$
$$x_b^+(t) = T_{i(t)}^b, \quad i \in \mathcal{I} \cup \mathcal{I}_r \tag{4c}$$
$$i(t) \in \mathcal{I} \cup \mathcal{I}_r : H_{i(t)} \begin{bmatrix} x_c(t) \\ x_b(t) \end{bmatrix} + J_{i(t)}u_c(t) \leq K_{i(t)}. \tag{4d}$$

The complete formulation would also require strict inequalities in the mode selection (4d), we skip them here for the compactness of notation. Note that (4) permits behaviors such as Zeno dynamics and non-persistent dynamics (i.e., not defined at all time instants $t$). This is done on purpose, as such behaviors may occur in LHA models (1) that we want to represent by (4).

## IV. Translation of LHA in PWA Form

Given a LHA (2), we want to construct a PWA (4) such that for any LHA trajectory $\{\chi(t), v(t)\}_{t \in \mathbb{R}_{0+}}$ of (2) there exists an exogenous input signal $\{u_c(t)\}_{t \in \mathbb{R}_{0+}}$, and a corresponding PWA trajectory $\{x_c(t), x_b(t)\}_{t \in \mathbb{R}_{0+}}$ of (4) with $\chi(t) = x_c(t)$, $v(t) = \mathrm{cod}^{-1}(x_b(t))$, $\forall t \in \mathbb{R}_{0+}$, for a given binary encoding function $\mathrm{cod}(\cdot)$.

### A. Continuous and Discrete States

The continuous states $x_c \in \mathbb{R}^{n_c}$ of the equivalent PWA model are the continuous states $X$ of the LHA, hence $n_c = n$.

The control modes $\mathbb{V} = \{v_1, \ldots, v_l\}$ are mapped into discrete states $x_b \in \{0,1\}^{n_b}$ by the encoding $\mathrm{cod} : \mathbb{V} \to \{0,1\}^{n_b}$, which associates a unique value $x_b \in \{0,1\}^{n_b}$ to each $v \in \mathbb{V}$. The inverse $\mathrm{cod}^{-1}\{0,1\}^{n_b} \to \mathbb{V}$ may be a partial function. Thus, for all $v \in \mathbb{V}$, $x_b = \mathrm{cod}(v)$ and for all $x_b \in \{0,1\}^{n_b}$ such that $\mathrm{cod}^{-1}(x_b)$ is defined, $v = \mathrm{cod}^{-1}(x_b)$. In this technical note we use the "one-hot"

[4]A combined representation of (4) based on differentials has been proposed in [16, Ch. 4]].

coding, namely the $i^{th}$ vertex $v_i$ is associated with $x_b = e_i$, where $e_i$ is the $i^{th}$ column of the identity matrix of dimension $l$, hence $n_b = l$.

When evolving in control mode $v_j$, the continuous state value $x_c$ of the LHA must satisfy the predicate $inv(v_j)$, namely $x_c \in \mathcal{IS}(j)$, where $\mathcal{IS}(j)$ is the *invariant set* for $v_j$. $\mathcal{IS}(j)$ is the set of all $x_c \in \mathbb{R}^{n_c}$ such that $[inv(v_j)|X \leftarrow x_c] = \mathrm{TRUE}$. Since the $inv$ predicate is defined by clauses composed of linear inequalities, $\mathcal{IS}(j)$ is the polyhedron described by the inequalities $L_j x_c \leq M_j$, where $L_j \in \mathbb{R}^{\xi_j \times n}$, $M_j \in \mathbb{R}^{\xi_j}$, and $\xi_j$ is the number of inequalities describing a (minimal) hyperplane representation of $\mathcal{IS}(j)$. Again, for the compactness of notation, we avoid distinguishing between strict and non-strict inequalities. The continuous dynamics associated to $v_j$ are defined by the zero-order linear differential inclusion (2). By introducing a constrained input $\mu(t) \in \mathbb{R}^{n_c}$ that models the uncertainty associated with the actual value of state derivatives, we transform (2) into

$$[\dot{x}_c(t)]_i = [\mu(t)]_i, \quad i = 1, \ldots, n \tag{5a}$$

$$\underline{p}_h^{(j)} \leq \sum_{k=1}^{N} q_{h,k}^{(j)} \cdot [\mu(t)]_k \leq \overline{p}_h^{(j)}, \quad h = 1, \ldots, r_j \tag{5b}$$

where (5a) can be expressed as $\dot{x}_c(t) = \mu(t)$, and (5b) as $\underline{p}_j \leq Q_j \mu(t) \leq \overline{p}_j$, where $Q_j \in \mathbb{R}^{r_j \times n}$, $[Q_j]_h^k = q_{h,k}^{(j)}$, and $\underline{p}_j, \overline{p}_j \in \mathbb{R}^{r_j}$, $[\underline{p}_j]_h = \underline{p}_h^{(j)}$, $[\overline{p}_j]_h = \overline{p}_h^{(j)}$. Due to the LHA dynamics, the obtained PWA dynamics are (piecewise) integral.

