Model-Predictive Control of Discrete Hybrid Stochastic Automata

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***Abstract*—This paper focuses on optimal and receding horizon control of a class of hybrid dynamical systems, called Discrete Hybrid Stochastic Automata (DHSA), whose discrete-state transitions depend on both deterministic and stochastic events. A finite-time optimal control approach "optimistically" determines the trajectory that provides the best tradeoff between tracking performance and the probability of the trajectory to actually execute, under possible chance constraints. The approach is also robustified, less optimistically, to ensure that the system satisfies a set of constraints for all possible realizations of the stochastic events, or alternatively for those having enough probability to realize. Sufficient conditions for asymptotic convergence in probability are given for the receding-horizon implementation of the optimal control solution. The effectiveness of the suggested stochastic hybrid control techniques is shown on a case study in supply chain management.**

***Index Terms*—Hybrid systems, model predictive control, optimization, stochastic systems.**

I. Introduction

MODERN automated systems are often constituted by interacting components of heterogenous continuous/discrete nature. It is indeed common to analyze and design systems in which some of the subsystems are physical processes described by equations involving continuous-valued variables, whilst some others are digital devices, whose dynamics are discrete-valued. Discrete dynamics are also used to model approximations of complex physical interactions such as impacts and stiction. Dynamical systems having such a hybrid continuous/discrete nature are named *hybrid systems* [1].

Several mathematical models were proposed in the last years for *deterministic* hybrid systems [2]–[4], that can be used for analysis of stability and other structural properties [5]–[7], identification [8], and for controller synthesis [9]–[11]. The drawback of such models is that no uncertainty is taken into account. Uncertain hybrid systems, and in particular *stochastic hybrid systems*, represent a difficult challenge [12]. Because of the hybrid nature of the dynamics, even simple questions such as the existence of solutions of the stochastic differential/difference equations and the characterization of the probability distribution functions are not easy to answer. Due to the heterogeneity of the hybrid dynamics, several different stochastic models have been proposed depending on the kind of dynamics (continuous, discrete, or both) affected by uncertainty.

In *Markov jump linear systems* [13], [14], the continuous dynamical equations of the system switch among different linear models, with jumps described by a Markov chain. This well analyzed model has the limitation that the discrete dynamics are not influenced by the continuous ones. A more complex stochastic hybrid model is the piecewise deterministic Markov process (PDMP) [15], that is a continuous-time system that interacts with a discrete-state stochastic system modeled as a continuous-time controlled Markov chain. Other stochastic models of hybrid systems were proposed in [16], [17], namely continuous-time *stochastic hybrid systems*, with uncertainty affecting only the continuous dynamics, and the more general *switching diffusion process* [18], [19], where both the discrete and continuous dynamics are affected by uncertainty.

The structural properties of some of these models have been analyzed in [20]–[22], and they have been applied in air traffic control [23], manufacturing systems [24], and communication networks [25]. More recently reachability analysis was applied in [26] as a control paradigm for stochastic hybrid systems, in order to maximize the probability that the system evolves into a desired safe region.

In this paper, we introduce a discrete-time stochastic hybrid model, denoted as *Discrete Hybrid Stochastic Automaton* (DHSA), tailored to the synthesis of optimization-based control algorithms. In DHSA, the uncertainty appears on the discrete component of the hybrid dynamics, in the form of stochastic events that, together with deterministic events, determine the transition of the discrete states. As a consequence, mode switches of the continuous dynamics become nondeterministic and uncertainty propagates also to continuous states. DHSA exhibit stochastic behaviors similar to PDMPs, when expressed in discrete time, and constitute a powerful modeling framework. For instance, unpredictable behaviors such as delays or faults in digital components, unexpected operating mode changes, and discrete approximations of continuous input disturbances can be modeled by DHSA. The main advantage of DHSA is that the number of possible values that the overall system state can have over a given bounded time interval is finite (although it may be large), so that the problem of controlling DHSA can be conveniently treated by numerical optimization. In particular, receding horizon control (RHC) algorithms can be synthesized

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for DHSA, leading to a model predictive control (MPC) design framework for stochastic hybrid systems. Thus, this paper extends to hybrid systems the RHC approach to stochastic control, developed mainly for linear systems [27]–[35] and, more recently, for Markov jump linear systems [36], [37].

The paper is organized as follows. Section II introduces DHSA and their properties. Section III defines two types of finite horizon stochastic optimal control problems based on DHSA: A control approach that uses stochastic information about the uncertainty to obtain an optimal trajectory whose probability of realization is known, and an extension of it that also ensures robust satisfaction of certain constraints. Both control approaches are evaluated in a case study in supply chain management in Section IV. In Section V, we study RHC strategies based on the proposed DHSA model and finite horizon optimal control problems, providing sufficient conditions for convergence in probability of the state for both vanishing and persistent disturbances.

II. Discrete Hybrid Stochastic Automaton

The Discrete Hybrid Automaton (DHA) introduced in [38] models hybrid dynamical systems that evolve in a deterministic way: For any given initial state and input sequence, the trajectories of the system are uniquely defined. A DHA can be automatically translated into an equivalent mixed logical dynamical model [9] by translating logic relations into mixed-integer linear inequalities [9], [38], [39]. Below, we extend the DHA to DHSA, that takes into account possible stochastic discrete-state transitions.

A. Model Formulation

A DHSA is composed by the four components shown in Fig. 1: the switched affine system, the event generator, the stochastic (nondeterministic) finite state machine, and the mode selector. The switched affine system satisfies the linear difference equations

$$x_c(k+1) = A_{i(k)}x_c(k) + B_{i(k)}u_c(k) + f_{i(k)} \tag{1}$$

in which $k \in \mathbb{Z}_{0+} \triangleq \{0, 1, \ldots\}$ is the discrete-time index, $i(k) \in \mathcal{I} \triangleq \{1, 2, \ldots, s\}$ is the current mode of the system, $x_c(k) \in \mathbb{R}^{n_c}$ and $u_c(k) \in \mathbb{R}^{m_c}$ are the vectors of continuous states and continuous exogenous inputs, respectively, at time k, and $\{A_i, B_i, f_i\}_{i\in\mathcal{I}}$, are constant matrices of suitable dimensions[1]. The event generator produces endogenous binary event signals $\delta_e(k) \in \{0,1\}^{n_e}$ defined by

$$\delta_e(k) = f_{\text{EG}}(x_c(k), u_c(k)) \tag{2}$$

where $f_{\text{EG}} : \mathbb{R}^{n_c+m_c} \to \{0,1\}^{n_e}$ is the event generation function defined as

$$[f_{\text{EG}}^j(x_c, u_c) = 1] \leftrightarrow [H_e^j x_c + J_e^j u_c + K_e^j \leq 0]$$

where $H_e \in \mathbb{R}^{n_e \times n_c}$, $J_e \in \mathbb{R}^{n_e \times m_c}$, $K_e \in \mathbb{R}^{n_e}$ are constant matrices defining linear threshold conditions, and the superscript j denotes the j-th row. The mode selector is defined by the Boolean function $f_{\text{MS}} : \{0,1\}^{n_b+m_b+n_e} \to \mathcal{I}$

[1]Equation (1) can be extended with an output equation $y_c(k) = C_{i(k)}x_c(k) + D_{i(k)}u_c(k) + g_{i(k)}$, $y_c \in \mathbb{R}^p$, where C_i, D_i, g_i are constant matrices [38].

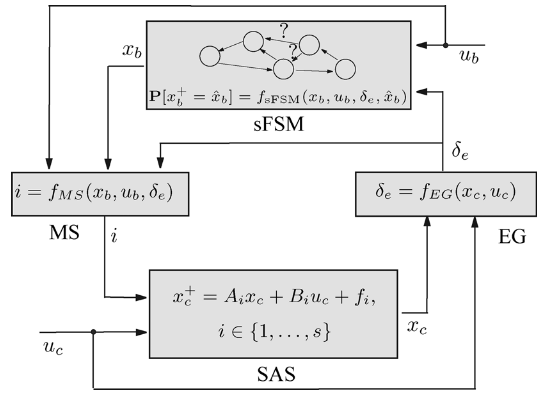

Fig. 1. Discrete hybrid stochastic automaton (DHSA). The superscript $^+$ denotes the successor at time $k+1$.

$$i(k) = f_{\text{MS}}(x_b(k), u_b(k), \delta_e(k)) \tag{3}$$

where $x_b \in \{0,1\}^{n_b}$ is the vector of binary states and $u_b \in \{0,1\}^{m_b}$ is the vector of exogenous binary inputs. In (3) we assume a "one-hot" encoding of the discrete state, hence $x_b \in \{\epsilon_1, \ldots, \epsilon_{n_b}\}$, where $\epsilon_j, j = 1, \ldots, n_b$, is the j^{th} unitary vector of $\mathbb{R}^{n_b}$. As a consequence, n_b is the number of the discrete-state values of the system.

Elements (1), (2), and (3) are the same as in DHA[2]. However, while in DHA the discrete dynamics are defined by the finite state machine (FSM)

$$x_b(k+1) = f_{\text{FSM}}(x_b(k), u_b(k), \delta_e(k)) \tag{4}$$

where $f_{\text{FSM}} : \{0,1\}^{n_b+m_b+n_e} \to \{0,1\}^{n_b}$ is a Boolean function defining the unique successor of the current state, in DHSA they are defined by the *stochastic FSM* (sFSM)

$$\mathbf{P}[x_b(k+1) = \hat{x}_b] = f_{\text{sFSM}}(x_b(k), u_b(k), \delta_e(k), \hat{x}_b) \tag{5}$$

where $f_{\text{sFSM}} : \{0,1\}^{2n_b+m_b+n_e} \to [0,1]$ and $\mathbf{P}[\cdot]$ denotes probability. Given $x_b(k)$, $\delta_e(k)$, and $u_b(k)$, only the probability distribution of $x_b(k+1) = \hat{x}_b$ is known. An example of sFSM is reported in Fig. 2. When the current state is $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ and $\delta_{e_1} = 1$ there is a probability p_1 that the next state is $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ and a probability p_2 that the next state is $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, with $p_1 + p_2 = 1$.

Definition 1: Given a binary state $x_b(k) = \bar{x}_b$, an exogenous binary input $u_b(k) = \bar{u}_b$, an endogenous vector of events $\delta_e(k) = \bar{\delta}_e$, we say that a discrete transition $\bar{x}_b \to \hat{x}_b$ to the successor state $x_b(k+1) = \hat{x}_b$ is *enabled* for $(\bar{u}_b, \bar{\delta}_e)$ if the probability $\mathbf{P}[(\bar{x}_b, \bar{u}_b, \bar{\delta}_e) \to \hat{x}_b] = f_{\text{sFSM}}(\bar{x}_b, \bar{u}_b, \bar{\delta}_e, \hat{x}_b) > 0$. An enabled transition is said *stochastic* if $\mathbf{P}[(\bar{x}_b, \bar{u}_b, \bar{\delta}_e) \to \hat{x}_b] <$

[2]The resets maps introduced in [38] can be straightforwardly included also in DHSA, so we do not explicitly consider them in this paper.

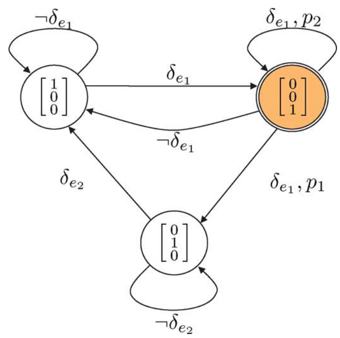

Fig. 2. Example of stochastic finite state machine with three possible state values [1 0 0]′, [0 1 0]′, [0 0 1]′, two events δ_{e_1}, δ_{e_2}, and two stochastic transitions with probabilities $p_1, p_2 = 1 - p_1$ enabled in [0 0 1]′.

1. Two or more transitions that are enabled for $(\bar{x}_b, \bar{u}_b, \bar{\delta}_e)$ are called *conflicting* on $(\bar{x}_b, \bar{u}_b, \bar{\delta}_e)$.

Definition 2: An sFSM (5) is *stochastically well-posed* if $\sum_{\hat{x}_b \in \{0,1\}^{n_b}} \mathbf{P}[(\bar{x}_b, \bar{u}_b, \bar{\delta}_e) \rightarrow \hat{x}_b] = 1$, for all $(\bar{x}_b, \bar{u}_b, \bar{\delta}_e) \in \{0,1\}^{n_b+m_b+n_e}$.

