Combined Design of Disturbance Model and Observer for Offset-free Model Predictive Control

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***Abstract*— This note presents a method for the combined design of an integrating disturbance model and of the observer (for the augmented system) to be used in offset-free model predictive controllers. A dynamic observer is designed for the original (non-augmented) system by solving an $\mathcal{H}_\infty$ control problem aimed at minimizing the effect of unmeasured disturbances and plant/model mismatch on the output prediction error. It is shown that, when offset-free control is sought, the dynamic observer is equivalent to choosing an integrating disturbance model and an observer for the augmented system. An example of a chemical reactor shows the main features and benefits of the proposed method.**

I. Introduction and Preliminary Results

Model Predictive Control (MPC) algorithms achieve offset-free control by introducing additional fictitious integrating disturbances in the system model. These are estimated at each sampling instant by feeding the prediction error (i.e., the difference between the measured and the predicted output) to a state estimator. Unlike other feedback controllers (e.g. PID) that achieve offset-free convergence by directly feeding back the integrated tracking error, in MPC the same goal is obtained as the result of the inherent integration of the prediction error while updating the estimated additional states. The effect of the estimated disturbances is cancelled by the MPC optimization and the controlled variables are tracked at their setpoint.

Industrial algorithms like DMC [1] use, for stable plants, the so-called "output disturbance model" in which a constant step disturbance is added to each output. Output disturbance models are also currently used in the state-space MPC algorithm of the Model Predictive Control Toolbox for Matlab [2]. A first theoretical analysis of the offset-free properties of such an augmentation, within the MPC framework, can be found in [3], where it is shown that a constant output disturbance guarantees offset-free performance for square systems without integrating modes. More general results were derived in [4] and [5], which present conditions that ensure detectability of the augmented system. In particular, the authors in [5] consider the following augmented system

$$\begin{bmatrix} x_{k+1} \\ d_{k+1} \end{bmatrix} = \begin{bmatrix} A & B_d \\ 0 & I \end{bmatrix} \begin{bmatrix} x_k \\ d_k \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u_k, \quad y_k = \begin{bmatrix} C & D_d \end{bmatrix} \begin{bmatrix} x_k \\ d_k \end{bmatrix}, \tag{1}$$

and show that a sufficient condition for (a subset of) the measured outputs to track the corresponding set-points with zero offset is that the system model is augmented with a number of integrating disturbances equal to the number of measured outputs, independently of the number of controlled variables, i.e., $\dim(d) = \dim(y)$. Moreover, the following condition:

$$\operatorname{rank} \begin{bmatrix} I - A & -B_d \\ C & D_d \end{bmatrix} = \dim(x) + \dim(d) \tag{2}$$

must hold in order for the augmented system to be detectable, and hence for an asymptotically stable observer to exist. They also show that the maximum number of integrating disturbances is equal to the number of measurements.

In principle any pair (B_d, D_d), referred to as "disturbance model", such that (2) holds is acceptable to achieve offset-free control, and the most common choice is $B_d = 0$, $D_d = I$. However, a number of different studies [4], [5], [6], [7], [8] emphasized that different disturbance models lead to different closed-loop performance in the presence of different unmeasured disturbances or plant/model mismatch. Therefore, a natural and interesting question is what kind of disturbance model should be used for a given plant. Still, the previous question is not completely well posed since the observer gain used for the augmented system plays an important role as well as the disturbance model itself. In fact, since two augmented systems (with the same number of integrators) are two different non-minimal realizations of the same input/output process, it is possible to show [9] that there exist filter gain matrices for each disturbance model that achieve the same input/output behavior.

In this paper we propose a criterion and a procedure for the simultaneous design of the disturbance model and of the observer gain for the augmented system which minimizes the effect of exogenous disturbances and of plant/model mismatch. This goal is achieved by designing a "dynamic" observer for the original system and by showing that, when offset-free control is enforced, such a dynamic observer is equivalent to choosing an integrating disturbance model and an observer gain for the augmented system.

