Data-driven synthesis of Robust Invariant Sets and Controllers

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***Abstract*— This paper presents a method to identify an uncertain linear time-invariant (LTI) prediction model for tube-based Robust Model Predictive Control (RMPC). The uncertain model is determined from a given state-input dataset by formulating and solving a Semidefinite Programming problem (SDP), that also determines a static linear feedback gain and corresponding invariant sets satisfying the inclusions required to guarantee recursive feasibility and stability of the RMPC scheme, while minimizing an identification criterion. As demonstrated through an example, the proposed concurrent approach provides less conservative invariant sets than a sequential approach.**

I. INTRODUCTION

The tube-based RMPC scheme of [1] is a popular method to design robust feedback controllers using LTI plant models

$$x(t+1) = Ax(t) + Bu(t) + w(t), \tag{1}$$

subject to state constraints $x \in \mathcal{X} \subset \mathbb{R}^{n_x}$, input constraints $u \in \mathcal{U} \subset \mathbb{R}^{n_u}$, and additive unknown but bounded disturbances $w \in \mathcal{W}$. Besides model (1), the RMPC scheme also requires a feedback gain K, and a Robust Positive Invariant (RPI) set [2] in which the state of the system $x(t+1) = (A+BK)x(t)+w(t)$ can be enforced to persistently belong.

Given a model $(A, B, \mathcal{W})$, the problem of computing constrained RPI sets has been well studied in the literature. We focus on polytopic RPI sets for their reduced conservativeness [2]. Given K, methods to compute tight invariant approximations of the minimal RPI (mRPI) sets were presented in [3], [4], [5]. They characterize the uncertainty tube that bounds the deviation of the actual state trajectory from a central nominal one. Similarly, methods to compute maximal PI (MPI) sets were presented in [6], [7]. They can be used as terminal sets in RMPC to guarantee feasibility and stability. It is known that improved RPI sets can be computed by also optimizing over K. In [8], [9], methods to compute RPI sets along with K were presented. Earlier approaches in [10], [11] optimize over K and reduce conservativeness in RMPC.

In order to identify a model $(A, B, \mathcal{W})$, physics-based, regression approaches and/or set-membership approaches [12], [13] can be used. Methods that take control design into account while performing system identification were presented in [14], [15]. It was demonstrated in [16] that if system identification can be combined with RPI set computation, then conservativeness in the computed RPI sets can be reduced. Motivated by this observation, we present a method to concurrently select a model $(A, B, \mathcal{W})$ and synthesize RPI sets for RMPC. Alternative methods that directly compute feedback controllers using an implicit plant description based on measured trajectories were presented in [17], [18], [19], [20], [21]. In [22], these methods were used to synthesize controllers that induce robust invariance in a given polyhedral set. However, these methods cannot be used directly to select a model and synthesize RPI sets optimized for RMPC synthesis.

Contribution: We consider a dataset of state-input measurements from a plant, and present a method to identify an LTI model (1), along with RPI sets suitable for RMPC synthesis. To this end, we characterize a set of models $(A, B, \mathcal{W})$ that can describe the plant behavior, and use nonlinear matrix inequality (NLMI)-based results from [9] on RPI set computation to formulate a NonLinear Program with Matrix Inequalities (NLPMI) that selects a model (1) along with suitable RPI sets and a corresponding feedback matrix K. We then present a method to solve the NLPMI based on a Sequential Convex Programming (SCP) approach that we tailor to preserve feasibility of the iterates and satisfy a cost decrease condition. Finally, we demonstrate the efficacy of the method using a simple numerical example.