*Lemma 1:* Given any time interval $\tau = [\underline{t}, \overline{t}]$ such that for all $t \in \tau$ the LHA control mode is $v(t) = v_j$ and no jump occurs, for any given constant vector $\alpha \in \mathbb{R}_+^l$, $[\alpha]_i < (1/2)$, $i \in \mathbb{Z}_{[1,l]}$ the dynamics

$$\dot{x}_c(t) = \mu(t) \tag{6a}$$

$$x_b^+(t) = \mathrm{cod}(v_j) \tag{6b}$$

$$\underline{p}_j \leq Q_j \mu(t) \leq \overline{p}_j \tag{6c}$$

$$L_j x_c(t) \leq M_j \tag{6d}$$

$$-\alpha \leq x_b(t) - \mathrm{cod}(v_j) \leq \alpha \tag{6e}$$

represents all the possible evolutions of the LHA state $\chi(t)$.

*Proof:* The proof follows by the construction of (6). For all $t \in \tau$, $x_b(t) = \mathrm{cod}(v_j)$ by (6e), (6d) enforces $[inv(v_j)|X \leftarrow x_c(t)] = \mathrm{TRUE}$, and by (5), (6c) enforces (2) from which $[flow(v_j)|\dot{X} \leftarrow \dot{x}(t)] = \mathrm{TRUE}$ follows. Finally, for any value of $\dot{\chi}(t)$ such that (2) holds, there exists a value $\mu(t)$ such that $\dot{x}_c(t) = \dot{\chi}(t)$ by (5). ■

The reason for using (6e) instead of $x_b - \mathrm{cod}(v_j) = 0$ is to define a full-dimensional polyhedron.

### B. Discrete Transitions

Dynamics (6) describe the trajectories of the continuous states of the LHA except for the (normally) zero-measure set of time instants $T = \{\underline{t}_0, \ldots, \underline{t}_n, \ldots\}$ at which the discrete state switches. We introduce additional PWA system modes to represent discrete state transitions and resets.

*Assumption 1:* For all $(v_i, v_j) \in \mathbb{V} \times \mathbb{V}$, $jump(v_i, v_j) \equiv enab(v_i, v_j) \wedge res(v_i, v_j)$, where the free variables in $enab$ are from $X$, while the free variables in $res$ are from $X \cup X'$. □

Assumption 1 states that the $jump$ predicate can be decomposed into the conjunction of two predicates. The $enab$ predicate, defining the enabling of the discrete transition, depends only on the current continuous state (i.e., $enab(v_i, v_j) ::= P_{i,j}^{enab}(X_1, \ldots, X_N)$). The $res$ predicate, defining the reset after the transition, depends on the current and on the successor continuous state (i.e., $res(v_i, v_j) ::= P_{i,j}^{res}(X_1, \ldots, X_N, X'_1, \ldots, X'_N)$). Since $jump$ is composed of linear inequalities, the same will be for both $enab$ and $res$. Assumption 1 is usually satisfied for realistic systems, where the enabling of the discrete transitions does not depend on the state after the transitions.

*Definition 1:* A transition $(v_i, v_j) \in \mathbb{E}$ is *enabled* at a state $x_c \in \mathbb{R}^{n_c}$ if $[enab(v_i, v_j)|X \leftarrow x_c] = \mathrm{TRUE}$. The *enabling set* of the transition $(v_i, v_j) \in \mathbb{E}$ is $\mathcal{ES}(i,j) = \{x_c \in \mathcal{IS}(i) : [enab(v_i, v_j)|X \leftarrow x_c] = \mathrm{TRUE}\}$. ■

When enabled, a discrete transition can occur at any time instant.

*Assumption 2:* For all $(v_i, v_j) \in \mathbb{E}$, for all $x_c \in \mathbb{R}^{n_c}$ such that $[enab(v_i, v_j)|X \leftarrow x_c] = \mathrm{TRUE}$ there exists $\bar{x}_c \in \mathbb{R}^{n_c}$ such that $[inv(v_j)|X \leftarrow \bar{x}_c] \wedge [res(v_i, v_j)|X \leftarrow x_c, X' \leftarrow \bar{x}_c]$. □

Assumption 2 requires that when a transition is enabled, there exists at least one feasible successor state after the reset. Also this condition is natural for real systems. However, while Assumption 1 is needed because of the transformation mechanism we introduce, Assumption 2 is only needed to ensure the system does not reach a deadlock in the middle of a transition.