II. Models and Observers

A. Plant and Internal Model

Consider the discrete-time linear time-invariant model of the plant

$$x_{k+1} = Ax_k + Bu_k + B_w w_k, \qquad y_k = Cx_k + D_w w_k, \tag{3}$$

in which $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is the input, $y \in \mathbb{R}^p$ is the measured output, and $w \in \mathbb{R}^q$ is a vector of unmeasured signals lumping the effects of all unmodelled disturbances and sources of mismatch between the linear model with matrices (A, B, C) and the real process. For instance, if the true plant is given by

$$x_{k+1} = f(x_k, u_k, d_k), \quad y_k = h(x_k, d_k),$$

where $d \in \mathbb{R}^\ell$ is a vector of disturbances, we can recover the model in (3) by defining $B_w w_k = f(x_k, u_k, d_k) - Ax_k - Bu_k$, $D_w w_k = h(x_k, d_k) - Cx_k$.

Without any loss of generality, we shall assume that $\dim(w) = n + p$ and that

$$B_w = \begin{bmatrix} I_n & 0 \end{bmatrix}, \quad D_w = \begin{bmatrix} 0 & I_p \end{bmatrix}. \tag{4}$$

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Also, let $y^c \in \mathbb{R}^r$ be the vector of *controlled variables* to be regulated on the corresponding set-point $\bar{y}^c$, where y^c is defined as the linear combination of the measured outputs

$$y^c := Hy. \tag{5}$$

Assumption 1 (General): A measurement of output y is available at each sampling time, (A, B) is stabilizable and (A, C) is detectable. Moreover, the following condition holds:

$$\text{rank} \begin{bmatrix} I - A & -B \\ HC & 0 \end{bmatrix} = n + r. \tag{6}$$

Remark 1: Condition (6) is necessary and sufficient for the controlled variables y^c to track an arbitrary setpoint without offset when the disturbance w is not present or known (see e.g. [5]). It clearly implies than $\dim(u) \geq \dim(y^c)$ and for square systems it is equal to assuming that the DC-gain from u to y^c is non-singular. ∎

Based on output measurements y, we use the internal model

$$\hat{x}_{k+1} = A\hat{x}_k + Bu_k + AB_v v_k, \qquad \hat{y}_k = C\hat{x}_k + D_v v_k, \tag{7}$$

to get an estimate $\hat{x}$ of the state vector and $\hat{y}$ of the model output, where v is taken as the output of the *dynamic* observer

$$\xi_{k+1} = A_L \xi_k + B_L e_k, \qquad v_k = C_L \xi_k + D_L e_k, \tag{8}$$

in which A_L, B_L, C_L and D_L are matrices of appropriate dimensions to be defined, and

$$e := y - \hat{y} \tag{9}$$

is the error between the measured output and the model output. Without loss of generality, we assume that $\dim(v) = n + p$ and

$$B_v = \begin{bmatrix} I_n & 0 \end{bmatrix}, \quad D_v = \begin{bmatrix} 0 & I_p \end{bmatrix}. \tag{10}$$

B. Connection with Offset-free Disturbance Models

By combining model (7) and observer (8) we obtain the following closed-loop augmented system:

$$\begin{bmatrix} \hat{x}_{k+1} \\ \xi_{k+1} \end{bmatrix} = \begin{bmatrix} A & AB_v C_L \\ 0 & A_L \end{bmatrix} \begin{bmatrix} \hat{x}_k \\ \xi_k \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u_k + \begin{bmatrix} AB_v D_L \\ B_L \end{bmatrix} e_k$$
$$e_k = (I + D_v D_L)^{-1} \left(y_k - \begin{bmatrix} C & D_v C_L \end{bmatrix} \begin{bmatrix} \hat{x}_k \\ \xi_k \end{bmatrix} \right), \tag{11}$$

where we assume that matrix D_L is synthesized in such a way that $(I + D_v D_L)^{-1}$ exists.