In order to demonstrate how a complex system can be modelled as a DHSA, we briefly introduce a supply chain case study, that will be discussed in details in Section IV. In a supply chain composed of production, storage, and retailer nodes, the accumulation of products in the inventories and of wear in the machines can be described by real-valued dynamics. The inventories are constrained by the storage capacity and when the machine wear increases above a danger threshold, breakdowns, that disrupt the production capability, are possible with a certain probability. The DHSA model of this system represents the real-valued wear and inventory dynamics by the SAS, the activation of the dangerous wear threshold by the EG, and the current condition of the production machines by the sFSM, whose state indicates whether a machine is working, risking breakdown, or broken. The MS is used to select the wear accumulation and the production dynamics according to the state of the sFSM.

B. Reformulation of DHSA as DHA With Uncontrollable Events (ueDHA)

A more explicit characterization of the uncertainty affecting the DHSA (1), (2), (3), (5) is needed for use in numerical optimization algorithms. The key idea is that an sFSM (5) having $l-1$ stochastic transitions can be equivalently represented by an FSM (4) by introducing a random binary input $w_i \in \{0,1\}$, called *uncontrollable event*, for each transition $i = 1, \ldots, l-1$. The i^{th} enabled stochastic transition $(\bar{x}_b, \bar{u}_b, \bar{\delta}_e) \rightarrow \hat{x}_b$ occurs if and only if a $w_i = 1$, with

$$p_i \triangleq \mathbf{P}[w_i = 1] = f_{\text{sFSM}}(\bar{x}_b, \bar{u}_b, \bar{\delta}_e, \hat{x}_b), \;\; i = 1, \ldots, l-1. \tag{6}$$

Let $w_l \in \{0,1\}$ be associated with deterministic transitions, that is, $w_l = 1$ whenever a transition from a binary state to another is deterministic, or, equivalently, no conflicting (stochastic) transitions exist. Given any $(x_b(k), u_b(k), \delta_e(k)) \in \{0,1\}^{n_b+m_b+n_e}$, let $I(x_b(k), u_b(k), \delta_e(k)) \subseteq \{1, \ldots, l\}$ denote the subset of indices of the uncontrollable events associated with the conflicting transitions on $(x_b(k), u_b(k), \delta_e(k))$. Let $\mathbb{W}(x_b(k), u_b(k), \delta_e(k)) \subseteq \{0,1\}^l$ be the set of vectors $w(k) = [w_1(k) \;\ldots\; w_l(k)]^T$ that satisfy the condition

$$[|I(x_b(k), u_b(k), \delta_e(k))| \geq 1] \rightarrow \tag{7a}$$

$$\left[\sum_{i \in I(x_b(k), u_b(k), \delta_e(k))} w_i(k) = 1\right] \tag{7b}$$

$$w_i(k) = 0, \quad \forall i \notin I(x_b(k), u_b(k), \delta_e(k)). \tag{7c}$$

Equation (7) imposes that: (*i*) when enabled conflicting transitions exist (i.e., $|I(x_b(k), u_b(k), \delta_e(k))| > 1$), then one and only one transition is taken ($\sum_i w_i(k) = 1$); (*ii*) if $|I(x_b(k), u_b(k), \delta_e(k))| = 1$ then the corresponding transition is deterministic, and in this case we assume without loss of generality that all deterministic transitions are associated with w_l, i.e., $I(x_b(k), u_b(k), \delta_e(k)) = \{l\}$; when $I(x_b(k), u_b(k), \delta_e(k)) = \emptyset$ no transition is defined (the system "hangs" at that particular discrete state). As an example, the sFSM represented in Fig. 2 can be associated with a FSM having additional uncontrollable events $w_1, w_2 \in \{0,1\}$ that affect the stochastic transitions in $\begin{bmatrix}0\\0\\1\end{bmatrix}$: transition $\begin{bmatrix}0\\0\\1\end{bmatrix} \rightarrow \begin{bmatrix}0\\0\\1\end{bmatrix}$ happens when $\delta_{e_1} \wedge w_2$ is true ("$\wedge$" denotes logic "and"), while transition $\begin{bmatrix}0\\0\\1\end{bmatrix} \rightarrow \begin{bmatrix}0\\1\\0\end{bmatrix}$ when $\delta_{e_1} \wedge w_1$ is true, w_1 and w_2 satisfy $\left[x_b = \begin{bmatrix}0\\0\\1\end{bmatrix} \wedge \delta_{e_1} = 1\right] \rightarrow [w_1 + w_2 = 1]$, $\mathbf{P}[w_1 = 1] = p_1$, and $\mathbf{P}[w_2 = 1] = p_2 = 1 - p_1$. Condition (7c) does not directly affect the evolution of the system, since the transitions that are not in the set $I(x_b(k), u_b(k), \delta_e(k))$ are not enabled and the value of the corresponding uncontrollable events does not affect the discrete state evolution. However, (7c) is enforced to correctly compute the transition probability.

More generally, an sFSM having $l-1$ stochastic transitions can be transformed into a deterministic automaton, denoted as *FSM with uncontrollable-events* (ueFSM), defined by

$$x_b(k+1) = f_{\text{ueFSM}}(x_b(k), u_b(k), \delta_e(k), w(k)) \tag{8}$$

where $w(k) = [w_1(k) \;\ldots\; w_l(k)]^T \in \mathbb{W}$ is the random vector of uncontrollable events at time k, $f_{\text{ueFSM}} : \{0,1\}^{n_b+m_b+n_e} \times \mathbb{W} \rightarrow \{0,1\}^{n_b}$ is derived from (5), and $\mathbb{W}$ from now is a short notation for $\mathbb{W}(x_b(k), u_b(k), \delta_e(k))$. The following proposition is immediate to prove.

Proposition 1: An sFSM (5) is stochastically well posed if and only if the components $w_i(k)$ of $w(k)$ in its equivalent ueFSM (8) are produced by an IID random binary number generator with probabilities $\mathbf{P}[w_i(k) = 1] = p_i$, $i = 1, \ldots, l$, $w(k) \in \mathbb{W}$, satisfying

$$[|I(x_b(k), u_b(k), \; \delta_e(k))| \geq 1] \rightarrow \sum_{i \in I(x_b(k), u_b(k), \delta_e(k))} p_i = 1. \tag{9}$$

Note that in case of deterministic transitions (9) implies that $p_i = 1$, which ensures that the only possible transition is always taken.

The notion of equivalent ueDHA to a given DHSA is formally defined below.

Definition 3: Given a DHSA (1), (2), (3), (5), its *equivalent* ueDHA is defined by (1), (2), (3), (8) with vectors $w \in \mathbb{W} \subseteq \{0,1\}^l$ satisfying (7) and generated according to (9).

We extend the definition of well-posedness given for deterministic hybrid systems in [40, Def. 1] to DHSA.

Definition 4: A DHSA (1), (2), (3), (5) is *well-posed* if its ueDHA equivalent is well-posed according to [40, Def. 1] as a deterministic DHA with inputs $(u(k), w(k)) \in \mathcal{U} \times \mathbb{W}$, $\mathcal{U} = \mathbb{R}^{m_c} \times \{0,1\}^{m_b}$ and the components $w_i(k)$ of $w(k)$ have probabilities p_i satisfying (9).

In the rest of the paper, we will assume that DHSA models are well-posed.

The ueDHA representation of the DHSA has two advantages. First, the uncertainty is associated with $w(k)$ so to that the probability of a state trajectory can be obtained as a function of the sequence of the corresponding uncontrollable events. Second, the ueDHA dynamics (1), (2), (3), (8) under the constraints (7) can be written in mixed logical dynamical (MLD) form as

$$x(k+1) = A^{\rm mld}x(k) + B_u^{\rm mld}u(k) + B_w^{\rm mld}w(k) + B_z^{\rm mld}z(k) + B_d^{\rm mld}\delta(k) \qquad (10a)$$

$$E_x^{\rm mld}x(k) + E_u^{\rm mld}u(k) + E_w^{\rm mld}w(k) \leq E_z^{\rm mld}z(k) + E_d^{\rm mld}\delta(k) + E_c^{\rm mld} \qquad (10b)$$

where $x = [x_c' \ x_b']'$, $u = [u_c' \ u_b']'$, and $\delta \in \{0,1\}^{n_d}$, $z \in \mathbb{R}^{n_z}$ are auxiliary vectors, whose value is uniquely assigned for any fixed $x \in \mathcal{X}$, $u \in \mathcal{U}$, $w \in \mathbb{W}$. The matrices in (10) are obtained from (1), (2), (3), (8) by automated procedures [38].

Given $x(0) \in \mathcal{X} \triangleq \mathbb{R}^{n_c} \times \{0,1\}^{n_b}$, $\mathbf{u} \triangleq \{u(k)\}_{k=0}^{N-1} \in \mathbf{U} \triangleq (\mathbb{R}^{m_c} \times \{0,1\}^{m_b})^N$, and $\mathbf{w} \triangleq \{w(k)\}_{k=0}^{N-1} \in \mathbf{W} \triangleq \mathbb{W}^N$, the probability of the state trajectory $\mathbf{x} = \{x(k)\}_{k=0}^{N}$, $x(k) \in \mathbb{R}^{n_c} \times \{0,1\}^{n_b}$, can be computed as follows. Consider the vector $p = [p_1 \ldots p_l]^T$ containing the probability coefficients of the transitions and

$$\begin{bmatrix} \pi(0) \\ \vdots \\ \pi(N-1) \end{bmatrix} = \begin{bmatrix} w^T(0) \\ \vdots \\ w^T(N-1) \end{bmatrix} \cdot p,$$

$$\pi(\mathbf{w}) = \pi(w(0), \ldots, w(N-1)) = \prod_{k=0}^{N-1} \pi(k). \qquad (11)$$

Each term $\pi(k)$ describes the probability of taking the transition defined by $w(k)$ at step k, $\pi(\mathbf{w})$ the probability of the complete trajectory $\mathbf{w}$ and hence, being p_i defined by (6), of the complete state trajectory $\mathbf{x}$ defined by $\mathbf{w}$ for the given $x(0)$ and $\mathbf{u}$.

C. Stochastic Mode Selector and Additive Stochastic Disturbances

Uncontrollable events can be easily included also in the mode selector function (3)

$$i(k) = f_{\rm sMS}(x_b(k), u_b(k), \delta_e(k), w_{ms}(k)), \qquad (12)$$

where $w_{ms}(k) \in \{0,1\}^S$ satisfies properties similar to (7), (9).

A stochastic mode selector (12) can be used to model additive stochastic quantized disturbances $\xi \in \mathbb{R}^{n_\xi}$ affecting the continuous dynamics. Consider the simplest case

$$x_c(k+1) = Ax_c(k) + E\xi(k) \qquad (13)$$

with ξ taking values in $\{\xi_1, \ldots, \xi_S\}$ with probabilities $\{p_1, \ldots, p_S\}$, $\sum_{i=1}^S p_i = 1$, and E is a constant matrix of suitable dimension. We introduce uncontrollable events $w_1, \ldots, w_S \in \{0,1\}$, $\sum_{i=1}^S w_i = 1$, with the same probabilities $\{p_i\}_{i=1}^S$, define the switched affine dynamics $A_i \triangleq A$, $f_i \triangleq E\xi_i$, $i = 1, \ldots, S$, and define (12) as $i(k) = [1 \ 2 \ \ldots \ S]w(k)$. The quantized disturbance ξ can be considered as a piecewise constant approximation of a given continuous disturbance ζ with probability distribution $p(\zeta)$, for instance by partitioning the domain of ζ into S cells $C_1, \ldots, C_S$, and defining $\xi_i \triangleq \int_{C_i} \zeta p(\zeta) d\zeta$, $p_i \triangleq \int_{C_i} p(\zeta) d\zeta$.

III. Finite Horizon Stochastic Optimal Control

A finite-time optimal control problem for a discrete-time dynamical system can always be reformulated as a finite-dimensional optimization problem, in which the optimization vector is the sequence $\mathbf{u} \in \mathbf{U}$ of control inputs and the constraints embed conditions on inputs and states that must be satisfied, such as bounds on inputs and states. If the system is stochastic, a sequence $\mathbf{w} \in \mathbf{W}$ of stochastic variables will also appear in the optimization problem.