Notation: $\mathcal{P}(A, b) := \{x \in \mathbb{R}^n : \ -b \leq Ax \leq b\}$ is a symmetric polytope, and $\mathcal{E}(Q, r) := \{x \in \mathbb{R}^n : \ x^\top Qx \leq r\}$ is an ellipsoid. The set of m dimensional positive vectors is denoted as $\mathbb{R}^m_+$, positive definite $m \times m$ diagonal matrices is denoted as $\mathbb{D}^m_+$, positive definite $m \times m$ symmetric matrices as $\mathbb{S}^m_+$. The symbols $\mathbf{1}$, $\mathbf{0}$ and $\mathbf{I}$ denote all-ones, all-zeros, and identity matrix, respectively. The set $\mathbb{I}_m^n := \{m, \cdots, n\}$ is the set of natural numbers between m and n. T_i and T_{ij} denote row i and element (i, j) of matrix $T \in \mathbb{R}^{n \times m}$, and $\|T\|_\infty := \max_{i \in \mathbb{I}_1^n} \sum_{j=1}^m |T_{ij}|$ is the ∞-norm of the matrix. We define $\|v\|_S^2 := v^\top Sv$, and use $*$ to represent symmetrically identifiable matrix entries. Given two compact sets $\mathcal{S}_1, \mathcal{S}_2 \subset \mathbb{R}^n$, the Minkowski sum is defined as $\mathcal{S}_1 \oplus \mathcal{S}_2 := \{x + y : x \in \mathcal{S}_1, y \in \mathcal{S}_2\}$, and the Minkowski difference as $\mathcal{S}_1 \ominus \mathcal{S}_2 := \{x : \{x\} \oplus \mathcal{S}_2 \subseteq \mathcal{S}_1\}$. We write $C_1\mathcal{P}(A_1, b_1) \oplus C_2\mathcal{P}(A_2, b_2) = [C_1 \ \ C_2]\mathcal{P}(\mathrm{diag}(A_1, A_2), [b_1^\top b_2^\top]^\top)$, where $\mathrm{diag}(A_1, A_2) := \begin{bmatrix} A_1 & \mathbf{0} \\ \mathbf{0} & A_2 \end{bmatrix}$ is a block-diagonal matrix.

II. PROBLEM FORMULATION

We briefly recall the tube-based RMPC scheme from [1]. Given system (1), consider the nominal model $\hat{x}(t+1) = A\hat{x}(t) + B\hat{u}(t)$, and parameterize the plant input as $u(t) = \hat{u}(t) + K(x(t) - \hat{x}(t))$, where K is a static feedback gain. Assuming that the feedback gain is stabilizing for (A, B), the state error $\Delta x := x - \hat{x}$ with dynamics $\Delta x(t+1) = (A+BK)\Delta x(t) + w(t)$ belongs to the RPI set $\Delta\mathcal{X}$

$$\text{if } \Delta x(0) \in \Delta\mathcal{X}, \quad \text{and} \quad (A+BK)\Delta\mathcal{X} \oplus \mathcal{W} \subseteq \Delta\mathcal{X}. \tag{2}$$

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Hence, x always belongs to the uncertainty tube with cross-section $\Delta\mathcal{X}$ around $\hat{x}$, i.e., $x(t) \in \hat{x}(t) \oplus \Delta\mathcal{X}, \forall t \geq 0$. The RMPC scheme then enforces $\hat{x} \in \mathcal{X} \ominus \Delta\mathcal{X}$ and $\hat{u} \in \mathcal{U} \ominus K\Delta\mathcal{X}$, and computes $\mathbf{z} := \{\hat{x}(t), \dots, \hat{x}(t+N), \hat{u}(t), \dots, \hat{u}(t+N-1)\}$ online given $x(t)$ by solving

$$\begin{aligned}
\min_{\mathbf{z}} \quad & \sum_{s=t}^{t+N-1} \left\| \begin{bmatrix} \hat{x}(s)^\top & \hat{u}(s)^\top \end{bmatrix}^\top \right\|_{H_\mathrm{Q}}^2 + \|\hat{x}(t+N)\|_{P_\mathrm{Q}}^2 \\
\text{s.t.} \quad & \hat{x}(s+1) = A\hat{x}(s) + B\hat{u}(s), \qquad s \in \mathbb{I}_t^{t+N-1}, \\
& \hat{x}(s) \in \mathcal{X} \ominus \Delta\mathcal{X},\ \hat{u}(s) \in \mathcal{U} \ominus K\Delta\mathcal{X}, \qquad s \in \mathbb{I}_{t+1}^{t+N-1}, \\
& x(t) \in \{\hat{x}(t)\} \oplus \Delta\mathcal{X}, \quad \hat{x}(t+N) \in \mathcal{X}_\mathrm{t},
\end{aligned} \tag{3}$$