Remark 2: The closed-loop system (11) is clearly equivalent to using a Luenberger observer for the augmented system

$$\begin{bmatrix} x_{k+1} \\ d_{k+1} \end{bmatrix} = \begin{bmatrix} A & B_d \\ 0 & A_L \end{bmatrix} \begin{bmatrix} x_k \\ d_k \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u_k, \quad y_k = \begin{bmatrix} C & D_d \end{bmatrix} \begin{bmatrix} x_k \\ d_k \end{bmatrix},$$

where $B_d = AB_v C_L$ and $D_d = D_v C_L$, with observer gain

$$L = \begin{bmatrix} L_1 \\ L_2 \end{bmatrix} = \begin{bmatrix} AB_v D_L (I + D_v D_L)^{-1} \\ B_L (I + D_v D_L)^{-1} \end{bmatrix}.$$

Moreover, if $A_L = I_p$ we obtain that (11) is a Luenberger observer for the offset-free augmented system (1). ∎

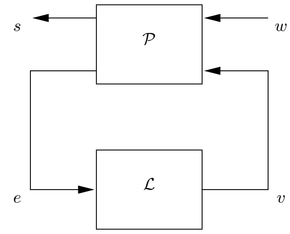

Fig. 1. Closed-loop (system and observer) block diagram

C. Problem Statement

In [5] it is shown that in order to achieve offset-free control of y^c (for instance of a subset of the measured outputs) the observer must be designed in a way that there is no steady-state error between the measured and the estimated value of *all* outputs, i.e., such that $\lim_{k\to\infty} y_k - \hat{y}_k = 0$, in the presence of arbitrary asymptotically constant disturbances. The rationale for this requirement is that any MPC algorithm (as the one described later in Section IV) is such that the predicted controlled variables asymptotically reach the desired setpoints, i.e., $\lim_{k\to\infty} H\hat{y}_k = \bar{y}^c$. Combining this fact with the previous requirement, it follows that:

$$\lim_{k\to\infty} y_k^c = \lim_{k\to\infty} Hy_k = \lim_{k\to\infty} H\hat{y}_k = \bar{y}^c.$$

A detailed proof can be found in [5, Th. 1]. Moreover, in many applications one is interested that some of the output variables lie within a given range, without even specifying a setpoint for such variables (or, equivalently, by zeroing the corresponding weight in the MPC performance index). In this case, since output constraints are imposed on $\hat{y}_k$ in the MPC optimization, ensuring $\lim_{k\to\infty} y_k - \hat{y}_k = 0$ provides better chances that output constraints are also fulfilled by the actual system outputs y_k.

We are ready now to state the main problem tackled in this paper:

Problem 1: Given plant model (3) with (A, C) is detectable, determine a disturbance model (B_d, D_d) and a matrix gain L such that the static observer (1) makes the predicted output-variable error $(y - \hat{y})$ converge to zero asymptotically for any asymptotically constant lumped disturbance w.

III. Method and Results

A. Observer Design Problem

We design the dynamic observer (8) by exploiting ideas from linear $\mathcal{H}_\infty$ control theory. Consider the block diagram depicted in Figure 1, and let

$$\delta := x - \hat{x}, \qquad s := e, \tag{12}$$

From (3) and (7) we obtain

$$\begin{aligned} \delta_{k+1} &= A\delta_k + B_w w_k - AB_v v_k \\ e_k &= C\delta_k + D_w w_k - D_v v_k \\ s_k &= C\delta_k + D_w w_k - D_v v_k \end{aligned} \tag{13}$$

which gives a complete description of block $\mathcal{P}$ in Figure 1. Equivalently, we can describe the block $\mathcal{P}$ in terms of discrete-time transfer matrices:

$$\begin{pmatrix} s \\ e \end{pmatrix} = \mathcal{P} \begin{pmatrix} w \\ v \end{pmatrix} = \begin{pmatrix} \mathcal{P}_{11} & \mathcal{P}_{12} \\ \mathcal{P}_{21} & \mathcal{P}_{22} \end{pmatrix} \begin{pmatrix} w \\ v \end{pmatrix}, \tag{14}$$

in which $\mathcal{P}_{ij}$ are transfer matrices whose definition is straightforward. The dynamic observer is described in transfer matrix form:

$$v = \mathcal{L}e, \tag{15}$$

where $\mathcal{L}$ is easily obtained from (8). Finally, we express the closed-loop transfer matrix from w to s as

$$\mathcal{G} = \mathcal{P}_{11} + \mathcal{P}_{12}\mathcal{L}(I - \mathcal{P}_{22}\mathcal{L})^{-1}\mathcal{P}_{21}. \tag{16}$$