When stochastic optimization algorithms are used to optimize the expected value of the performance criterion, usually a "scenario enumeration" approach is employed [41]. This amounts to consider a discrete set of disturbances and to explicitly enumerate all of them in the optimization problem. However, when the optimal control horizon gets large, the problem becomes easily intractable as the number of scenarios grows exponentially. Scenario enumeration can be applied to DHSA by enumerating the realizations of $\mathbf{w} \in \mathbf{W}$. However, the exponential growth of the number of scenarios will affect the DHSA as well (see the discussion at the end of Section III-A).

For the above reason, in this paper we avoid minimizing average performance and consider instead the problem of choosing the input profile that optimizes the most favorable situation, under penalties and hard constraints on the probability of the disturbance realization that determines such a favorable situation. Such an "optimistic" approach is detailed next.

A. Finite Horizon Optimal Control Setup

Consider the convex *performance index* $J_d : \mathbf{U} \times \mathbf{W} \times \mathbf{R} \times \mathcal{X} \to \mathbb{R}$ defined as

$$J_d(\mathbf{u}, \mathbf{w}, \mathbf{r}, x(0)) = \sum_{k=0}^{N-1} L(x(k), u(k), r(k)) \qquad (14)$$

where for DHSA $\mathcal{X} \equiv \mathbb{R}^{n_c} \times \{0,1\}^{n_b}$, $r(k) = [r_x(k)' \ r_u(k)']'$ are reference signals for state (r_x) and input (r_u) trajectories[3], $\mathbf{r} = \{r(k)\}_{k=0}^{N-1} \in \mathbf{R} \equiv \mathcal{X}^N \times \mathbf{U}$ and we assume that $L(r_x, r_u, r) = 0$, for all references $r = [r_x' \ r_u']'$.

[3]References on output trajectories can also be included similarly.

Typical example of the *stage cost* that satisfy such assumptions are $L(x,u,r) = \|Q(x-r_x)\|_\infty + \|R(u-r_u)\|_\infty$ where Q,R are full rank matrices, or $L(x,u,r) = (x-r_x)^T Q(x-r_x) + (u-r_u)^T R(u-r_u)$ where $Q \geq 0$, $R > 0$ are positive (semi)definite matrices.

Next, consider the *probability cost* $J_p : \mathbf{W} \to \mathbb{R}$ defined as

$$J_p(\mathbf{w}) = \ln \frac{1}{\pi(\mathbf{w})} = -\ln\Big(\pi(\mathbf{w})\Big). \tag{15}$$

The smaller the probability $\pi(\mathbf{w})$ of a disturbance realization $\mathbf{w}$, the larger is the probability cost, so that trajectories that realize rarely are penalized. The most desirable situation is to obtain a trajectory with high performance (small J_d) and high probability (small J_p). Hence, we define as *objective function*

$$J(\mathbf{u},\mathbf{w},\mathbf{r},x(0)) = J_d(\mathbf{u},\mathbf{w},\mathbf{r},x(0)) + q_p J_p(\mathbf{w}) \tag{16}$$

where $J : \mathbf{U} \times \mathbf{W} \times \mathbf{R} \times \mathcal{X} \to \mathbb{R}$, and $q_p \in (0, +\infty)$ is the *probability weight* trading off between optimism (performance) and realism (chance, i.e., likelihood of the predicted trajectory). The coefficient q_p must be greater than 0 to account for the probability of the trajectory in (16), since as will be shown later, this is important for convergence of the control algorithm.

In order to completely eliminate trajectories that realize rarely from the set of feasible solutions, we also consider the *chance constraint* [27], [28], [34]

$$\pi(\mathbf{w}) \geq \tilde{p} \tag{17}$$

where $\tilde{p} \in [0,1]$ is called *probability bound*. Chance constraint (17) enforces that $\mathbf{w}$, and hence the corresponding trajectory $\mathbf{x}$, realizes with probability at least $\tilde{p}$. More general constraints on $\mathbf{w}$ could be imposed. The problem of optimally controlling a DHSA with respect to (16) subject to (17) is formulated through its equivalent ueDHA.

1) Problem 1 (Stochastic Hybrid Optimal Control, SHOC):

$$\min_{\mathbf{u},\mathbf{w},\mathbf{d},\mathbf{z}} \quad J_d(\mathbf{u},\mathbf{w},\mathbf{r},x(0)) + q_p J_p(\mathbf{w}) \tag{18a}$$
$$\text{s.t.} \quad \text{ueDHA dynamics (10)} \tag{18b}$$
$$g(\mathbf{u},\mathbf{w},x(0)) \leq 0 \tag{18c}$$
$$\pi(\mathbf{w}) \geq \tilde{p} \tag{18d}$$

where $\mathbf{d} = \{\delta(k)\}_{k=0}^N$, $\mathbf{z} = \{z(k)\}_{k=0}^N$ are the sequence of auxiliary vectors in (10), (18c) models constraints on the closed-loop system and the probabilities p_i, $i = 1,\ldots,l$, of $\mathbf{w}$ satisfy (9).

In order to formulate (18) as a mixed-integer linear/quadratic program, we need to transform (15) and (17) into linear functions of $\mathbf{w}$. We assume that $g : \mathbf{U} \times \mathbf{W} \times \mathcal{X} \to \mathbb{R}^q$ can be expressed through mixed-integer linear inequalities [39] (see later in (23)). J_d can be dealt with as described in [42] for the deterministic case (see also (23)).

Consider a DHSA whose transition probabilities are collected in vector $p = [p_1 \ldots p_l]^T$, and consider the equivalent ueDHA with uncontrollable events $w = [w_1 \ldots w_l]^T$. By (11), $\pi(\mathbf{w}) = \prod_{k=0}^{N-1} \prod_{i=1}^{l} \pi_i(k)$, where $\pi_i(k)$ represents the contribution on the trajectory probability of the stochastic transition i at step k, and is defined by

$$\pi_i(k) = \begin{cases} 1, & \text{if } w_i(k) = 0 \\ p_i, & \text{if } w_i(k) = 1. \end{cases} \tag{19}$$

Equivalently, $\pi_i(k) = 1 + (p_i - 1)\, w_i(k)$. Hence

$$J_p(\mathbf{w}) = -\sum_{k=0}^{N-1} \sum_{i=1}^{l} \ln \pi_i(k). \tag{20}$$

For all $\pi(\mathbf{w}) > 0$

$$\ln \pi(\mathbf{w}) = \ln \prod_{i,k} \pi_i(k) = \sum_{i,k} \ln \pi_i(k) = \sum_{i,k} \ln(1 + (p_i - 1)\, w_i(k)).$$

Since for $w_i(k) \in \{0,1\}$, $\ln \pi_i(k) = w_i(k) \ln(p_i)$, for all $\mathbf{w} \in \mathbf{W}$,

$$\ln \pi(\mathbf{w}) = \sum_{k=0}^{N-1} \sum_{i=1}^{l} w_i(k)\, \ln(p_i)\,. \tag{21}$$

Thus, (20) is expressed as a linear function of $\mathbf{w}$, and (17) as a linear constraint on $\mathbf{w}$.

The solution of (18) is a pair $(\mathbf{u}^*, \mathbf{w}^*)$, where $\mathbf{u}^*$ is the optimal control sequence for the predicted sequence $\mathbf{w}^*$ of uncontrollable events that respects all the dynamical and operational constraints, the chance constraints, and that represents the best tradeoff between performance and likelihood of the predicted trajectory. Note that, since random event probabilities are a function of the system state, dynamics prediction ($\mathbf{w}^*$) and control action selection ($\mathbf{u}^*$) have to be performed together.

Since only $\mathbf{u}^*$ can be decided and actuated, the trajectory may be different from the one provided by (18), unless the realization of the stochastic events is equal to $\mathbf{w}^*$. The larger q_p, the more the prediction likelihood is important in the optimization problem, hence trajectories where more likely $\mathbf{w}$ will coincide with $\mathbf{w}^*$ will be preferred, at the expense of a possibly diminished performance. Chance constraint (17) ensures that $\mathbf{P}[\mathbf{w} = \mathbf{w}^*] \geq \tilde{p}$, or, in other words, that the actual state evolution of the system is different from the expected optimal one with probability at most $1 - \tilde{p}$. The more restrictive is (17), the more trajectories are eliminated a priori, which may ease the solution of the optimization problem (18). However, if $\tilde{p}$ is set too large, many trajectories are eliminated by (17), and (18) may be infeasible or result in poor performance (14).

Remark 1: while useful for notational purposes, for practical purposes w_l, is removed from (1), since its contribution to (15) is always zero. Condition (7c) is also removed from (1), since if the transition associated with w_i is not enabled, the value of w_i does not affect the trajectory, while setting $w_i = 1$ causes an additional cost. Thus, the optimal solution of (18) is guaranteed to have $w_i^* = 0$ for all w_i associated to nonenabled transitions.

As discussed in [9], in finite horizon optimal control of DHA the optimizer of the associated mixed integer program is composed of the input vector $\mathbf{u}$, $\mathbf{d}$, $\mathbf{z}$. For Problem 1, the uncontrollable event vector $\mathbf{w}$ is also included. Let N_u, N_d, N_z, N_w be the size of those vectors, respectively, N_c be the number of constraints. The size of these vectors is indeed proportional to the horizon N. Along the horizon, the possible scenarios are 2^{N_w}, i.e., all the possible realizations of $\mathbf{w}$. Note that infeasible scenario pruning is not possible a priori, since the scenario realization depends on the control input, which is a decision variable. Thus, if a scenario enumeration approach is used for controlling the DHSA, the size of the optimizer is $N_u + (N_d + N_z) \times 2^{N_w}$, and the number of constraints is $(N_c) \times 2^{N_w}$. In fact, $\mathcal{W}$ is fixed by the scenario, while $\mathbf{d}$ and $\mathbf{z}$ have to be duplicated for each scenario (since the logical expressions are possibly different), and the constraints enforced for all scenarios. In the SHOC problem there are $N_u + (N_d + N_z) + N_w$ variables and N_c constraints, a much simpler problem than a scenario enumeration approach would require.

B. Robust Constraint Handling

The SHOC approach does not ensure that constraints are satisfied when the actual disturbance realization $\mathbf{w}$ differs from $\mathbf{w}^*$. Henceforth, the SHOC approach can only be used when a possible violation of (18c) is not critical. On the other hand, for safety critical constraints, it may be necessary to satisfy (18c) for any disturbance realization, or at least for disturbance realizations having enough probability to happen.

Definition 5: Given a DHSA, an initial condition $x(0) \in \mathcal{X}$ and an input sequence $\mathbf{u} \in \mathbf{U}$, we say that a constraint $g(\mathbf{u}, \mathbf{w}, x(0)) \leq 0$ is *robustly satisfied in probability* if it is satisfied for all $\mathbf{w} \in \mathbf{W}$ such that $\pi(\mathbf{w}) > p_s$, $0 \leq p_s < 1$. We say that the constraint is *robustly satisfied* if $p_s = 0$, that is, if it is satisfied for all $\mathbf{w} \in \mathbf{W}$ that can realize.

Problem (1) is extended to robustly satisfy constraint (18c).

1) Problem 2 (Robustified SHOC, RSHOC):

$$\min_{\mathbf{u},\mathbf{w},\mathbf{d},\mathbf{z}} \quad J_d(\mathbf{u}, \mathbf{w}, \mathbf{r}, x(0)) + q_p J_p(\mathbf{w}) \tag{22a}$$
$$\text{s.t.} \quad \text{ueDHA dynamics (10)} \tag{22b}$$
$$g(\mathbf{u}, \mathbf{w}, x(0)) \leq 0 \tag{22c}$$
$$\pi(\mathbf{w}) \geq \tilde{p} \tag{22d}$$
$$g(\mathbf{u}, \mathbf{w}_\omega, x(0)) \leq 0, \quad \forall \mathbf{w}_\omega \in \mathbf{W} : \pi(\mathbf{w}_\omega) > p_s b \tag{22e}$$

where $p_s \geq 0$. Compared to Problem 1, Problem 2 requires in (22e) that the optimal input $\mathbf{u}^*$ is such that constraint $g(\mathbf{u}, \mathbf{w}, x(0)) \leq 0$ is robustly satisfied for all the admissible values of stochastic events $\mathbf{w}$ that have a certain probability to realize (or all of them), while still optimizing the input sequence $\mathbf{u}$ and the predicted disturbance trajectory $\mathbf{w}$ as in Problem 1. Note that $\tilde{p}$ in (17) is a lower bound on the probability of the computed optimal solution, while p_s in (22e) defines an upper bound on the probability of the disturbance sequences against which robust constraints are enforced.