where $\mathcal{X}_\mathrm{t}$ is the terminal set. We assume that the set $\Delta\mathcal{X}$ is *small* enough for feasibility of Problem (3), i.e.,

$$\Delta\mathcal{X} \subset \mathcal{X},\ K\Delta\mathcal{X} \subset \mathcal{U}, \tag{4}$$

and $\mathcal{X}_\mathrm{t}$ is a PI set for $\hat{x}(t+1) = (A+BK)\hat{x}(t)$ that satisfies

$$(A+BK)\mathcal{X}_\mathrm{t} \subseteq \mathcal{X}_\mathrm{t} \subseteq \mathcal{X} \ominus \Delta\mathcal{X}, \quad K\mathcal{X}_\mathrm{t} \subseteq \mathcal{U} \ominus K\Delta\mathcal{X}. \tag{5}$$

Then $\Omega_N := \{x : (3) \text{ is feasible with } x(t) = x\}$ is such that for each $x(t) \in \Omega_N$, there recursively exists an optimal solution $\mathbf{z}_* := \{\hat{x}_*(t), \dots, \hat{x}_*(t+N), \hat{u}_*(t), \dots, \hat{u}_*(t+N-1)\}$ [1, Proposition 2]. Then, the input $u(t) := \hat{u}_*(t) + K(x(t) - \hat{x}_*(t))$ is applied to the plant. Moreover, if $(H_\mathrm{Q}, P_\mathrm{Q})$ are such that P_Q is the solution of the Discrete Algebraic Riccati equation (DARE) formulated using H_Q for the system (A, B), and K is corresponding optimal feedback gain, $\Delta\mathcal{X}$ is exponentially stable from every $x \in \Omega_N$ [1, Theorem 1].

Problem description: We consider a plant with dynamics $x(t+1) = f_\mathrm{tr}(x(t), u(t), v(t))$ that is subject to bounded inputs $u(t)$ and unknown but bounded disturbances $v(t) \in \mathcal{V}_\mathrm{tr} \subset \mathbb{R}^{n_v}$. Assuming that f_tr and $\mathcal{V}_\mathrm{tr}$ are unknown, we collect a dataset $\mathcal{D} := \{x_\mathrm{D}(t), u_\mathrm{D}(t),\ t \in \mathbb{I}_1^T\}$ of state-input measurements from the plant. Using $\mathcal{D}$, we propose a method to compute $(A, B, \mathcal{W}, K, \Delta\mathcal{X}, \mathcal{X}_\mathrm{t})$ that satisfy (1), (2), (4), (5) required for RMPC synthesis, while optimizing some criterion. In the sequel, we assume that we are interested in modeling the plant in a bounded subset $\Phi_* \subset \mathbb{R}^{n_x+n_u}$ of state-input vectors $[x^\top\ u^\top]^\top$. If f_tr is open-loop stable, then Φ_* can also represent the set of all possible state-input vectors that are reachable by the plant. Then, we assume that the set $\Phi_T := \{[x_\mathrm{D}(t)^\top\ u_\mathrm{D}(t))^\top]^\top, t \in \mathbb{I}_1^T\}$ of measured state-input vectors is a subset of Φ_*, i.e., $\Phi_T \subseteq \Phi_*$. We also define $\mathcal{J}_* := \left\{ \begin{bmatrix} x \\ u \\ x_+ \end{bmatrix} : \begin{matrix} x_+ = f_\mathrm{tr}(x, u, v), \\ \forall\ [x^\top\ u^\top]^\top \in \Phi_*,\ \forall v \in \mathcal{V}_\mathrm{tr} \end{matrix} \right\}$. Since Φ_*, $\mathcal{V}_\mathrm{tr}$ and u are assumed to be bounded, $\mathcal{J}_*$ is also a bounded set. Finally, we denote the measured subset built using $\mathcal{D}$ as $\mathcal{J}_T := \{[x_\mathrm{D}(t)^\top\ u_\mathrm{D}(t)^\top\ x_\mathrm{D}(t+1)^\top]^\top,\ t \in \mathbb{I}_1^{T-1}\} \subseteq \mathcal{J}_*$.