We wish to design $\mathcal{L}$ such that the following goals are achieved:

1) The closed-loop system from w to s is asymptotically stable, i.e., the transfer matrix $\mathcal{G}$ has all poles in the open unit circle.
2) The DC-gain $\mathcal{G}(1)$ from w to s is zero.

In addition to these basic requirements, we wish to design $\mathcal{L}$ such that the effect of w on s is minimized in some sense. For instance, given two stable transfer matrices $\mathcal{W}_o$ and $\mathcal{W}_i$ and a positive scalar γ, we may want to design $\mathcal{L}$ such that

$$\|\mathcal{W}_o \mathcal{G} \mathcal{W}_i\|_\infty < \gamma, \tag{17}$$

in which $\|\cdot\|_\infty$ denotes the $\mathcal{H}_\infty$ norm (see e.g. [10]).

Remark 3: For given transfer matrices $\mathcal{W}_o$ and $\mathcal{W}_i$, there exists a positive scalar $\gamma_{\min}$ such that:

$$\gamma_{\min} = \min_{\mathcal{L}} \|\mathcal{W}_o \mathcal{G} \mathcal{W}_i\|_\infty < \infty. \tag{18}$$

■

Remark 4: Satisfaction of (17) implies that $\mathcal{G}$ is asymptotically stable. Moreover, if we let

$$\mathcal{W}_o = \frac{1}{z-1} I_p, \tag{19}$$

then $\mathcal{G}(1) = 0$. ■

B. Technical Results

Lemma 1: Suppose that Assumption 1 holds, assume that $\mathcal{W}_o$ is given by (19) and that $\mathcal{W}_i$ is a strictly stable transfer matrix. Then, any dynamic observer (8), with $\dim(\xi) = p$, that satisfies (17) is such that

$$A_L = I_p. \tag{20}$$

Proof: Since $\mathcal{G}(1) = 0$, we have that $0 = \bar{e} = \mathcal{G}(1)\bar{w}$ for any $\bar{w} \in \mathbb{R}^{n+p}$. From (8) and (13), after some straightforward algebraic manipulations, this is equivalent to:

$$\begin{bmatrix} I-A & -AB_vC_L \\ C & D_vC_L \\ 0 & I-A_L \end{bmatrix} \begin{bmatrix} \bar{\delta} \\ -\bar{\xi} \end{bmatrix} = \begin{bmatrix} B_w \\ -D_w \\ 0 \end{bmatrix} \bar{w}. \tag{21}$$

Since $\bar{w} \in \mathbb{R}^{n+p}$ can be arbitrary and because of (4), the right hand side of (21) is a vector with the first $n+p$ components arbitrary and the last p components equal to zero. Hence, (21) can have a solution for any $\bar{w}$ if and only if the following conditions hold:

$$\operatorname{rank} \begin{bmatrix} I-A & -AB_vC_L \\ C & D_vC_L \end{bmatrix} = n+p, \quad A_L = I. \tag{22}$$

■

Theorem 1: Under the assumptions of Lemma 1, given a dynamic observer (8) with $\dim(\xi) = p$ that satisfies (17), let

$$B_d = AB_vC_L, \quad D_d = D_vC_L, \tag{23}$$

and

$$L_1 = AB_vD_L(I + D_vD_L)^{-1}, \qquad L_2 = B_L(I + D_vD_L)^{-1}. \tag{24}$$

Then:

i) Condition (2) holds.
ii) The matrix

$$\mathcal{A}_{\text{cl}} = \begin{bmatrix} A & B_d \\ 0 & I \end{bmatrix} - \begin{bmatrix} L_1 \\ L_2 \end{bmatrix} \begin{bmatrix} C & D_d \end{bmatrix} \tag{25}$$