By the techniques of Section III-A and of [9], (22) can be formulated as

$$\min_{\mathbf{u},\mathbf{w},\lambda} \ \frac{1}{2} [\mathbf{u}' \quad \mathbf{w}' \quad \lambda'] H \begin{bmatrix} \mathbf{u} \\ \mathbf{w} \\ \lambda \end{bmatrix} + (x'(0)F_1 + F_2) \begin{bmatrix} \mathbf{u} \\ \mathbf{w} \\ \lambda \end{bmatrix} + \frac{1}{2} [x(0)' \quad \mathbf{r}'] Y \begin{bmatrix} x(0) \\ \mathbf{r} \end{bmatrix} \tag{23a}$$
$$\text{s.t.} \quad A_u \mathbf{u} + A_w \mathbf{w} + A_\lambda \lambda \leq b + Sx(0) \tag{23b}$$
$$G_u \mathbf{u} + G_w \mathbf{w} + G_\lambda \lambda \leq b_g + S_g x(0) \tag{23c}$$
$$\pi(\mathbf{w}) \geq \tilde{p} \tag{23d}$$
$$\underbrace{\begin{bmatrix} A_u \\ G_u \end{bmatrix} \mathbf{u} + \begin{bmatrix} A_w \\ G_w \end{bmatrix} \mathbf{w}_\omega + \begin{bmatrix} A_\lambda \\ G_\lambda \end{bmatrix} \lambda(\mathbf{w}_\omega) \leq \begin{bmatrix} b \\ b_g \end{bmatrix} + \begin{bmatrix} S \\ S_g \end{bmatrix} x(0)}_{\forall \mathbf{w}_\omega \in \mathbf{W} \text{ such that } \pi(\mathbf{w}_\omega) > p_s,} \tag{23e}$$

where $\mathbf{w} \in \mathbf{W}$ is the predicted sequence of uncontrollable events, $\mathbf{w}_\omega \in \mathbf{W}$ is any other sequence of uncontrollable events with probability to realize greater than p_s, (23a)–(23b) are the reformulation of the performance index (22a) and of the dynamics (22b), (23c) is the mixed-integer equivalent reformulation of constraint (18c), and (23e) the reformulation of (22c). For simplicity of notation, λ (resp., $\lambda(\mathbf{w}_\omega)$) collects $\mathbf{z}$ and $\mathbf{d}$ associated[4] to the DHSA dynamics evolving from $x(0)$ by $\mathbf{u}$ and $\mathbf{w}$ (resp., $\mathbf{w}_\omega$).

Because of the quantified constraints (23e), (23) cannot be directly formulated as a mixed integer program. As the possible values of $(\mathbf{w}_\omega, \lambda(\mathbf{w}_\omega))$ are finite, it would be possible to expand the quantified constraints in groups of normal constraints, one for each realization of $\mathbf{w}_\omega \in \mathbf{W}$, as in scenario enumeration. However, as observed earlier, the obtained mixed-integer problem would be intractable in most practical cases.

In general, only certain sequences $\mathbf{w}_\omega$ lead to constraint violation for a fixed $\mathbf{u} = \mathbf{u}^*$. We propose a procedure to enumerate only such potentially dangerous sequences $\mathbf{w}_\omega$. The procedure is based on the interaction between a "partially" robustly constrained optimal control problem to get a candidate solution $\tilde{\mathbf{u}}_i$, and a reachability problem aiming at determining whether an event sequence $\mathbf{w} \in \mathbf{W}$ with probability larger than p_s exists that violates (18c), for the given $x(0) \in \mathcal{X}$ and control input $\tilde{\mathbf{u}}_i$. Such a reachability problem is solved by the mixed-integer feasibility program

$$\min_{\mathbf{w},\lambda} \quad 0 \tag{24a}$$
$$\text{s.t.} \quad A_u \tilde{\mathbf{u}}_i + A_w \mathbf{w} + A_\lambda \lambda \leq b + Sx(0) \tag{24b}$$
$$\pi(\mathbf{w}) > p_s \tag{24c}$$
$$\bigvee_{i=1}^{q} g^i(\tilde{\mathbf{u}}_i, \mathbf{w}, x(0)) > 0 \tag{24d}$$

where $\bigvee$ represent the logical "or" and g^i denotes the i-th component of function g. When $g^i(\mathbf{u}, \mathbf{w}, x(0)) > 0$ is a (mixed integer) linear inequality in $\mathbf{u}^*$, $\mathbf{w}$, $x(0)$, (24d) can be expressed through mixed logical/linear inequalities by associating to each g^i a binary variable $[\delta_j^o = 1] \leftrightarrow [g^i(\tilde{\mathbf{u}}_i, \mathbf{w}, x(0)) > 0]$ (which can be transformed to mixed integer linear inequalities by using for instance the big-M technique [39]) and by introducing the constraint $\sum_{j=1}^{q} \delta_j^o \geq 1$. If (24) is infeasible, then constraint (23e) is satisfied. On the other hand, any feasible solution $\bar{\mathbf{w}}$ of (24) provides a counterexample to (23e). Further trajectories $\mathbf{w}$ violating (23e) can be iteratively found (if they exist) by simply adding the "no good" cut $\mathbf{w} \neq \bar{\mathbf{w}}$ in problem (24) [44].

[4]In case infinity norms are used in the performance index J_d, λ also includes additional variables required to carry on the optimization, and an additional term linear in $\mathbf{r}$ needs to be added in (23b), and $H, Y = 0$ [43]; when quadratic forms [9] are used, $F_2 = 0$.

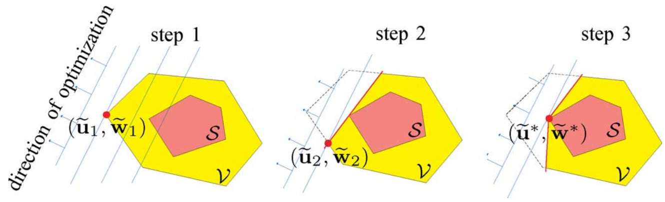

Fig. 3. Geometric interpretation of Algorithm III.1: optimal solution $(\tilde{\mathbf{u}}^*, \tilde{\mathbf{w}}^*)$ found by only introducing two cuts on $\mathcal{V}$, without the need of explicitly characterizing $\mathcal{S}$.

Algorithm III.1: Robustified stochastic hybrid optimal control

1.$i \leftarrow 0$; $\mathcal{P}_1 \leftarrow$ SHOC problem **(1)**;

2.do

2.1.$i \leftarrow i+1$

2.2.Solve $\mathcal{P}_i$ and let $\tilde{\mathbf{u}}_i$ be the corresponding optimal input sequence ($\tilde{\mathbf{u}}_i \leftarrow \emptyset$ if $\mathcal{P}_i$ is infeasible);

2.3.if $\tilde{\mathbf{u}}_i \neq \emptyset$

2.3.1.Solve the reachability analysis problem **(24)** and let $\tilde{\mathbf{w}}_i$ be the solution ($\tilde{\mathbf{w}}_i \leftarrow \emptyset$ if **(24)** is infeasible);

2.3.2.if $\tilde{\mathbf{w}}_i \neq \emptyset$ then $P_{i+1} \leftarrow \mathcal{P}_i$ with additional variables λ_i and additional constraints

$$\begin{aligned} A_u\mathbf{u} + A_w\tilde{\mathbf{w}}_i + A_\lambda\lambda_i &\leq b \\ G_u\mathbf{u} + G_w\tilde{\mathbf{w}}_i + G_\lambda\lambda_i &\leq b_g + S_g x(0) \end{aligned} \quad (25)$$

while $\tilde{\mathbf{u}}_i \neq \emptyset$ and $\tilde{\mathbf{w}}_i \neq \emptyset$;

3.$\mathbf{u}^* \leftarrow \tilde{\mathbf{u}}_i$.

Algorithm III.1 is used to solve problem (23) and is based on the iterative solution of the optimal control problem $\mathcal{P}_i$, whose dimension increases at each iteration i of step 2.3.2., looking for a candidate solution $\tilde{\mathbf{u}}_i$, and of a verification problem, whose number of decision variables remains constant, that looks for a stochastic event sequence $\tilde{\mathbf{w}}_i$ that leads to constraint violation when $\mathbf{u} = \tilde{\mathbf{u}}_i$. Both problems can be solved via mixed integer linear/quadratic programming. The dimension of the optimal control problem increases until no dangerous stochastic sequences $\tilde{\mathbf{w}}_i$ are found. The λ_i variables and the constraints (25) are added to enforce constraint with respect to $\tilde{w}_i$, even if a different trajectory $\mathbf{w}$ is optimizing $\mathcal{P}_i$. Algorithm III.1 terminates in finite time because each new sequence $\tilde{\mathbf{w}}_i$ determined at an iteration of step 2.3.1. is different from the previous ones $\tilde{\mathbf{w}}_1, \ldots, \tilde{\mathbf{w}}_{i-1}$ by virtue of (25), and because the number of admissible stochastic event sequences $\mathbf{w} \in \mathbf{W}$ is finite. Termination with $\mathbf{u}^* = \emptyset$ occurs if (23) is infeasible.

Algorithm III.1 has the geometrical interpretation depicted in Fig. 3. Let $\mathcal{V}$ be the set of input sequences $\mathbf{u}$ that fulfill constraints (23b)–(23d), and let $\mathcal{S}$ be the set of input sequences $\mathbf{u}$ that in addition satisfy (23e), that is, $\mathcal{V}$ is the feasibility set of the SHOC problem while $\mathcal{S}$ is the feasibility set of the RSHOC problem, where clearly $\mathcal{S} \subseteq \mathcal{V}$. The information extracted from (24) is used to "cut away" part of $\mathcal{V}$, without cutting $\mathcal{S}$. The iterative procedure continues until the optimal $\mathbf{u} \in \mathcal{S}$, so that $\mathbf{u}$ also solves (23). Thanks to Algorithm III.1, an explicit characterization of $\mathcal{S}$, which might require a large number of constraints, is in general avoided. Nevertheless, it must be noted that Algorithm III.1 still has a combinatorial complexity. We finally remark that it is straightforward to generalize the algorithm to robustly enforce only a subset of the constraints.

C. Model Predictive Control of DHSA

Both the SHOC and RSHOC problems are open-loop and finite horizon optimal control problems. To use them in practical applications, a receding horizon closed-loop control strategy is needed to ensure state-feedback and unlimited operations over time. We achieve such features by exploiting the optimal control problems presented in the previous sections in a MPC setup.

Let $\mathbf{x}^*(x(t)) = [x^*(0|x(t)) \ldots x^*(N|x(t))]$ be the optimal trajectory obtained by solving open-loop optimal control problem (18) from $x(t)$, $\mathbf{u}^*(x(t)) = [u^*(0|x(t)) \ldots u^*(N-1|x(t))]$ and $\mathbf{w}^*(x(t)) = [w^*(0|x(t)) \ldots w^*(N-1|x(t))]$ be the corresponding optimal input profile. The MPC policy for DHSA (*dhsa-MPC*) is defined by Algorithm III.2.