In standard PWA systems, mode switches are deterministic events that occur when the system state crosses the boundaries of the currently active region. Nondeterminism can be obtained by introducing further additional inputs acting as disturbances, that affect the hyperplanes defining the PWA region boundaries.

*Example 1:* Consider the system shown in Fig. 1 with one-dimensional continuous state $x_c \in \mathbb{R}$ and initial value $x_c(t_0) = 0$. The state evolves in mode $v_j$, with dynamics $\dot{x}_c = \rho$, where $\rho > 0$ is a given scalar. The control mode can switch for all $x_c \in [x_m, x_M]$, and the invariant set for this mode is $\mathcal{IS} = \{x_c \in \mathbb{R} : 0 \leq x_c \leq x_M\}$. Let us introduce an additional input $w \in \mathbb{R}$. The partitions of the expanded PWA system are defined in the lifted $(x_c, w)$-space, $\mathbb{R}^2$. The region $j$ in the expanded PWA system is defined by $\mathcal{P}_j = \{(x_c, w) \in \mathbb{R}^2 : 0 \leq x_c < x_m + w, 0 \leq w \leq w_M\}$, where $w_M$ is a constant such that $w_M > x_M - x_m$. In this example, let $w(t) = \bar{w} \leq x_M - x_m$ for all $t \in [\underline{t}_0, \overline{t}_0]$, where $\overline{t}_0$ is the instant at switch occurs, that is $x_c(\overline{t}_0) = x_m + \bar{w}$. For different values of $\bar{w}$ the mode switch occurs at different state values, covering the whole range of possible values $[x_m, x_M]$. See for instance the two trajectories $A$ and $B$ shown in Fig. 1. When $\mathcal{P}_j$ is projected back onto the state space $\mathbb{R}$, the partition can be decomposed into a region (thin line) in which the system certainly does not switch, and a region (thick line) in which the system is enabled to switch. Note that in the general case $w(t)$ will not be constant. ■

Consider a LHA in control mode $v_j$ and let $\mathcal{IS}(j)$ be defined as in (6d). Consider a set of functions $\sigma_j : \mathbb{Z}_{0+} \rightarrow \mathbb{Z}_+$, defining a relation between the set of outgoing transitions from $v_j \in \mathbb{V}$ and the set $\mathbb{Z}_{[1,l_j]}$, where $l_j$ is the number of outgoing transitions from $v_j$. In detail, for a control mode $v_j$, $\sigma_j(h) = i$ associates a unique positive index $i \in \mathbb{Z}_{[1,l_j]}$ to the transition $(v_j, v_h) \in \mathbb{E}$. Let $\sigma_j^{-1} : \mathbb{Z}_{[1,l_j]} \rightarrow \mathbb{Z}_{0+}$ be the inverse mapping that associates the second vertex index to a transition index.

Let the enabling set $\mathcal{ES}(j,i)$ of transition $(v_j, v_i) \in \mathbb{E}$ be defined by the $m_{j,i}$ linear inequalities $H_{j,i} x_c \geq k_{j,i}$ where $H_{j,i}$ is a matrix, $k_{j,i}$ is a vector, both with $m_{j,i}$ rows, and where we avoid distinguishing between strict and non-strict inequalities, for the compactness of notation. The transition $(v_j, v_i)$ can occur at $t \in \mathbb{R}_{0+}$ such that $H_{j,i} x_c(t) \geq k_{j,i}$, but not necessarily at the smallest $t$ such that this holds, as defined by the LHA semantics.[5]

Consider the set of continuous states associated to control mode $v_j$ such that all the linear inequalities in $enab(v_j, v_{\sigma_j^{-1}(i)})$, for all $i \in \mathbb{Z}_{[1,l_j]}$, are not satsfied

$$H_{j,i} x_c < k_{j,i}, \quad i \in \mathbb{Z}_{[1,l_j]} \tag{7a}$$

$$L_j x_c \leq M_j. \tag{7b}$$

The continuous dynamics cannot change while $x_c$ satisfies (7). In order to represent nondeterminism in discrete transitions by using a PWA model, we introduce other additional continuous input vectors $\{w_{j,i}\}_{i=1}^{l_j}$ to relax constraints (7a) in $H_{j,i} x_c < k_{j,i} + w_{j,i}$. The effect

[5]Transitions that occur as soon as enabled are called *urgent events* [3], [4]. If the PWA transitions were associated to regions described by $H_{j,i} x_c \geq k_{j,i}$, they would occur as soon as the LHA conditions were enabled, becoming urgent events.