is strictly Hurwitz.
iii) For any disturbance sequence such that $\lim_{k\to\infty} w_k = \bar{w} \in \mathbb{R}^{n+p}$, we have that

$$\lim_{k\to\infty} [y_k - (C\hat{x}_k + D_d\xi_k)] = 0. \tag{26}$$

Proof: Result i) follows trivially from (22), (23). In order to prove result ii) we first rewrite $\mathcal{G}$ from (8), (13), (23) and (24) in the state-space form

$$\begin{aligned} \begin{bmatrix} \delta_{k+1} \\ -\xi_{k+1} \end{bmatrix} &= \left\{ \begin{bmatrix} A & B_d \\ 0 & I \end{bmatrix} - \begin{bmatrix} L_1 \\ L_2 \end{bmatrix} \begin{bmatrix} C & D_d \end{bmatrix} \right\} \begin{bmatrix} \delta_k \\ -\xi_k \end{bmatrix} \\ &\quad + \begin{bmatrix} B_w - L_1D_w \\ -L_2D_w \end{bmatrix} w_k \end{aligned} \tag{27}$$

Since $\mathcal{G}$ is a strictly stable transfer matrix, result ii) follows from (27) and (25). Moreover, since $\mathcal{G}(1) = 0$ we have that $\lim_{k\to\infty} e_k = 0$ for any disturbance sequence such that $\lim_{k\to\infty} w_k = \bar{w}$. Thus, result iii) follows by simply noticing that

$$y_k - \begin{bmatrix} C & D_d \end{bmatrix} \begin{bmatrix} \hat{x}_k \\ \xi_k \end{bmatrix} = (I + D_vD_L)e_k. \tag{28}$$

■

C. Design Procedure

We exploit the theoretical results of the previous sections now to design the dynamic observer $\mathcal{L}$ from the $\mathcal{H}_\infty$ control problem defined in (18), which can be efficiently solved as an LMI convex optimization problem [11]. The proposed procedure is outlined below.

Procedure 1: Given the system matrices (A, C), (B_w, D_w) as in (4), (B_v, D_v) as in (10), the outer filter as in (19):

1) Choose a non-negative scalar α and define the inner filter as the following stable transfer matrix:

$$\mathcal{W}_i = \frac{(1+\alpha)z - \alpha}{z} I_{n+p}. \tag{29}$$

2) Solve (18) and compute a minimal realization of order p the solution $\mathcal{L}$.
3) Define the disturbance model matrices (B_d, D_d) from (23) and the corresponding observer matrices from (24).

Remark 5: The maximal state dimension of $\mathcal{L}$ is equal to that of the plant (including the outer and inner filters), i.e., $2(n+p)$. Conditions for reduced order design are discussed in [11]. Nonetheless, a large number of tests on random systems, performed using the Robust Control Toolbox (version 3) of Matlab (version 7.0sp1), suggests the conjecture that the minimal realization of $\mathcal{L}$ always has a state dimension of p. ■

Remark 6: The filter defined in (29) corresponds to the difference equation $w_k^f = w_k + \alpha(w_k - w_{k-1})$ in which w_k^f is the filtered disturbance. Hence, α can be regarded as a tuning parameter which can be varied to ensure low sensitivity of the dynamic observer to high frequency disturbances (e.g., to measurement noise). ■

IV. MPC ALGORITHM

We now describe in details the "overall" model predictive control algorithm used in this work, which is based on the three main modules depicted in Figure 2.

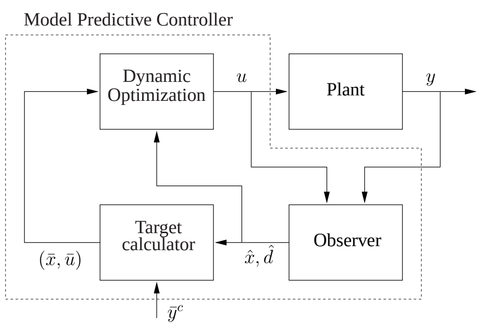

Fig. 2. Model predictive controller scheme ($\bar{y}^c$ = desired setpoint for controlled variables)

Notation: we denote by $(\cdot)_k$ the actual value of variable $(\cdot)$ at time k, by $(\cdot)_{j|k}$ (with $j \geq k$) the predicted value of $(\cdot)$ at time j based on the measurements up to time k.