Algorithm III.2: dhsa-MPC algorithm

1.$t \leftarrow 0$;

2.while (TRUE)

2.1.at time t, measure (or estimate) $x(t)$;

2.2.solve SHOC problem **(18)** (or RSHOC problem **(23)**) where $x(0) = x(t)$ and obtain $\mathbf{u}^*(x(t)) = [u^*(0|x(t)) \ldots u^*(N-1)|x(t))]$, $\mathbf{w}^*(x(t)) = [w^*(0|x(t)) \ldots w^*(N-1)|x(t))]$;

2.3.discard $\mathbf{w}^*$ and apply input $u(t) = u^{MPC}(x(t)) \triangleq u^*(0|x(t))$;

2.4.$t \leftarrow t+1$;

end

Indeed, the application of MPC for DHSA control is straightforward once the SHOC/RSHOC problems are defined, where the only major difference from a standard MPC algorithm is that part of the decision variables ($\mathbf{w}^*(x(t))$) are discarded. However, the optimization problems defined by SHOC and RSHOC are different from both classical nominal and robust MPC problems. Accounting for the stochastic nature of the dynamics, SHOC searches for the best tradeoff between trajectory likelihood and performance, and RSHOC also guarantees robust constraint satisfaction. However, differently from robust MPC approaches, RSHOC does not optimize the worst case performance, so that it avoids excessive conservativeness of the control action, and avoids solving min-max problems that in the present hybrid system context would be extremely complex. The theoretical properties of the proposed dhsa-MPC scheme will be analyzed in Section V.

IV. A Case Study in Supply Chain Management

We show the effectiveness of techniques discussed in Section III on a problem of supply chain management. Successful examples of optimization-based control of supply chains exist in the literature, see for instance [45]. We consider a supply chain that distributes two product types A, B, and that is composed of three production nodes P_i, $i \in \{1,2,3\}$, two storage nodes I_h, $h \in \{1,2\}$, and one retailer node R. The products are fractionable, i.e., their quantities are measured by real numbers. During each control step, any P_i can produce at most one type of products in a fixed quantity ϑ_{ij}, $j \in \{A,B\}$, and ship it to one storage node only. Also, P_1 can produce only type A, P_3 only type B, while P_2 both types. Moreover, a percentage of the items of type A shipped from each storage node to the supplier may be returned to the corresponding storage node. The percentage of shipped items from h that is returned is represented as a discrete stochastic disturbance $\xi_h \in \Xi_h$ where Ξ_h is a discrete set with cardinality ϖ_h, and for each $\xi_h^s \in \Xi_h$, $s = 1, \ldots, \varpi_h$, $p_{\xi_h}^{(s)}$ is the corresponding probability.

The storage nodes provide the supplier with the requested amount of products to be sold, so that the dynamics of the product j stored at node h is

$$I_{hj}(k+1) = I_{hj}(k) + \sum_{i=1}^{3} \vartheta_{ij} u_{hij}(k) - (1-\beta_j \xi_h) v_{hj} \quad (26a)$$

$$y_j = \sum_{h=1}^{2} (1-\beta_j \xi_h) v_{hj}, \quad h \in \{1,2\} \quad (26b)$$

where $u_{hij} = 1$ if and only if producer i is producing and shipping type j to storage h, $u_{hij} \in \{0,1\}$, $\sum_{h=1,2,\ j=A,B} u_{hij} \leq 1$, and $v_{hj} \in \mathbb{R}$ are the amount of product of type j sent from storage h to the retailer, and coefficients $\beta_A = 1$, $\beta_B = 0$. Each storage node I_h stores products of both types and has a limited storage space γ_h, where the occupation of product types is considered to be the same and normalized to 1. Hence, $0 \leq I_{hA} + I_{hB} \leq \gamma_h$. Similarly, the products of type j provided from I_h to R are limited by ν_{hj}, hence $0 \leq v_{hj} \leq \nu_{hj}$. The objective of the supply chain planning system is to meet as much as possible the product demands of products type A and B at the retailer, ϱ_{rA}, ϱ_{rB}, respectively, that is to have $y_j = v_{1j} + v_{2j}$ as close as possible to ϱ_{rj}, $j \in \{A,B\}$. Note that the demand can be exceeded, a situation which is not desirable since case the retailer is forced to remove the excess products by trading them at low price.

In addition, P_1 and P_3 accumulate wear when producing. The wear dynamics is described by

$$\psi_i(k+1) = \alpha_i \psi_i(k) + (1-\alpha_i) \sum_{h=1,2,\ j=A,B} u_{hij}(k) \quad (27)$$

where $\alpha_i \in (0,1)$, $i \in \{1,3\}$, is a coefficient related to maintenance frequency. When the wear level $\psi_i > \overline{\psi}_i$ there is a probability $p_i^{(b)}$ that P_i breaks. When this happens, P_i cannot produce anymore until its wear crosses the lower threshold $\underline{\psi}_i$.

The system is modelled as a DHSA with six continuous states (the products at the storage nodes I_{hj}, $h \in \{1,2\}$, $j \in \{A,B\}$, and the producers' wear ψ_i, $i \in \{1,3\}$), eight binary inputs (the allowed producer-to-storage node-shipping paths u_{hij}), and four continuous input variables (the quantities v_{hj}, provided from the storage nodes to the retailer).

An automaton is associated to each of the producer nodes P_1, P_3, representing different states of wear: $normal$ (N), when no break can occur; $danger$ (D), where if $\psi_i(k+1) > \overline{\psi}_i$, breakdown occurs with a probability $p_i^{(b)}$; and *breakdown* (B), where the production at the node is blocked until $\psi_i < \underline{\psi}_i$. The automata structure is reported in Fig. 2 where $N = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $D = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $\delta_{e_1}^i = 1$ if $\psi_i > \overline{\psi}_i$, $\delta_{e_2}^i = 1$ if $\psi_i < \underline{\psi}_i$, $p_1^i = p_i^{(b)}$, $p_2^i = 1 - p_i^{(b)}$. Note that the automata describing P_1 and P_3 discrete dynamics are independent, so that the full discrete state of the system is the couple $(x_b^{(1)}, x_b^{(3)})$ for a total of 9 possible combinations. In order to represent the system as an ueDHA, 4 uncontrollable events, two for each producer P_i, $i = 1,3$, are added to represent the uncertain transitions in the automata. Also, 6 uncontrollable events are added, 3 for each storage node, to represent the discretized probability distributions of item returns Ξ_h. Hence, 10 uncontrollable events are added in total.

The control problem is to make each product at the retailer track the reference demand, minimizing

$$J = \sum_{k=0}^{N-1} \|Q_y(y(k) - \varrho(k))\|_\infty + \|Q_u u(k)\|_\infty + q_p J_p(w)$$

where $y = \begin{bmatrix} y_A \\ y_B \end{bmatrix}$, $\varrho = \begin{bmatrix} \varrho rA \\ \varrho rB \end{bmatrix}$, $u \in \mathbb{R}^4 \times \{0,1\}^{12}$ is the complete input vector, $w \in \{0,1\}^{10}$ is the vector of uncontrollable events, and the horizon is $N = 6$. State and input constraints are added to enforce the mutual exclusivity relations of production, limits on the quantities stored at the supplier, and limits on the items shipped to the retailer, and $p_i^{(b)} = 0.2$. The weight matrix penalizing demand tracking errors is $Q_y = \begin{bmatrix} 100 & 0 \\ 0 & 100 \end{bmatrix}$, while the input weight matrix is a diagonal matrix that weighs v_{11} and v_{12} by 4, v_{21} and v_{22}, and all the production input by 10. The probability cost J_p is constructed as described in Section V, we have set $\tilde{p} = 0.25^N$ and simulations for different values of q_p are shown. The optimization problem is converted to a mixed-integer quadratic problem with 126 continuous variables, 204 discrete variables, and 1536 mixed-integer linear inequalities. The simulations were executed on a 2-MHz Pentium-IV PC with 2 GB of RAM running Matlab 7.0 and Cplex 9.0 for solving the optimization problems.

Figs. 4 and 5 show the solution of the optimal control problem for a constant value of the reference demand, for the cases $q_p = 10^{-2}$ and $q_p = 10^2$, respectively. For a small value of q_p, the controller decides to track the reference as fast as possible, even at the risk of breakdowns. When q_p is increased, the controller acts more cautiously avoiding such a risk. The effect of the weights are even more visible in the simulations reported in Fig. 6 where the SHOC problem is used in the dhsa-MPC strategy, and the uncontrollable events that affect the system evolution are generated randomly according to the corresponding probabilities. The reference is constant over the prediction horizon and equal to the current demand.

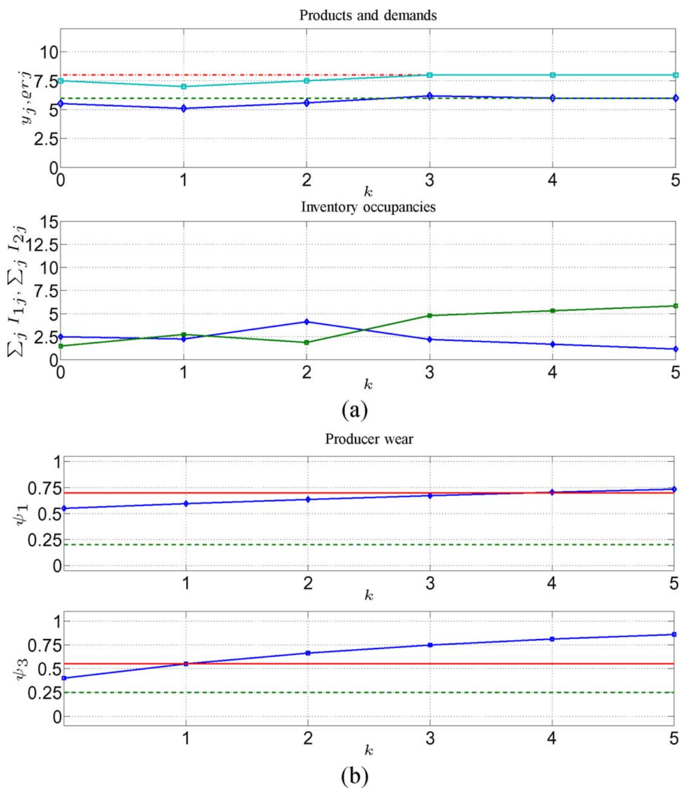

Fig. 4. Supply chain management problem: SHOC solution for $q_p = 10^{-2}$. (a) Upper plot: demand (dashed) and product (solid) at the retailer. Lower plot: product occupancy at supplier nodes. (b) Wear dynamics at producer nodes, and thresholds $\overline{\psi}_i$, $\underline{\psi}_i$. P_1 (up), P_3 (down).

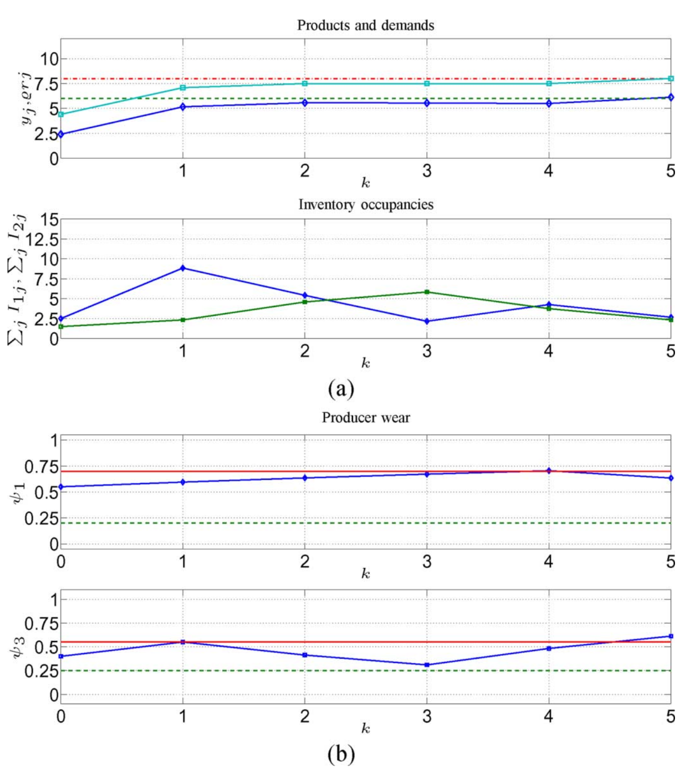

Fig. 5. Supply chain management problem: SHOC solution for $q_p = 10^{2}$. (a) Upper plot: demand (dashed) and product (solid) at the retailer. Lower plot: product occupancy at supplier nodes. (b) Wear dynamics at producer nodes, and thresholds $\overline{\psi}_i$, $\underline{\psi}_i$. P_1 (up), P_3 (down).