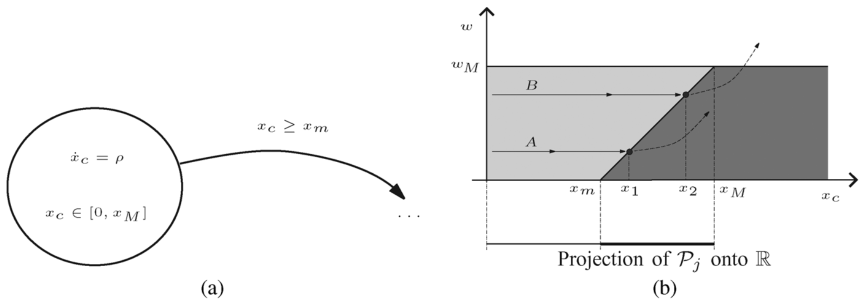


Fig. 1. Uncertain LHA transitions can be represented by a PWA system with additional disturbance inputs. (a) LHA with an uncertain transition. (b) Equivalent PWA representation.

of $[w_{j,i}]_\ell \in [0, +\infty)$ is to enlarge the half-space $[H_{j,i}]_\ell x_c \le [k_{j,i}]_\ell$ by $[w_{j,i}]_\ell / \|[H_{j,i}]_\ell\|_2$. In this way, we lift the dimension of the PWA partition and we obtain polyhedra that are not overlapping. The projection back onto the $x_c$-space of the set

$$H_{j,i} x_c < k_{j,i} + w_{j,i}, \quad i \in \mathbb{Z}_{[1,l_j]} \tag{8a}$$

$$L_j x \le M_j, w_{i,j} \ge 0 \tag{8b}$$

is clearly $\mathcal{IS}(j)$. By fixing $w_{j,i} = 0$ for all $i \in \mathbb{Z}_{[1,l_j]}$, (8) describes (7), while by setting $w_{j,i} > 0$ for all $i \in \mathbb{Z}_{[1,l_j]}$, (8) describes a set in the $x$-space contained in between the set (7) and $\mathcal{IS}(j)$. $\mathcal{IS}(j)$ is recovered by setting $w_{j,i} = \infty$ (or large enough, see [16, Sec. 4.3.4]).

In the lifted PWA space we can define equivalent discrete dynamics of LHA. Introduce $l_j + 1$ PWA modes $\mathcal{P}_{f,j}$, $f \in \mathbb{Z}_{[0,l_j]}$, associated to the LHA control mode $v_j$, where

$$\mathcal{P}_{j,0} = \Big\{ (x_c, x_b, w) \in \mathbb{R}^{n+l+l_j} :$$

$$H_{j,i} x_c < k_{j,i} + w_{j,i}, i \in \mathbb{Z}_{[1,l_j]} \tag{9a}$$

$$L_j x_c \le M_j, \quad w_{j,i} \ge 0, i \in \mathbb{Z}_{[1,l_j]} \tag{9b}$$

$$-\alpha \le x_b - \mathrm{cod}(v_j) \le \alpha \Big\} \tag{9c}$$

and for $f = 1, \ldots$, $\mathcal{P}_{j,f} = \Big\{ (x_c, x_b, w) \in \mathbb{R}^{n+l+l_j} :$

$$H_{j,f} x_c \ge k_{j,f} + w_{j,f} \tag{10a}$$

$$H_{j,i} x_c < k_{j,i} + w_{j,i}, i \in \mathbb{Z}_{[1,l_j]} \setminus f \tag{10b}$$

$$L_j x_c \le M_j, \quad w_{j,i} \ge 0, i \in \mathbb{Z}_{[1,l_j]} \tag{10c}$$

$$-\alpha \le x_b - \mathrm{cod}(v_j) \le \alpha \Big\}. \tag{10d}$$

The above sets $\mathcal{P}_{j,f}$, $f \in \mathbb{Z}_{[0,l_j]}$, are non-overlapping polyhedral cells associated to the original control mode $v_j$ of the LHA. $\mathcal{P}_{j,0}$ represents the situation in which the current mode remains active, while $\mathcal{P}_{j,f}, f \in \mathbb{Z}_{[1,l_j]}$, represents the occurrence of the transition enabled by $H_{j,f} x_c \ge k_{j,f}$. Inequalities (9a), (10a), (10b) impose constraints on the selection of $w_{i,j}$ for a given state $x_c$, and as a consequence the dynamics are not globally defined in the space of states and disturbances. The reason is that LHA transitions only occur within a given set, which restricts the choice of the disturbance input for a given state, similarly to (5b).