Remark 1: Given (A, B, K), matrices $(H_\mathrm{Q}, P_\mathrm{Q})$ satisfying the DARE can be computed using the procedure in [23]. Hence, our approach involves tuning the MPC scheme. Combining our approach with other MPC tuning methods such as [24] is a subject of future research. □

III. Identification based on invariant sets

To compute $(A, B, \mathcal{W}, K, \Delta\mathcal{X}, \mathcal{X}_\mathrm{t})$ using the dataset $\mathcal{D}$ by optimization, we first characterize the set of models $(A, B, \mathcal{W})$ that are suitable to model the underlying plant.

1) ***Characterization of feasible models***: We consider LTI models of the form in (1) to model the underlying plant, with the disturbance set parametrized as $\mathcal{W} := \mathcal{P}(F, d), d \in \mathbb{R}_+^{m_w}$. We assume for simplicity that the matrix F is fixed a priori. Then, an LTI model (A, B, d) is suitable for RMPC synthesis if it can capture all possible state transitions of the plant as

$$\begin{aligned} f_\mathrm{tr}(x, u, v) \in \{Ax + Bu\} \oplus \mathcal{P}(F, d), \\ \forall\ [x^\top\ u^\top]^\top \in \Phi_*, \quad \forall\ v \in \mathcal{V}_\mathrm{tr}. \end{aligned} \tag{6}$$

Defining the prediction error $\zeta(A, B, z) := x_+ - Ax - Bu$ with $z := [x^\top\ u^\top\ x_+^\top]^\top$, a model (A, B, d) satisfies (6) if and only if $\zeta(A, B, z) \in \mathcal{P}(F, d), \forall z \in \mathcal{J}_*$ by definition of $\mathcal{J}_*$. Hence, $\Sigma_* := \{(A, B, d) : \zeta(A, B, z) \in \mathcal{P}(F, d),\ \forall\ z \in \mathcal{J}_*\}$ characterizes the set of all models (A, B, d) satisfying (6). However, we cannot construct Σ_* since we only access the measured subset $\mathcal{J}_T \subseteq \mathcal{J}_*$. Hence, we instead characterize

$$\Sigma_T(\theta_T) := \left\{ \begin{pmatrix} A, \\ B, \\ d \end{pmatrix} : \begin{matrix} \zeta(A, B, z) \in \mathcal{P}(F, d - \kappa_T(A, B)\mathbf{1}) \\ \kappa_T(A, B) = \|F[-A\ \ -B\ \ \mathbf{I}]\|_\infty \theta_T, \\ d > \kappa_T(A, B)\mathbf{1},\ \forall\ z \in \mathcal{J}_T \end{matrix} \right\}$$

using $\mathcal{J}_T$, where $\theta_T := \mathbf{d}_\infty(\mathcal{J}_*, \mathcal{J}_T)$ is the Hausdorff distance between the sets $\mathcal{J}_T$ and $\mathcal{J}_*$ in ∞-norm, and is given by $\mathbf{d}_\infty(\mathcal{J}_*, \mathcal{J}_T) := \max_{z_* \in \mathcal{J}_*} \min_{z \in \mathcal{J}_T} \|z - z_*\|_\infty$ since the inclusion $\mathcal{J}_T \subseteq \mathcal{J}_\infty$ holds for every $T > 0$.