When $q_p = 10^2$ the controller tries to avoid breakdowns, hence during certain periods it is not able to entirely meet the retailer demand. When $q_p = 10^{-2}$ the controller tries to aggressively meet the demand, which results in a better tracking, but also in a higher risk of breakdowns, with consequent impossibility to meet the retailer demand. The simulations are repeated five times in Fig. 7. As expected the solution obtained for $q_p = 10^{-2}$ exhibits higher variance, since the low probability weight does not avoid low probability predictions, which have a good chance to lead to unexpected transitions. The computation of the solution along these simulations took (on average) 3.28 s, with 23.9 s is the worst case.

Finally, the RSHOC algorithm is applied, where it is required that the total inventory is $I = \sum_{h=1,2,\ j=A,B} I_{hj} \geq 4$, always. The solution of the robustified optimal control problem for $p_s = 0$ with $q_p = 10^{-4}$ is obtained after three iterations (hence, three optimal control and three verification problems are solved, for a total of 13.68 s). The simulation of the worst case, the one where the producers break down as soon as they are at risk and there are no returns, $\xi_h = 0, h = 1, 2$, is compared with the SHOC solution in Fig. 8, where it can be noticed that the robustified control succeeds in robustly enforcing the constraint. By setting $p_s = 0.3^N$, only two iterations are needed to find the solution, while the drawback is that the system is not robust to all disturbance realizations.

V. Convergence Properties of DHSA Control

In this section, we provide sufficient conditions for asymptotic convergence in probability of a DHSA controlled by the MPC Algorithm III.2. We separate the problem of obtaining convergence of a deterministic system and the problem of obtaining convergence of a system affected by stochastic disturbances. The first can be addressed by well known results of asymptotic convergence of hybrid MPC [9], [11] and is therefore not handled here. Instead, by exploiting the convergence theory of Markov chains [46], we focus on the second issue showing that the convergence properties of the underlying deterministic MPC scheme are preserved despite the stochastic disturbances. We first highlight some immediate results.

1) Result 1: The DHSA is a discrete-time controlled Markov process defined over the hybrid state-space $\mathcal{X}$, that is, there exists a function $G : \mathcal{X} \times \mathcal{U} \times \mathcal{X} \to [0, 1]$, such that $\mathbf{P}[x(t+1) = \hat{x}] = G(x(t), u(t), \hat{x})$.

Result 1 immediately follows from (2), (5), that uniquely define the probability of the successor binary state as a function of x_c, x_b, u_c, and u_b.

2) Result 2: The DHSA in closed-loop with dhsa-MPC is a discrete-time Markov process defined on the hybrid state space $\mathcal{X}$, that is, there exists a function $G^{MPC} : \mathcal{X} \times \mathcal{X} \to [0, 1]$, such that $\mathbf{P}[x(t+1) = \hat{x}] = G^{MPC}(x(t), \hat{x})$.

Result 2 follows by noticing that the MPC policy $u^{MPC}(x)$ is a static state feedback.

We extend to DHSA the following definitions for Markov chains [46].

Definition 6: Given a DHSA, a set $\mathcal{B} \subseteq \mathcal{X}$ is *recurrent* if the asymptotic probability to reach $\bar{x}$ from $\bar{x}$ itself (possibly in infinite time) is 1. $\mathcal{B} \subseteq \mathcal{X}$ is *positive recurrent* if the average

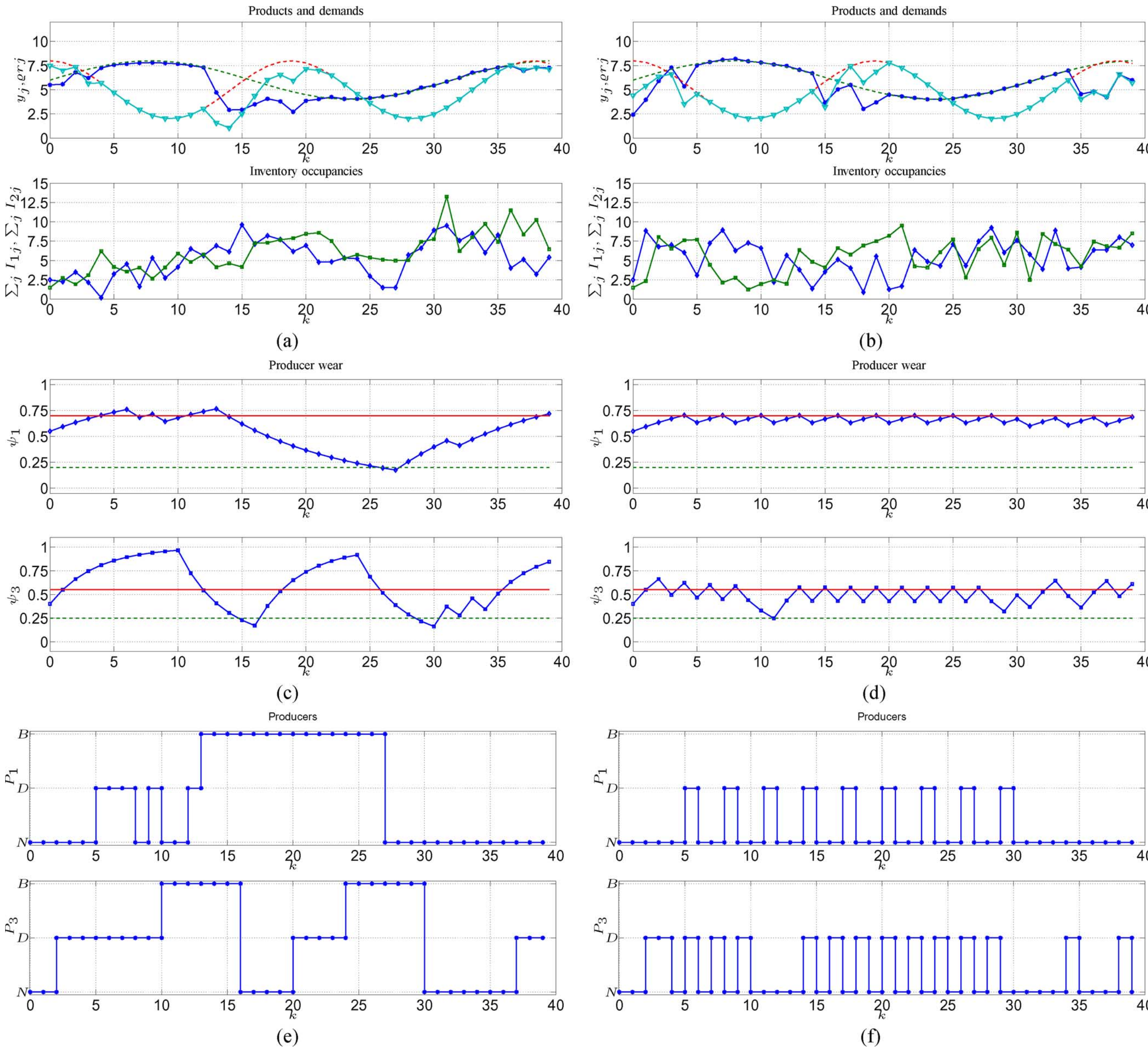

Fig. 6. Supply chain management problem: SHOC algorithm executed in receding horizon mode, with $q_p = 10^{-2}$ (left), $q_p = 10^2$ (right). (a) Case $q_p = 10^{-2}$. Upper plot: demand (dashed) and product (solid) at the retailer. Lower plot: product occupancy at supplier nodes. (b) Case $q_p = 10^2$. Upper plot: demand (dashed) and product (solid) at the retailer. Lower plot: product occupancy at supplier nodes. (c) Case $q_p = 10^{-2}$. Wear dynamics at producer nodes, and thresholds $\overline{\psi}_i$, $\underline{\psi}_i$. P_1 (up), P_3 (down). (d) Case $q_p = 10^2$. Wear dynamics at producer nodes, and thresholds $\overline{\psi}_i$, $\underline{\psi}_i$. P_1 (up), P_3 (down). (e) Case $q_p = 10^{-2}$. Producers discrete state. P_1 (up), P_3 (down). (f) Case $q_p = 10^2$. Producers discrete state. P_1 (up), P_3 (down).

time needed to reach $\mathcal{B}$ from any $x \in \mathcal{B}$ is finite. A possible case is $\mathcal{B} = \{\bar{x}\}$.

The following is the convergence in probability notion used here, where $\|\cdot\|$ is any norm.

Definition 7 ([47]): A sequence of random vectors $\{x(t)\}_{t=0}^{\infty}$, $x \in \mathcal{X}$, $t \in \mathbb{Z}_{0+}$ converges in probability to a (possibly random) vector $\bar{x}$ if $\forall \varepsilon > 0, \lim_{t\to\infty} \mathbf{P}[\|x(t) - \bar{x}\| > \varepsilon] = 0$.

Let $\mathcal{R}(\bar{x}, N) \subseteq \mathcal{X}$ be the set of states from which the state $\bar{x}$ is reachable within N steps. Define $\tilde{p}$ such that $0 \le \tilde{p} \le \min_{x\in\mathcal{R}(\bar{x},N)}\{\mathbf{P}[\mathcal{T}(x,\bar{x})]\} < 1$, where $\mathcal{T}(x,\bar{x})$ is the state trajectory along N steps with maximum probability from state x to $\bar{x}$ computed with respect to $\mathbf{u}$, $\mathbf{w}$. Let $\mathcal{X}_s \subseteq \mathcal{X}$ be the set of states x for which problem (18) is feasible from $x(0) = x$, and let the initial state be $x(t) \in \mathcal{R}(\bar{x}, N) \cap \mathcal{X}_s$.

A. Convergence of MPC Policy Based on SHOC or RSHOC

For convergence analysis, given a dhsa-MPC problem we define the *related deterministic MPC strategy (D-dhsa-MPC)* as the case where both u and w are manipulated variables. The D-dhsa-MPC is a standard hybrid MPC problem, whose difference with the dhsa-MPC is that at Step 2.3. of Algorithm III.2 the D-dhsa-MPC ideally applies both u and w. We call *deterministic behavior* of the closed-loop DHSA, the (hypothetic) state trajectory generated by D-dhsa-MPC. The *associated probability* of the D-dhsa-MPC trajectory is computed by (21) and the generated sequence $\{w(k)\}$. The deterministic behavior is the evolution obtained if the MPC prediction is correct, i.e., $w(t) = w^*(0|x(t))$, for all $t \in \mathbb{Z}_{0+}$. We call *deviations* the evolutions that are different from the deterministic behavior.

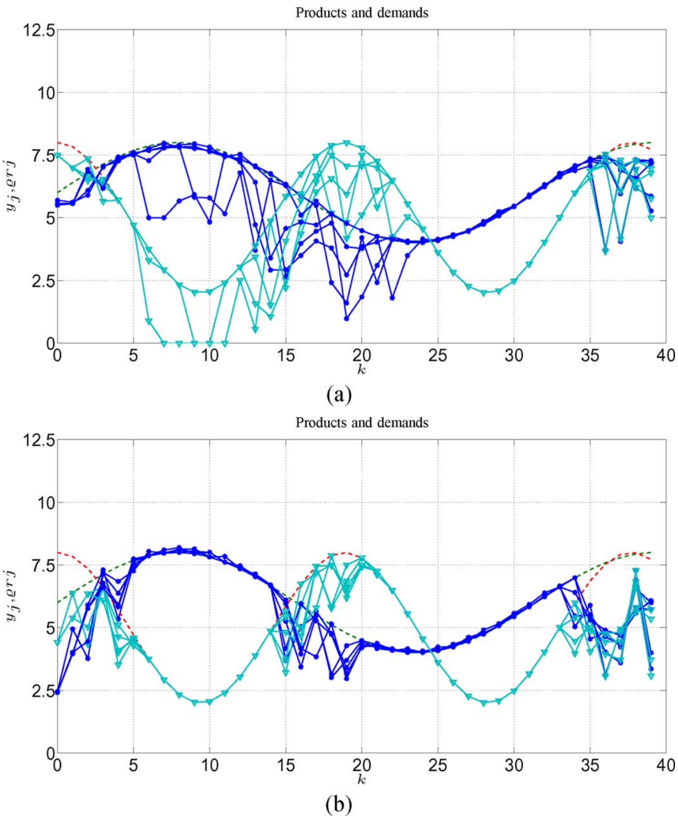

Fig. 7. Supply chain management problem: multiple simulations of SHOC algorithm executed in receding horizon mode, with $q_p = 10^{-2}$ and $q_p = 10^2$. (a) Case $q_p = 10^{-2}$. Demand (dashed) and product (solid) at the retailer. (b) Case $q_p = 10^2$. Demand (dashed) and product (solid) at the retailer.