Sets $\mathcal{P}_{j,f}$ allow the PWA system to represent discrete transitions $(v_j, v_{\sigma_j^{-1}(f)})$ that occur at any state value $x_c \in \mathcal{ES}(j, \sigma_j^{-1}(f))$. Consider a transition $(v_j, v_{\sigma_j^{-1}(f)})$, let $\phi_j = \sum_{i=1}^{l_j} m_{j,i}$, $w_j \in \mathbb{R}_{0+}^{\phi_j}$ be the vector collecting all $w_{j,i}$, $i \in \mathbb{Z}_{[1,l_j]}$, and $\bar{w}_j \in \mathbb{R}_{0+}^{\phi_j}$ be such that $(\bar{x}_c, \mathrm{cod}(v_j), \bar{w}_j) \in \mathcal{P}_{j,f}$. Let $\underline{t}$ be the time instant at which the evolution in control mode $v_j$ begins, and $\bar{t} > \underline{t}$ be the time instant such that $x_c(\bar{t}) = \bar{x}_c$. Thus, assuming that $x_c(t) \in \mathcal{IS}(j)$ and $(x_c(t), x_b(t), w_j(t)) \in \mathcal{P}_{j,0}$, for all $t \in [\underline{t}, \bar{t}]$, if $w_j(\bar{t}) = \bar{w}_j$, at time $\bar{t}$ the system enters $\mathcal{P}_{j,f}$.

*Lemma 2:* At any given time $t$, let the LHA control mode be $v_j$, the continuous state be $x_c(t)$, and $[inv(v_j)|X \leftarrow x_c(t)] = \mathrm{TRUE}$. Then for all $(v_j, v_{\sigma_j^{-1}(f)}) \in \mathbb{E}$ it holds that *(i)* $\exists w_j(t) \in \mathbb{R}_{0+}^{\phi_j}$ such that $(x_c(t), \mathrm{cod}(v_j), w_j(t)) \in \mathcal{P}_{j,0}$; *(ii)* $\exists w_j(t) \in \mathbb{R}_{0+}^{\phi_j}$, $i \in \mathbb{Z}_{[1,l_j]}$ such that $(x_c(t), \mathrm{cod}(v_j), w_j(t)) \in \mathcal{P}_{j,f}$ if and only if $[enab(v_j, v_{\sigma_j^{-1}(f)})|X \leftarrow x_c(t)] = \mathrm{TRUE}$.

*Proof:* Set $w_j(t) = \infty$. Since $[inv(v_j)|X \leftarrow x_c(t)] = \mathrm{TRUE}$ and $x_b(t) = \mathrm{cod}(v_j)$, all the inequalities that define $\mathcal{P}_{j,0}$ are satisfied for $(x_c(t), \mathrm{cod}(v_j), w_j(t))$. Let $[enab(v_j, v_{\sigma_j^{-1}(f)})|X \leftarrow x_c(t)] =$ FALSE, then $H_{j,f} x_c(t) < k_{j,f}$. Thus, since $w_j \ge 0$, for all values of $w_j$, $H_{j,f} x_c(t) < k_{i_j} + w_j$, hence $(x_c(t), \mathrm{cod}(v_j), w_j(t)) \notin \mathcal{P}_{j,f}$, for $f \in \mathbb{Z}_{[1,l_j]}$. Let $[enab(v_j, v_{\sigma_j^{-1}(f)})|X \leftarrow x_c(t)] = \mathrm{TRUE}$, which means $H_{j,f} x_c(t) \ge k_{j,f}$. Define $w_j(t)$ such that $w_{j,i}(t) = \infty$ for all $i \in \mathbb{Z}_{[1,l_j]} \setminus f$, and $w_{j,f}(t) = 0$. For this choice of $w_j(t)$, $(x_c(t), \mathrm{cod}(v_j), w_j(t)) \in \mathcal{P}_{j,f}$. ■

As a consequence of Lemma 2 the space $(x_c, x_b, w_j)$ is partitioned into regions in which the jumps to different control modes are either enabled or disabled. By associating to the regions the discrete state update equation

$$x_b^+(t) = \begin{cases} \mathrm{cod}(v_j) & \text{if } (x_c(t), x_b(t), w_j(t)) \in \mathcal{P}_{j,0} \\ \mathrm{cod}\left(v_{\sigma_j^{-1}(f)}\right) & \text{if } (x_c(t), x_b(t), w_j(t)) \in \mathcal{P}_{j,f}, \\ & f \in \mathbb{Z}_{[1,l_j]} \end{cases} \tag{11}$$

when discrete transitions are enabled, $w_j(t)$ selects whether the jump occurs or not, by selecting the active region. The sets $\mathcal{P}_{j,i}$ partition the $x_c \times x_b \times w_j$ space into regions in which the system behavior is deterministically defined: either it performs a continuous state evolution, or it performs a discrete transition. When the active region is $\mathcal{P}_{j,0}$, the discrete state $x_b$ remains constantly equal to $\mathrm{cod}(v_j)$, while when the system is in $\mathcal{P}_{j,f}$, $f \in \mathbb{Z}_{[1,l_j]}$, the discrete state changes to $\mathrm{cod}(v_{\sigma_j^{-1}(f)})$. Hence, the successor discrete state $x_b^+(t)$ is defined by a region-dependent constant. The regions $\mathcal{P}_{j,f}$ are also used to model resets

$$x_c^+(t) = S_{j,f} x_c + T_{j,f} \quad \text{if} \quad (x_c(t), x_b(t), w_j(t)) \in \mathcal{P}_{j,f}, \quad j \in \mathbb{Z}_{[1,l]}, f \in \mathbb{Z}_{[1,l_j]}. \tag{12}$$