Assumption 1: (a) $\Sigma_T(\theta_T) \neq \emptyset$; ($b$) For all scalars $\theta > 0$, there exists some $\tilde{T} < \infty$ such that $\mathbf{d}_\infty(\mathcal{J}_*, \mathcal{J}_{\tilde{T}}) \leq \theta$. □

Assumption 1 implies that as $T \to \infty$, the set $\mathcal{J}_*$ is densely covered by $\mathcal{J}_T$: this is an assumption on the persistence of excitation of inputs, and bound-exploring property of the disturbances acting on the underlying plant [15, Section 3.2].

Theorem 1: If Assumption 1 holds, then (6) holds for all models $(A, B, d) \in \Sigma_T(\theta_T)$. □

Proof: We show that $\zeta(A, B, z) \in \mathcal{P}(F, d), \forall z \in \mathcal{J}_*, \forall (A, B, d) \in \Sigma_T(\theta_T)$. For any $(A, B, d) \in \Sigma_T(\theta_T)$, clearly $\zeta(A, B, z) \in \mathcal{P}(F, d - \kappa_T(A, B)\mathbf{1}) \subset \mathcal{P}(F, d), \forall z \in \mathcal{J}_T$. By definition of the Hausdorff distance, for every remaining $\bar{z} \in \mathcal{J}_* \setminus \mathcal{J}_T := \{\tilde{z} : \tilde{z} \in \mathcal{J}_*, \tilde{z} \notin \mathcal{J}_T\}$, $\exists\ z \in \mathcal{J}_T$ such that $||z - \bar{z}||_\infty \leq \theta_T$. Then, for any $(A, B, d) \in \Sigma_T(\theta_T)$,

$$\begin{aligned} F\zeta(A, B, \bar{z}) &= F(\zeta(A, B, \bar{z}) - \zeta(A, B, z) + \zeta(A, B, z)) \\ &\leq F[-A\ \ -B\ \ \mathbf{I}](\bar{z} - z) + d - \kappa_T(A, B)\mathbf{1} \\ &\leq ||F[-A\ \ -B\ \ \mathbf{I}]||_\infty \theta_T \mathbf{1} + d - \kappa_T(A, B)\mathbf{1} = d, \end{aligned}$$

where the second step follows from definition of $\Sigma_T(\theta_T)$, and third step from the definition of ∞-norm, the Cauchy-Schwarz inequality and $||\bar{z} - z||_\infty \leq \theta_T$. Using similar arguments, the condition $-d \leq F\zeta(A, B, \bar{z})$ follows, thus concluding that $\zeta(A, B, \bar{z}) \in \mathcal{P}(F, d), \forall\ \bar{z} \in \mathcal{J}_* \setminus \mathcal{J}_T$. ■

Theorem 1 implies that every $(A, B, d) \in \Sigma_T(\theta_T) \subseteq \Sigma_*$ is a feasible model for RMPC synthesis. However, $\Sigma_T(\theta_T)$ cannot be constructed from data since θ_T is unknown. To tackle this issue, we follow the standard approach of inflating the disturbance set using some parameter (e.g., [15]): we propose to select some $\hat{\theta}_T > 0$, and approximate $\Sigma_T(\theta_T)$ with $\hat{\Sigma}_T := \Sigma_T(\hat{\theta}_T)$ under the following assumption.