Assumption 1: The D-dhsa-MPC is converging and the optimal cost $J^*(x(t)) \leq J^*(x(t-1)) - L(x(t-1), u(t-1), r)$, for all $t \in \mathbb{Z}_{0+}$, where $r = \begin{bmatrix} \bar{x} \\ \bar{u} \end{bmatrix}$ and $\bar{u} \in \mathbb{R}^{m_c} \times \{0,1\}^{m_b}$ is the equilibrium input corresponding to $\bar{x}$.

As D-dhsa-MPC is a standard (deterministic) hybrid MPC, Assumption 1 is usually satisfied by using terminal state constraints and defining cost weight matrices in the objective function as reported in [9], [11]. In what follows, we will enable the validity of Assumption 1 by using the terminal state constraint, which yields converging dynamics for hybrid systems modelled as DHA by [9], [38].

We prove asymptotic convergence in probability to the target state according to Definition 7 by showing that the probability that the desired behavior realizes at each step $t \in \mathbb{Z}_{0+}$ (i.e., that $w(t) = w^*(0|x(t))$,for all $t \in \mathbb{Z}_{0+}$) is finite and that, by summation along all the possible trajectories that are executed, the cumulative probability tends asymptotically to 1.

The key step is to prove that the probability of each D-dhsa-MPC trajectory is finite. Constraint(17) is formulated along a finite horizon, which, in a receding horizon approach, shifts at every control step, hence it cannot guarantee finiteness of the trajectory probability along an infinite horizon.

Assumption 2: The following conditions hold:

2.1 the terminal state constraint $x(N) = \bar{x}$ is added to (18);

2.2 $\tilde{p}$ in (17) satisfies $0 \leq \tilde{p} \leq \min_{x \in \mathcal{R}(\bar{x},N)}\{\mathbf{P}[\mathcal{T}(x,\bar{x})]\} < 1$;

2.3 for all $x \in \mathcal{R}(\bar{x},N) \cap \mathcal{X}_s$, for all $w \in \mathbb{W}$, $f_{\text{ueDHA}}(x, u^{MPC}(0|x), w) \in \mathcal{R}(\bar{x},N) \cap \mathcal{X}_s$, where f_{ueDHA} is the function that describes the ueDHA state update, i.e, (1), (2), (3), (7), (8);

2.4 the equilibrium $\bar{x}$ is not affected by stochastic events, $\forall w \in \mathbb{W}$, $f_{\text{ueDHA}}(\bar{x}, \bar{u}, w) = \bar{x}$;

2.5 the optimal performance index is zero for $x = \bar{x}$.

Assumption 2.3 can be removed if RSHOC is used, since the feasibility of the RSHOC problem at the initial instant implies Assumption 2.3. Assumptions 2.4, 2.5 ensure that the target state will never be left, once it is reached. Assumptions 2.1–2.5 also imply Assumption 1, as by the terminal constraint convergence of the deterministic hybrid MPC can be proved [9]. Assumption 2.4 will be relaxed in the following sections.

Lemma 1: Let Assumptions 1 and 2 hold. For any initial state $x(t) \in \mathcal{R}(\bar{x},N) \cap \mathcal{X}_s$ the probability of the trajectory of the D-dhsa-MPC to realize is finite and lower-bounded by a real positive number $p^{MPC}(x(t))$.

Proof: See Appendix I. ■

From the proof of Lemma 1, it is easy to verify that if the state belongs to a bounded set, a finite uniform bound $p^{MPC} = \min_{x \in \mathcal{X}} p^{MPC}(x)$ exists.

Assumption 3: The state x of the DHSA always remains into a bounded set $\mathcal{X}_B \subset \mathcal{X}_s \bigcap \mathcal{R}(\bar{x},N)$.

If not intrinsically guaranteed by the system dynamics, Assumption 3 can be enforced for instance by adopting the RSHOC approach (22) with $p_s = 0$ to enforce the state constraint $x(t) \in \mathcal{X}_B$ robustly.

Theorem 1: Under the assumptions of Lemma 1 and Assumption 3, the state of the DHSA in closed-loop with dhsa-MPC converges in probability to the target $\bar{x}$.

Proof: By Lemma 1, the fact that by Assumption 2.3, $x(t) \in \mathcal{R}(\bar{x},N) \cap \mathcal{X}_s$, for all $t \in \mathbb{Z}_{0+}$ and that by Assumption 3, $x(t)$ is bounded for all $t \in \mathbb{Z}_{0+}$, the probability of the complete D-dhsa-MPC trajectory to realize is $\hat{p}_0 \geq p^{MPC} > 0$, hence $\mathbf{P}[w(t) = w^*(0|x(t)), \forall t \in \mathbb{Z}_{0+}] = \hat{p}_0 \geq p^{MPC}$. By Assumption 1, p^{MPC} is also the lower bound on the probability of $x(t)$ asymptotically converging to $\bar{x}$ without any deviation. Accordingly, $\mathbf{P}[\exists t \in \mathbb{Z}_{0+}, w(t) \neq w^*(0|x(t))] = 1 - \hat{p}_0$. Let $t_1 \in \mathbb{Z}_{0+}$ be such that $w(t) = w^*(0|x(t))$, $\forall t < t_1$, $w(t_1) \neq w^*(0|x(t_1))$, hence at t_1 the first deviation occurs. One may see $x(t_1)$ as the initial state of a new D-dhsa-MPC trajectory, again with probability of converging $\hat{p}_1 \geq p^{MPC}$. Thus, the probability of converging after *at most* one deviation is $\hat{p}_0 + (1-\hat{p}_0)\hat{p}_1$. For $i \in \mathbb{Z}_{0+}$, let

$$P_i = \mathbf{P}[\exists j \in \mathbb{Z}_{0+}, j \leq i : w(t) \neq w^*(0|x(t)) \text{ if } t \in \{t_h\}_{h=1}^{j}, \; w(t) = w^*(0|x(t)) \text{ if } t \in \mathbb{Z}_{0+} \backslash \{t_h\}_{h=1}^{j}]$$

hence P_i is the probability of experiencing at most i deviations before convergence to $\bar{x}$.

We prove convergence by induction. For $i = 0$, we have $P_0 \geq p^{MPC} = \sum_{j=0}^{0} p^{MPC}(1-p^{MPC})^j$. Assume that

$$P_{i-1} \geq \sum_{j=0}^{i-1} p^{MPC}(1-p^{MPC})^{i-1}. \tag{28}$$

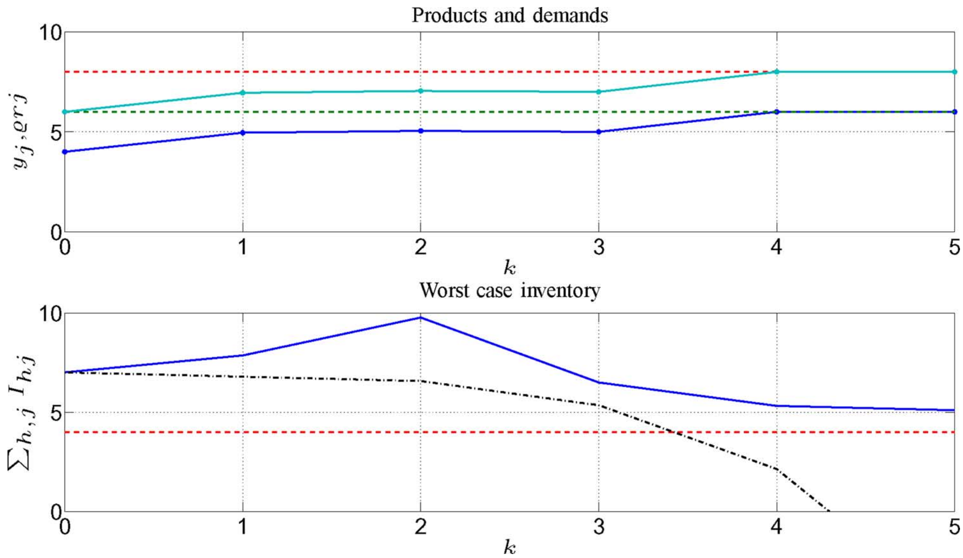

Fig. 8. Solution and worst-case simulation of the RSHOC strategy for $q_p = 10^{-4}$. Upper plot, RSHOC solution: demand (dashed) and product (solid) at the retailer. Lower plot: worst case simulation, cumulative inventory $\sum_{h=1,2,\ j=A,B} I_{hj}$ (solid), lower limit (dashed), and behavior of the SHOC solution (dash-dot).

Then

$$\begin{aligned}P_i &= P_{i-1} + \hat{p}_{i-1}(1 - P_{i-1}) \geq P_{i-1} + p^{MPC}(1 - P_{i-1})\\ &= P_{i-1}(1 - p^{MPC}) + p^{MPC}\end{aligned}$$

where $\hat{p}_{i-1} \geq p^{MPC}$ is the probability that after the $(i-1)^{th}$ deviation, the trajectory converges without further deviations. By the induction assumption (28)

$$\begin{aligned}P_i &\geq \sum_{j=0}^{i-1} p^{MPC}(1 - p^{MPC})^{i+1} + p^{MPC}\\ &= \sum_{j=1}^{i} p^{MPC}(1 - p^{MPC})^{j} + p^{MPC}(1 - p^{MPC})^{0}\\ &= \sum_{j=0}^{i-1} p^{MPC}(1 - p^{MPC})^{j}\end{aligned}$$

and thus we have $\sum_{j=0}^{i} p^{MPC}(1 - p^{MPC})^j \leq P_i \leq 1$. Since

$$\lim_{i\to\infty} \sum_{j=0}^{i} p^{MPC}(1 - p^{MPC})^j = 1 \tag{29}$$

we conclude that $\lim_{t\to\infty} \mathbf{P}[x(t) = \bar{x}] = 1$. ■

B. Persistent Disturbances: Recurrence of the Target State

In this section, we relax Assumption 2.4 by allowing stochastic transitions from the target state $\bar{x}$. Accordingly, we assume that $(\bar{x}, \bar{u})$ is an equilibrium only if $w = \bar{w}$, $0 < \mathbf{P}[w = \bar{w}] < 1$. As $\mathbf{P}[w = \bar{w}] < 1$, $J_p(\mathbf{w}) \neq 0$, for $\mathbf{w} = \{\bar{w}, \ldots, \bar{w}\}$. Since to achieve convergence (16) for $(u, w, x(0)) = (\bar{u}, \bar{w}, \bar{x})$ must be zero, we modify the cost function by introducing w_{l+1} that is enabled only at $(x, u) = (\bar{x}, \bar{u})$ and we replace $\ln p_{l+1}$ by 0 in (21).

Lemma 2: Let $(\bar{u}, \bar{w})$ be an equilibrium input for $\bar{x}$ for the ueDHA, $f_{\text{ueDHA}}(\bar{x}, \bar{u}, \bar{w}) = \bar{x}$. By remapping the DHSA into a modified ueDHA with $w_{l+1} = \bar{w}$ enabled only for $(x, u) = (\bar{x}, \bar{u})$, and by replacing $\ln p_{l+1}$ with 0 in (21), under the assumptions of Theorem 1 except for Assumption 2.4 there exists a finite a lower bound p^{MPC} of the probability that the D-dhsa-MPC trajectory reaches the target state without deviations.

The proof of Lemma 2 and the computation of the lower bound can be obtained similarly to Lemma 1, by considering the probability of *reaching* the target state without accounting for *remaining* in it. However, contrarily to Lemma 1 , Lemma 2 does not account for the evolution *after* the target state is reached. Indeed, once at $\bar{x}$ the probability to remain in $\bar{x}$ depends on $\mathbf{P}[w = \bar{w}] < 1$.

Theorem 2: Let $(\bar{u}, \bar{w})$ be an equilibrium input for the ueDHA dynamics (1), (2), (3), (7), (8) for $x = \bar{x}$. Under the assumptions of Lemma 2, the target state $\bar{x}$ is a recurrent state for the Markov process obtained by the DHSA in closed loop with the dhsa-MPC.