### C. Equivalent PWA Reformulation

The PWA dynamics can be obtained by collecting the continuous dynamics (6a) and the constraints (6c), the resets (12), the discrete dynamics (11), and the constraints (9), (10) that define the switching conditions on $x_c$, $x_b$ and $w_j$, $j \in \mathbb{Z}_{[1,l]}$. Let $w \in \mathbb{R}^\phi$, $\phi = \sum_{j=1}^{l} \phi_j$, be the vector that collects $w_j$, $j \in \mathbb{Z}_{[1,l]}$, let $\ell = (j,i)$, $j \in \mathbb{Z}_{[1,l]}$, and $i \in \mathbb{Z}_{[0,l_j]}$. Define $\mathcal{L} \triangleq \{\ell = (j,0) : j \in \mathbb{Z}_{[1,l]}\}$, $\mathcal{L}_r \triangleq \{\ell = (j,i) : j \in \mathbb{Z}_{[1,l]}, i \in \mathbb{Z}_{[0,l_j]} : (v_j, v_{\sigma_j^{-1}(i)}) \in \mathbb{E}\}$. The PWA model (4) equivalent to the given LHA is

$$\dot{x}_c(t) = \mu(t), \ \ell \in \mathcal{L} \tag{13a}$$

$$x_c^+(t) = S_\ell x_c(t) + T_\ell, \ \ell \in \mathcal{L}_r \tag{13b}$$

$$x_b^+(t) = b_\ell, \ \ell \in \mathcal{L} \cup \mathcal{L}_r \tag{13c}$$

$$\underline{p}_j \le Q_j \mu(t) \le \overline{p}_j, \ j \in \mathbb{Z}_{[1,l]} \tag{13d}$$

$$(x_c(t), x_b(t), w_j(t)) \in \mathcal{P}_\ell \tag{13e}$$

where $b_{(j,0)} = \mathrm{cod}(v_j)$, and $b_{(j,i)} = \mathrm{cod}(v_i)$ for $i \in \mathbb{Z}_{[1,l_j]}$. For $\ell \in \mathcal{L}$, (13a) models the continuous flow evolution of the LHA while in control mode $v_j$, and, for $\ell \in \mathcal{L}_r$, (13b) models the reset following the LHA jump from control mode $v_j$ to $v_{\sigma_j^{-1}(i)}$. The full state of the PWA (13) is $x = \begin{bmatrix} x_c \\ x_b \end{bmatrix} \in \mathbb{R}^{n_c} \times \{0,1\}^l$, and the full (disturbance) input is $u_c = \begin{bmatrix} \mu \\ w \end{bmatrix} \in \mathbb{R}^{n+\phi}$. Only modes $\ell = (j,i) \in \mathbb{E}$ are defined in (13), hence the number of partitions is $|\mathbb{V}| + |\mathbb{E}|$, where $|\cdot|$ denotes set cardinality. Because of (9c), (10d), whenever $x_b(t) = \mathrm{cod}(v_j)$ for any $i \neq j$ and $h \in \mathbb{Z}_{[0,l_i]}$, there exists no value $w \geq 0$ such that $\mathcal{P}_{i,h}$ is active.

*Theorem 1:* For any state trajectory $\{\chi(t), v(t)\}_{t\in\mathbb{R}_{0+}}$ of the LHA (1), there exists a trajectory of vectors $\mu(t)$ and $w(t)$ such that the trajectory $\{x_c(t), x_b(t)\}_{t\in\mathbb{R}_{0+}}$ of PWA (13) is $x_c(t) = \chi(t)$, $x_b(t) = \mathrm{cod}(v(t))$, $\forall t \in \mathbb{R}_{0+}$.

*Proof:* Lemma 1 ensures that between two consecutive discrete transitions $\mu(t) = \dot{\chi}(t)$, and hence $\dot{x}_c(t) = \dot{\chi}(t)$. Lemma 2 ensures that whenever $v(t) = v_i$ and $[inv(v_i)|X \leftarrow x_c(t)] = \mathrm{TRUE}$, there exists $w(t)$ such that $(x_c(t), \mathrm{cod}(v(t)), w(t)) \in \mathcal{P}_{i,0}$ and hence the discrete state can remain constant. If at time $t$, $v(t) = v_i$, and the LHA performs the jump $jump(v_i, v_{\sigma_i^{-1}(j)})$, it must hold $[enab(v_i, v_{\sigma_i^{-1}(j)})|X \leftarrow x_c(t)] = \mathrm{TRUE}$, and Lemma 2 ensures that there exists $w(t)$ such that $(x_c(t), \mathrm{cod}(v(t)), w(t)) \in \mathcal{P}_{i,j}$. Thus, for such value of $w(t)$, the PWA system state is updated by (11) and (12) to $x_b^+(t) = \mathrm{cod}(v_{\sigma_i^{-1}(j)})$ and $x_c^+(t) = S_{i,j}x_c(t) + T_{i,j}$, where $[res(v_i, v_{\sigma_i^{-1}(j)})|X \leftarrow x_c(t), X' \leftarrow S_{i,j}x_c(t) + T_{i,j}]$ =TRUE, which is the PWA state equivalent to the LHA state after the jump. ■