Assumption 2: $\hat{\theta}_T \geq \theta_T = \mathbf{d}_\infty(\mathcal{J}_*, \mathcal{J}_T)$. □

Under Assumption 2, we have $\hat{\Sigma}_T \subseteq \Sigma_T(\theta_T)$. Hence, every $(A, B, d) \in \hat{\Sigma}_T$ is suitable for RMPC synthesis. We assume in the sequel that Assumption 2 is satisfied by some user-defined $\hat{\theta}_T$. We then encode $\hat{\Sigma}_T$ with linear constraints as

$$\hat{\Sigma}_T = \left\{ \begin{pmatrix} A, \\ B, \\ d \end{pmatrix} : \begin{matrix} \zeta(A,B,z) \in \mathcal{P}(F, d - \lambda\hat{\theta}_T \mathbf{1}), d > \lambda\hat{\theta}_T\mathbf{1}, \\ -\mathcal{Z} \leq F[-A \ \ -B \ \mathbf{I}] \leq \mathcal{Z}, \mathcal{Z} \geq \mathbf{0}, \\ \Sigma_{j=1}^{2n_x+n_u} \mathcal{Z}_{ij} \leq \lambda, \forall\ i \in \mathbb{I}_1^{m_w}, \forall\ z \in \mathcal{J}_T \end{matrix} \right\}$$

using the definition of ∞-norm for matrices, where $\mathcal{Z} \in \mathbb{R}_+^{m_w \times (2n_x+n_u)}$ is a slack variable matrix. We reiterate that since Assumption 2 cannot be verified directly using data, robustness guarantees with respect to the underlying plant can only be provided in theory. However, if Assumption 1(b) holds, the distance $\theta_T \to 0$ for large T. Hence, guessing some $\hat{\theta}_T \approx 0$ can satisfy Assumption 2 for large datasets. Moreover, the validity of a given $\hat{\theta}_T$ can be checked by verifying the existence of a model $(A,B,d) \in \hat{\Sigma}_T$ explaining a validation dataset. On the other hand, computation of a $\hat{\theta}_T$ satisfying Assumption 2 is a fundamental issue in data-driven methods: while statistical techniques such as, e.g., bootstrapping can be used, the development of such methods is a future research subject.

Remark 2: (a) In [15], an optimal LTI model set is first computed, from which a model is selected and then feedback controllers are synthesized. We combine all three phases in the current work; (b) In [19], the closed-loop dynamics of an unknown LTI plant with a known disturbance set is characterized in terms of the measured dataset, and parametrized by unknown but bounded disturbance sequences. Then, a controller is synthesized for all feasible LTI models. We instead use a model-dependent disturbance set. While the assumption of a known disturbance set is as strict as Assumption 2, comparison with [19] is a subject of future research. □

2) ***Robust PI set design:*** We will now compute a feedback gain K and corresponding invariant sets $\Delta\mathcal{X}$ and $\mathcal{X}_{\mathrm{t}}$ for some $(A,B,d) \in \hat{\Sigma}_T$. To this end, we parametrize the RPI set as $\Delta\mathcal{X} = \mathcal{P}(\underline{P}, \underline{b}), \underline{b} \in \mathbb{R}_+^{\underline{m}}$, the PI terminal set as $\mathcal{X}_{\mathrm{t}} = \mathcal{P}(\bar{P}, \bar{b}), \bar{b} \in \mathbb{R}_+^{\bar{m}}$, and assume that the constraint sets are $\mathcal{X} = \mathcal{P}(V^x, v^x), v^x \in \mathbb{R}_+^{m_x}$, and $\mathcal{U} = \mathcal{P}(V^u, v^u), v^u \in \mathbb{R}_+^{m_u}$. Then, for some $(A,B,d) \in \hat{\Sigma}_T$, if $(K, \underline{P}, \underline{b}, \bar{P}, \bar{b})$ satisfies

$$(A+BK)\mathcal{P}(\underline{P}, \underline{b}) \oplus \mathcal{P}(F,d) \subseteq \mathcal{P}(\underline{P}, \underline{b}), \tag{7a}$$

$$(A+BK)\mathcal{P}(\bar{P}, \bar{b}) \subseteq \mathcal{P}(\bar{P}, \bar{b}), \tag{7b}$$

$$\mathcal{P}(\underline{P}, \underline{b}) \oplus \mathcal{P}(\bar{P}, \bar{b}) \subseteq \mathcal{X}, \tag{7c}$$

$$K\mathcal{P}(\underline{P}, \underline{b}) \oplus K\mathcal{P}(\bar{P}, \bar{b}) \subseteq \mathcal{U}, \tag{7d}$$

it can be used to synthesize the RMPC scheme (since (7a) implies (2), (7b) implies (5), (7c)-(7d) imply the constraint inclusions in (4)-(5)). We encode (7a)-(7d) using Theorem 2.