The proof of Theorem 2 is identical to the proof of Theorem 1, where the probability of *reaching* the target state is computed using the bound obtained in Lemma 2.

Lemma 3: Consider the D-dhsa-MPC closed loop, any given $\eta > 0$, and $\mathcal{B}_\eta = \{x \in \mathcal{X} : L(x, \bar{u}, r) \leq \eta\} \bigcap \mathcal{X}_B$, where $r = \begin{bmatrix}\bar{x}\\ \bar{u}\end{bmatrix}$. Under the assumptions of Lemma 2, if for all $x \in \mathcal{X}$, $r \in \mathcal{X} \times \mathcal{U}$, $L(x, \bar{u}, r) \leq L(x, u, r)$ for all $u \in \mathcal{U}$, for any $x(0) \in \mathcal{X}_B$ there exists $\bar{K}_\eta \geq 0$ such that $x(t) \in \mathcal{B}_\eta$ for some $t \leq \bar{K}_\eta$.

Proof: See Appendix I. ■

Commonly used cost functions, including the ones Section III-A, satisfy the conditions of Lemma 3. Using Lemma 3, positive recurrence of DHSA in closed-loop with dhsa-MPC can be proven.

Theorem 3: Under the assumptions of Lemma 3, for all $\eta > 0$ the set $\mathcal{B}_\eta = \{x \in \mathcal{X} : L(x, \bar{u}, r) \leq \eta\} \bigcap \mathcal{X}_B$, where $r = \begin{bmatrix}\bar{x}\\ \bar{u}\end{bmatrix}$, contains the target state $\bar{x}$ and is positive recurrent for a DHSA in closed-loop with a dhsa-MPC controller.

Proof: $\bar{x} \in \mathcal{B}_\eta$ follows by the assumed property of the stage cost $L(\bar{x},\bar{u},r) = 0$. We need to show that the average time $\mathbf{E}[T(x)]$ to reach $\mathcal{B}_\eta$ from any state $x \in \mathcal{X}_B$ is finite. Consider two time instants $t_1, t_2 \in \mathbb{Z}_{0+}$, $t_2 > t_1$, and such that $x_b(t_1) = x_b(t_2) = \bar{x}_b$, $x_b(t) \neq \bar{x}_b$, $\forall t, t_1 < t < t_2$. For $t_1 < t < t_2$, let $i \in \mathbb{Z}_{0+}$ be such that there exist time instants $\{t_j\}_{j=1}^i$, $t_1 \le t_j \le t_2$, $j = 1, \ldots, i$, such that $w(t) \neq w^*(0|x(t))$ if $t \in \{t_j\}_{j=1}^i$, $w(t) = w^*(0|x(t))$ otherwise. Hence, i is the number of deviations from the optimal disturbance realization during the recursion. Let

$$\sigma_i = \mathbf{P}[w(t) \neq w^*(0|x(t)) \text{ if } t \in \{t_j\}_{j=1}^i, \; w(t) \neq w^*(0|x(t)) \text{ if } t_1 \le t \le t_2, t \notin \{t_j\}_{j=1}^i]$$

the probability of having exactly i deviations, and let $T_i(x(t_1))$ be the stochastic variable indicating the time needed to reach $\mathcal{B}_\eta$ from $x(t_1)$ after exactly i deviations. The mean recurrence time (i.e., the mean time to go from $x(t_1)$ to $\mathcal{B}_\eta$) can be expressed as $\mathbf{E}[T(x(t_1))] = \sum_{i=0}^\infty \sigma_i \mathbf{E}[T_i(x(t_1))]$ and by Lemma 3 $\mathbf{E}[T_i(x(t_1))] \le (i+1)\bar{K}_\eta$. Thus, for any initial state $x(t_1) \in \mathcal{X}_B$, $\mathbf{E}[T(x(t_1))] \le \bar{K}_\eta \sum_{i=0}^\infty (i+1)\sigma_i$. Let p^{MPC} be the lower bound of the probability computed in Lemma 2 to reach the target state without any deviation. We know that $p^{MPC} \le \sigma_0 \le 1$, and an upper bound on the probability of reaching $\mathcal{B}_\eta$ with i unexpected events is $\sigma_i \le (1 - p^{MPC})^i$. This is obvious since to reach the target after exactly i deviations, i deviations must occur and their probability is lower bounded by $1 - p^{MPC}$. The average recursion time of the target state $\bar{x}$ is therefore

$$\mathbf{E}[T(x(t_1))] \le \bar{K}_\eta \sum_{i=0}^\infty (i+1)(1-p^{MPC})^i = \bar{K}_\eta \left(\frac{1}{p^{MPC}}\right)^2$$

a finite value that indicates positive recurrence. ■

VI. Conclusions

This paper has proposed a discrete-time model of stochastic hybrid systems that allows the numerical solution of optimal control problems where performance and probability are traded off, possibly under chance constraints and/or robust constraints. Fault-tolerant control and stochastic reachability analysis problems can be also addressed by slightly modifying the proposed methodology.

Although the computational complexity of the approach increases with the number of sources of uncertainties and the complexity of the hybrid model, the example presented in this paper has shown that even in the case of a system with a multidimensional state space, a large input dimension, and multiple sources of uncertainty, the solution is computed in a few seconds.

Under suitable and usually not very restrictive assumptions on the cost function and on the constraints, closed-loop properties of asymptotic convergence in probability in the case of vanishing disturbances, and of positive recurrence in the case of persistent disturbances, are guaranteed.

TABLE I
Numerical Values Used in the Supply Chain Example

Variables	Values
ϑ_{ij}	$\vartheta_{1A} = 4$, $\vartheta_{2A} = 6$, $\vartheta_{2B} = 7$, $\vartheta_{3B} = 3$
γ_h	$\gamma_1 = 14$, $\gamma_2 = 14$
ν_{hj}	$\nu_{1A} = 10$, $\nu_{1B} = 10$, $\nu_{2A} = 10$, $\nu_{2B} = 10$
α_i	$\alpha_1 = 0.9$, $\alpha_2 = 0.75$
$\overline{\psi}_i$	$\overline{\psi}_1 = 0.7$, $\overline{\psi}_3 = 0.55$
$\underline{\psi}_i$	$\underline{\psi}_1 = 0.2$, $\underline{\psi}_3 = 0.25$
$p_i^{(b)}$	$p_1^{(b)} = 0.3$, $p_3^{(b)} = 0.1$
ξ_h	$\xi_1 \in \{0, 0.035, 0.02\}$, $\xi_2 \in \{0, 0.035, 0.02\}$
$p_\xi^{(h)}$	$p_\xi^{(1)} = \{0.2, 0.7, 0.1\}$, $p_\xi^{(2)} = \{0.2, 0.7, 0.1\}$

Appendix
Proofs of Lemmas

Proof of Lemma 1: Given $x(t) \in \mathcal{R}(\bar{x}, N) \cap \mathcal{X}_s$, because of the terminal constraint in Assumption 2.1 (see also [9]), and Assumption 2.4

$$J^*(x(t+1)) \le J^*(x(t)) - L(x(t), u^*(0|x(t)), r) - q_p J_p^*(0|x(t)) \quad (30)$$

where $J^*(x)$ is the optimal cost from x, and $J_p^*(0|x) = -\ln(\mathbf{P}[w(0) = w^*(0|x)])$ indicates the probability cost of the first step of open-loop optimal control from x. Thus, by Assumptions 2.1, 2.4, 2.5, and by Assumption 1, it holds that

$$J_d^*(x(t+1)) + q_p J_p^*(x(t+1)) \le \left(J_d^*(x(t)) - L(x(t)\, u^{MPC}(x(t)), r)\right) + q_p\left(J_p^*(x(t)) - J_p^*(0|x(t))\right). \quad (31)$$

The terms in the right hand side of (31) are the deterministic cost and the probability cost to continue with the trajectory planned at the previous step of the MPC algorithm, and extended to remain on the target state $\bar{x}$. Let

$$\Delta_1 \triangleq q_p(J_p^*(x(t+1)) - J_p^*(x(t)) + J_p^*(0|x(t))). \quad (32)$$

Because of (31), $0 \le J_d^*(x(t+1)) \le (J_d^*(x(t)) - L(x(t), u(0|x(t)), r)) - \Delta_1$. Since $J_d(x(t)) \ge 0$ and $L(x(t), u(0|x(t)), r)) \ge 0$, $\forall t \in \mathbb{Z}_{0+}$, along l steps of the trajectory $\sum_{i=1}^l \Delta_i \le J_d^*(x(t))$. Since

$$\Delta_i = -q_p\Big(\ln \mathbf{P}[\mathbf{x}^*(x(t+i))] - \ln \mathbf{P}[\mathbf{x}^*(x(t+i-1))] + \ln \pi^*(0|x(t+i-1))\Big) \quad (33)$$

by recursive summations, we obtain

$$\sum_{i=1}^l \Delta_i = q_p\left(\sum_{i=1}^l J_p^*(0|x(t+i-1)) + J_p^*(x(t+l)) - J_p^*(x(t))\right)$$

and thus

$$\ln \mathbf{P}[\mathbf{x}_l^{MPC}(x(t))] + \ln \mathbf{P}[\mathbf{x}^*(x(t+l))] \ge -\frac{J^*(x(t))}{q_p} \quad (34)$$

where $\mathbf{x}_l^{MPC}(x(t)) = [x(t), x(t+1), \ldots, x(t+l)]$ is the actual closed-loop MPC state trajectory from $x(t)$ along l steps. Thus, by setting

$$p^{MPC}(x) \triangleq \left(e^{J^*(x)/q_p}\right)^{-1}$$

by (34) it follows that

$$\mathbf{P}\Big[[\mathbf{x}_l^{MPC}(x(t))\,\mathbf{x}^*(x(t+l))]\Big] \geq p^{MPC}(x(t)),\ \forall\, l \in \mathbb{Z}_{0+} \tag{35}$$

is finite and lower bounded. Hence, for $l \to \infty$ the complete D-dhsa-MPC closed-loop trajectory, which by Assumption 1 converges to the target state, has a finite probability $p^{MPC}(x(t))$ to realize. □

Proof of Lemma 3: By Assumption 3, the trajectory of the D-dhsa-MPC closed loop is such that $\forall t \in \mathbb{Z}_{0+}$, $x(t) \in \mathcal{X}_B$. Hence by the definition of $\mathcal{B}_\eta$, along the trajectories generated by the D-dhsa-MPC control strategy, $L(x,\bar{u},r) \leq \eta, \forall x \in \mathcal{B}_\eta$, and $L(x,\bar{u},r) \geq \eta$, $\forall x \notin \mathcal{B}_\eta$. Let $x(0)$ be the initial state and assume by contradiction that for all $t \geq 0$, $x(t) \notin \mathcal{B}_\eta$. By (30)

$$\begin{aligned} J^*(x(t+1)) &\leq J^*(x(t)) - L(x(t),u^{MPC}(t),r) - J_p(0|x(t)) \\ &\leq J^*(x(t)) - L(x(t),\bar{u},r) \leq J^*(x(t)) - \eta \end{aligned}$$

and therefore $J^*(x(t)) \leq J^*(0) - \eta t$. Since $J^*(0)$ is bounded, there exists a finite $\bar{k}_{x(0)} \in \mathbb{Z}_{0+}$ such that

$$\begin{aligned} J^*(x(\bar{k}_{x(0)})) &= J_p^*(x(\bar{k}_{x(0)})) + \sum_{i=0}^{N-1} L\big(x^*(\bar{k}_{x(0)}+i) \\ & u^*(\bar{k}_{x(0)}+i),r\big) \leq J^*(0) - \eta\bar{k}_{x(0)} < \eta. \end{aligned}$$

Hence, $L(x(\bar{k}_{x(0)}),\bar{u},r) \leq L(x^*(\bar{k}_{x(0)}),\ u^*(\bar{k}_{x(0)}),r) \leq \eta$, which implies $x(\bar{k}_{x(0)}) \in \mathcal{B}_\eta$, a contradiction. By choosing $\bar{K}_\eta \triangleq \lceil (1/\eta)\max_{x(0)\in\mathcal{X}_B} J^*(x(0))\rceil$, where the maximum exists because we are considering a bounded state-space, and $\lceil\cdot\rceil$ denotes roundoff to the smallest greater or equal integer, the lemma is proven. □

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