By similar arguments, it is straightforward to prove that for any trajectory $\{x_c(t), x_b(t), \mu(t), w(t)\}$ of the PWA system (13) well defined on a time interval $\tau = [\underline{t}, \bar{t}]$, there exists a LHA trajectory $\{\chi(t), v(t)\}_{t\in\tau}$, such that $x_c(t) = \chi(t)$, $v(t) = \mathrm{cod}^{-1}(x_b(t))$, and that if at time $t$ there are no $\mu(t)$, $w(t)$ such that the PWA trajectory is defined, then the LHA trajectory is deadlock.

The proposed techniques model the nondeterminism of LHA by appropriately defining auxiliary vectors $\mu$ and $w$ which act as disturbances on the PWA dynamics and regions, respectively. The dimension of $\mu$ is $n$ and in [16, Sec. 4.3.4] a procedure to make the dimension of $w$ to be $\bar{\phi} = \max_{j:v_j\in\mathbb{V}} \phi_j$ is discussed. Also, a way to suitably define upper bounds on $w$ and to model nondeterministic resets is described in [16, Sec. 4.3.4].

Finally, we summarize the procedure for obtaining the equivalent PWA of a given LHA: (*i*) map the continuous states of the PWA model on the continuous states of the LHA; introduce function $\mathrm{cod}(\cdot)$, to map control modes of the LHA to the discrete states of the PWA; (*ii*) for each $v_i \in \mathbb{V}$ build the invariant set (6d) and the continuous dynamics (2) by collecting the inequalities in the predicates $inv(v_i)$ and $flow(v_i)$, respectively; (*iii*) for any $v_i \in \mathbb{V}$, for any $(v_i, v_j) \in \mathbb{E}$ build set (8) from the predicate $enab(v_i, v_j)$, in $jump(v_i, v_j)$, then build the PWA partitions (9), (10) from (6e), (8); (*iv*) associate the continuous state update (6a), (6c), the discrete state update (11), and the resets (12) to partitions (9), (10) to obtain (13). An example of a classical LHA translated into PWA through the proposed technique is reported in in [16, Sec. 4.3.5].

## V. Discussion on the Opposite Transformation and Conclusions

In this technical note we have provided a constructive way to convert linear hybrid automata into a piecewise affine model with piecewise-integral continuous dynamics. Since PWA systems allows for affine state dynamics, the result can be extended to more complex classes of hybrid automata, where the continuous flow depends linearly on the current state.

The conversion of a standard piecewise affine system in LHA form is possible and relatively simple if (*i*) the PWA is defined over a bounded domain, (*ii*) it has bounded inputs, and (*iii*) it has partitions that are independent of the input

$$\dot{x}(t) = A_{i(t)}x(t) + B_{i(t)}u(t) + f_{i(t)} \tag{14a}$$

$$i(t) \in \mathcal{I} : H_{i(t)}x(t) \leq K_{i(t)}, x \in \mathbb{R}^{n_c} \tag{14b}$$

$$u \in \mathcal{U} \triangleq \{u \in \mathbb{R}^m : \underline{u} \leq u \leq \bar{u}\}. \tag{14c}$$