A. Observer

At each sampling time, given the measured output y_k and a previous estimate of the augmented state, we compute

$$\begin{aligned} \hat{x}_{k|k} &= \hat{x}_{k|k-1} + L_x(y_k - C\hat{x}_{k|k-1} - D_d\hat{d}_{k|k-1}) \\ \hat{d}_{k|k} &= \hat{d}_{k|k-1} + L_d(y_k - C\hat{x}_{k|k-1} - D_d\hat{d}_{k|k-1}), \end{aligned} \tag{30}$$

in which $L_x = B_v(D_L - C_L B_L)(I + D_v D_L)^{-1}$, $L_d = B_L(I + D_v D_L)^{-1}$.

Remark 7: It is trivial to show that (L_x, L_d) satisfy

$$\begin{bmatrix} L_1 \\ L_2 \end{bmatrix} = \begin{bmatrix} A & B_d \\ 0 & I \end{bmatrix} \begin{bmatrix} L_x \\ L_d \end{bmatrix},$$

where (L_1, L_2) are given in (24). Moreover, the results of Theorem 1 imply that (30) is an asymptotically stable offset-free filter, i.e., such that $\lim_{k\to\infty} y_k - C\hat{x}_{k|k-1} - D_d\hat{d}_{k|k-1} = 0$ whenever the "lumped" plant disturbance w_k in (3) reaches a constant (yet unknown) value. ■

B. Constrained Target Calculation

At each sampling time, given the current estimate of the disturbance and the setpoint for the controlled variables $\bar{y}^c$, we compute the state and input targets that drive the controlled variables to their setpoints, in the presence of constraints, by solving the following quadratic program (QP):

$$(\bar{u}_k, \bar{x}_k) = \arg\min_{u_s, x_s} \quad u_s^T \bar{R} u_s \tag{31a}$$

subject to

$$\begin{bmatrix} I-A & -B \\ HC & 0 \end{bmatrix} \begin{bmatrix} x_s \\ u_s \end{bmatrix} = \begin{bmatrix} B_d\hat{d}_{k|k} \\ \bar{y}^c - HD_d\hat{d}_{k|k} \end{bmatrix} \tag{31b}$$

$$u_{\min} \leq u_s \leq u_{\max} \tag{31c}$$

$$y_{\min} \leq Cx_s + D_d\hat{d}_{k|k} \leq y_{\max}, \tag{31d}$$

in which $\bar{R}$ is a positive definite matrix of appropriate dimensions.

Remark 8: The QP (31) may be infeasible because of the presence of output inequality constraints (31d) and because of the equality constraint $H(Cx_s + D_d\hat{d}_{k|k}) = \bar{y}^c$. The former case of infeasibility can be addressed by softening the output inequality constraints (see e.g. [12]). The latter case of infeasibility instead means that, given the current disturbance estimate, offset-free control of y^c to the setpoint is *not* possible because the input constraints are overly stringent. In such case we can solve a second QP aimed at minimizing the offset [13]. ■

C. Constrained Dynamic Optimization

At each sampling time, given the current state and input targets, we compute an optimal finite-horizon input sequence $\pi_k^* = (u_{k|k}^*, u_{k+1|k}^*, ..., u_{k+N-1|k}^*)$ as the solution of the following optimization problem:

$$\begin{aligned} \pi_k^* = \arg\min_{\pi_k} \sum_{j=k}^{k+N-1} &\Big\{ \|\hat{x}_{j|k} - \bar{x}_k\|^2_{C^TQC} \\ &+ \|u_{j|k} - \bar{u}_k\|^2_R \Big\} + \|\hat{x}_{k+N|k} - \bar{x}_k\|^2_P \end{aligned} \tag{32a}$$