In order to obtain an equivalent LHA, define $X_h = [x]_h$, $h = 1,\ldots,n$, and associate control mode $v_i$ to PWA mode $i \in \mathcal{I}$. For all $i \in \mathcal{I}$, let $[\bar{\dot{x}}_i]_h = \max_{x\in\mathcal{P}_i, u\in\mathcal{U}}[\dot{x}]_h$, $[\underline{\dot{x}}_i]_h = \min_{x\in\mathcal{P}_i, u\in\mathcal{U}}[x]_h$, where $\mathcal{P}_i = \{x \in \mathbb{R}^{n_c} : H_i x \leq K\}$. For all $i, j \in \mathcal{I}$ let $\mathcal{E}_{i,j} = \mathrm{cl}(\mathcal{P}_i) \cap \mathrm{cl}(\mathcal{P}_j)$, where cl indicates the set closure, hence $\mathbb{E} = \{(i,j) : \mathcal{E}_{i,j} \neq \emptyset\}$, and $\mathcal{E}_{i,j} = \{x \in \mathbb{R}^{n_c} : \Psi_{i,j}x \leq \Phi_{i,j}\}$ for some $\Psi_{i,j} \in \mathbb{R}^{\gamma_{i,j}\times n}$, $\Phi_{i,j} \in \mathbb{R}^{\gamma_{i,j}}$. Then, $P_i^{flow} = (\underline{\dot{x}}_i \leq \dot{X} \leq \bar{\dot{x}}_i)$, $P_i^{inv} = (H_i X \leq K_i)$, $P_{i,j}^{jump} = (\Psi_{i,j}X \leq \Phi) \wedge (X' = X)$. It can be shown that every trajectory of the PWA is generated by the obtained LHA, while for the opposite to hold, $A_i = 0$, $\forall i \in \mathcal{I}$, is also needed. Conditions (*i*)–(*iii*) can be further relaxed.

## References


[1] T. Henzinger, "The theory of hybrid automata," in *Proc. 11th Annu. IEEE Symp. Logic Comp. Sci. (LICS'96)*, New Brunswick, NJ, 1996, pp. 278–292.

[2] J. Lygeros, K. Johansson, S. Simic, J. Zhang, and S. Sastry, "Dynamical properties of hybrid automata," *IEEE Trans. Autom. Control*, vol. 48, no. 1, pp. 2–17, Jan. 2003.

[3] T. Henzinger, P. Ho, and H. Wong-Toi, "HyTech: A model checker for hybrid systems," *Int. J. Software Tools Technol. Transfer*, vol. 1, no. 1–2, pp. 110–122, 1997.

[4] J. Bengtsson, K. G. Larsen, F. Larsson, P. Pettersson, Y. Wang, and C. Weise, "New generation of UPPAAL," in *Int. Workshop Software Tools Technol. Transfer*, June 1998, pp. 1–9.

[5] E. Sontag, "Nonlinear regulation: The piecewise linear approach," *IEEE Trans. Autom. Control*, vol. AC-26, no. 2, pp. 346–358, Apr. 1981.

[6] W. Heemels, B. de Schutter, and A. Bemporad, "Equivalence of hybrid dynamical models," *Automatica*, vol. 37, no. 7, pp. 1085–1091, Jul. 2001.

[7] M. Johansson and A. Rantzer, "Computation of piece-wise quadratic Lyapunov functions for hybrid systems," *IEEE Trans. Autom. Control*, vol. 43, no. 4, pp. 555–559, Apr. 1998.

[8] A. Bemporad and M. Morari, "Control of systems integrating logic, dynamics, and constraints," *Automatica*, vol. 35, no. 3, pp. 407–427, 1999.

[9] L. Habets, P. Collins, and J. Van Schuppen, "Reachability and control synthesis for piecewise-affine hybrid systems on simplices," *IEEE Trans. Autom. Control*, vol. 51, no. 6, pp. 938–948, Jun. 2006.

[10] A. Bemporad, "Efficient conversion of mixed logical dynamical systems into an equivalent piecewise affine form," *IEEE Trans. Autom. Control*, vol. 49, no. 5, pp. 832–838, May 2004.

[11] A. Bemporad, S. Di Cairano, and J. Júlvez, "Event-based model predictive control and verification of integral continuous-time hybrid automata," in *Hybrid Systems: Computation and Control*. New York: Springer-Verlag, 2006, pp. 93–107.

[12] S. D. Cairano, A. Bemporad, and J. Júlvez, "Event-driven optimization-based control of hybrid systems with integral continuous-time dynamics," *Automatica*, vol. 45, pp. 1243–1251, 2009.

[13] A. Bemporad and N. Giorgetti, "Logic-based solution methods for optimal control of hybrid systems," *IEEE Trans. Autom. Control*, vol. 51, no. 6, pp. 963–976, Jun. 2006.

[14] R. Alur, C. Courcoubetis, T. Henzinger, and P. H. Ho, "Hybrid automata: An algorithmic approach to the specification and verification of hybrid systems," in *Hybrid Systems*. Berlin, Germany: Springer, 1993, pp. 209–229.

[15] D. van Beek, A. Pogromsky, H. Nijmeijer, and J. Rhooda, "Convex equations and differential inclusions in hybrid systems," in *Proc. 43th IEEE Conf. Decision Control*, 2004, pp. 1424–1429.

[16] S. Di Cairano, "Model Predictive Control of Hybrid Dynamical Systems: Stabilization, Event-Driven, and Stochastic Control" Ph.D. dissertation, Dip. Ing. Informazione, Universita' di Siena, Siena, Italy, 2008 [Online]. Available: http://phd.dii.unisi.it/people/tesi/172_stefano_dicairano.pdf