subject to:

$$\hat{x}_{j+1|k} = A\hat{x}_{j|k} + Bu_{j|k} + B_d\hat{d}_{k|k} \tag{32b}$$

$$u_{\min} \leq u_{j|k} \leq u_{\max} \tag{32c}$$

$$y_{\min} \leq C\hat{x}_{j|k} + D_d\hat{d}_{k|k} \leq y_{\max}, \tag{32d}$$

in which $\|x\|^2_Y = x^T Y x$, Q and R are positive definite matrices, and P is a positive semi-definite matrix. Common choices for P are the solution of a Lyapunov equation as in [13] or the solution of a Riccati equation as in [14]. The latter is used in this work.

Then, the current input is chosen as the first component of the optimal input sequence π_k^*:

$$u_k := u_{k|k}^*, \tag{33}$$

and the augmented state estimate for the next sampling time is computed accordingly:

$$\hat{x}_{k+1|k} = A\hat{x}_{k|k} + Bu_k + B_d\hat{d}_{k|k}, \qquad \hat{d}_{k+1|k} = \hat{d}_{k|k}. \tag{34}$$

V. APPLICATION EXAMPLE

A. Results

A simulated jacketed continuous stirred tank reactor (CSTR) in which an exothermic irreversible reaction $\mathrm{A} \rightarrow \mathrm{B}$ occurs [15], [5] is chosen as an example to show the main features of the proposed method. The system has two manipulated variables (coolant temperature, outlet flow rate) and three measured outputs (molar concentration of A, reactor temperature, tank level), and a discrete-time state space realization is reported in [5]. The second and third outputs (reactor temperature and the tank level) are the controlled variables.

Remark 9: Although only two variables must be controlled without offset, in order to achieve such a goal it is necessary, in general, to augment the system model with three integrating states [5]. It is also important to remark that the system is only Lyapunov stable, because it has one pole equal to 1 (associated to the tank level). Hence, it is easy to see that the output disturbance model ($B_d = 0$, $D_d = I_3$) cannot be used because it violates condition (2). ■

We compare four MPC regulators designed with the following tuning parameters: $N = 20$, $Q = \left[\begin{smallmatrix} 0.1 & 0 \\ 0 & I_2 \end{smallmatrix}\right]$, $R = \bar{R} = 0.1I_2$, $-u_{\min} = u_{\max} = [2\ 20]^T$. The four controllers only differ from each other in the disturbance model and observer used, as detailed below.

- MPC 0 uses a "mixed" input/output disturbance model (a disturbance enters the second input channel, and two disturbances enter the second and third output channels) with a steady-state Kalman filter as observer.
- MPC 1 uses an "optimal" disturbance model and estimator obtained for $\alpha = 0.1$.
- MPC 2 uses an "optimal" disturbance model and estimator obtained for $\alpha = 1.0$.
- MPC 3 uses an output disturbance model in which two disturbances enter the second and third output channels with a steady-state Kalman filter as observer.

We compare the behavior of the four MPC controllers in the rejection of a sequence of step disturbances. The first step appears at $t = 5$ on the inlet flow rate up to $t = 50$. Then, at time $t = 50$ three steps appear superimposed on the second input channel and the second and third output channels, in accordance with the disturbance model used by MPC 0.

Closed-loop results (controlled and manipulated variables) are reported in Figure 3 for the case of noise-free measurements and in Figure 4 for the case of noisy measurements (output noise is uniformly distributed in $\pm[0.001,\ 0.050,\ 0.050]$).

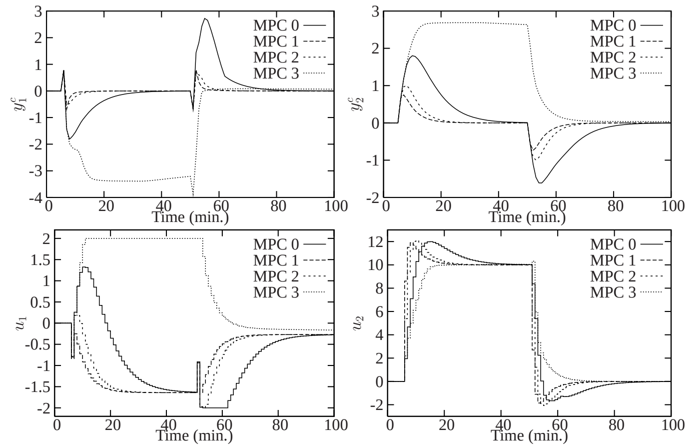

Fig. 3. CSTR: closed-loop results without noise

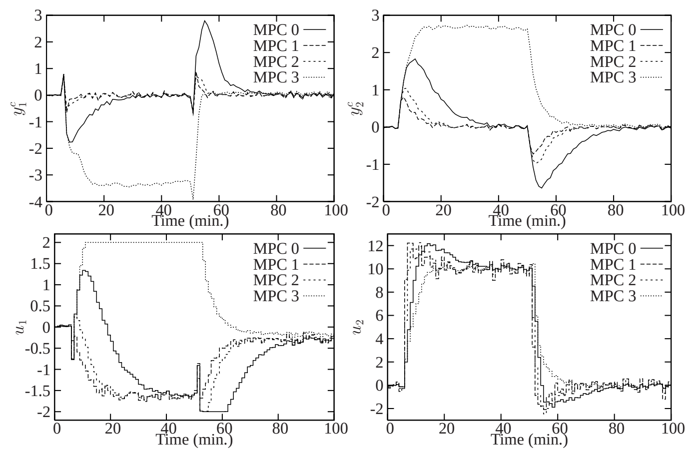

Fig. 4. CSTR: closed-loop results in the presence of noise

B. Brief Discussion

The results presented above clearly show that the disturbance model and the observer used for the augmented system affect significantly the closed-loop performance of MPC in the presence of plant/model mismatch and unmeasured disturbances. In Figure 3 we can see that MPC 1 and MPC 2, based on disturbance models and observers designed with the proposed method, reject the unmeasured disturbances much more efficiently than MPC 0, even during the second phase of the rejection ($t \geq 50$) when the actual disturbance is consistent with the disturbance model used by MPC 0. Most likely the reason for the worse behavior of MPC 0 is that it is based on a fixed-structure observer (steady-state Kalman filter), while the design of MPC 1 and MPC 2 exploits its freedom in choosing the observer gain. Indeed, the existence result of [9] proves that an observer gain exists for MPC 0 that provides the same behavior of MPC 1 or MPC 2.

Notice that this performance improvement is associated to a "better" (not a "larger") input usage. In fact, this occurs because the output prediction errors go to zero much more quickly for MPC 1 and MPC 2 than for MPC 0. We can also see, as expected [5], that MPC 3 is not able to guarantee offset-free control in the two controlled variables because it uses a disturbance model with two integrating disturbances only. Figure 4 shows that efficient disturbance rejection and low sensitivity to output noise is achieved by the proposed method. It is also interesting to notice the effect of the tuning parameter α on the closed-loop performance: the lower α, the more effective the estimator in rejecting disturbances (but also the more sensitive to output and process noise). Hence, by simply varying this single tuning parameter, one can trade off between effectiveness in obtaining offset-free control and low sensitivity to noise.

VI. Conclusions

In this paper we proposed a novel method to design a disturbance model and its associated observer for offset-free model predictive control. This objective was achieved by synthesizing a "dynamic" observer for the nominal system and by showing that, when offset-free control is required, this is equivalent to choosing an integrating disturbance model and a static observer gain for the augmented system. The dynamic observer was designed by solving an appropriate $\mathcal{H}_\infty$ control problem, aimed at minimizing the effect of external unmeasured disturbances (and plant/model mismatch) on the output prediction. There is a single scalar parameter to choose in the proposed design method, which trades off between the aggressiveness in the rejection of disturbances and the resiliency to output and process noise. A simple application example showed the effectiveness of the proposed method and the benefits that can be achieved with respect to other more common choices of disturbance models and observers.

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