Theorem 2: [9, Theorem 2] For some $C \in \mathbb{R}^{n \times n^c}$, $M^c \in \mathbb{R}^{m^c \times n}$, $b^c \in \mathbb{R}_+^{m^c}$, $M^0 \in \mathbb{R}^{m^0 \times n}$, $b^o \in \mathbb{R}_+^{m^o}$, the inclusion $C\mathcal{P}(M^c, b^c) \subseteq \mathcal{P}(M^0, b^0)$ holds if $\forall\ i \in \mathbb{I}_1^{m^0}, \exists L_{[i]}^c \in \mathbb{D}_+^{m^c}$ such that $\begin{bmatrix} 2b_i^0 - b^{c\top} L_{[i]}^c b^c & M_i^0 C \\ * & M^{c\top} L_{[i]}^c M^c \end{bmatrix} \succ 0.$ □

Remark 3: The condition in Theorem 2 is necessary and sufficient for the inclusion $C\mathcal{P}(M^c, b^c) \subseteq \mathcal{P}(M^0, b^0)$ if a non-strict inequality $\succeq$ is used. However, we only use the sufficiency property given by $\succ$ for numerical robustness. □

Hence, (7a) $\Longleftarrow \forall i \in \mathbb{I}_1^{\underline{m}}, \exists \underline{D}_{[i]} \in \mathbb{D}_+^{\underline{m}}, W_{[i]} \in \mathbb{D}_+^{m_w}$ s.t.

$$\begin{bmatrix} 2\underline{b}_i - \underline{b}^\top \underline{D}_{[i]} \underline{b} - d^\top W_{[i]} d & \underline{P}_i & \underline{P}_i(A+BK) \\ * & F^\top W_{[i]} F & \mathbf{0} \\ * & * & \underline{P}^\top \underline{D}_{[i]} \underline{P} \end{bmatrix} \succ 0, \tag{8}$$

(7b) $\Longleftarrow \forall\ i \in \mathbb{I}_1^{\bar{m}}, \exists\ \bar{D}_{[i]} \in \mathbb{D}_+^{\bar{m}}$ s.t.

$$\begin{bmatrix} 2\bar{b}_i - \bar{b}^\top \bar{D}_{[i]} \bar{b} & \bar{P}_i(A+BK) \\ * & \bar{P}^\top \bar{D}_{[i]} \bar{P} \end{bmatrix} \succ 0, \tag{9}$$

(7c) $\Longleftarrow \forall\ i \in \mathbb{I}_1^{m_x}, \exists\ \underline{S}_{[i]} \in \mathbb{D}_+^{\underline{m}}, \bar{S}_{[i]} \in \mathbb{D}_+^{\bar{m}}$ s.t.

$$\begin{bmatrix} 2v_i^x - \underline{b}^\top \underline{S}_{[i]} \underline{b} - \bar{b}^\top \bar{S}_{[i]} \bar{b} & V_i^x & V_i^x \\ * & \underline{P}^\top \underline{S}_{[i]} \underline{P} & \mathbf{0} \\ * & * & \bar{P}^\top \bar{S}_{[i]} \bar{P} \end{bmatrix} \succ 0, \tag{10}$$

(7d) $\Longleftarrow \forall\ i \in \mathbb{I}_1^{m_u}, \exists\ \underline{R}_{[i]} \in \mathbb{D}_+^{\underline{m}}, \bar{R}_{[i]} \in \mathbb{D}_+^{\bar{m}}$ s.t.

$$\begin{bmatrix} 2v_i^u - \underline{b}^\top \underline{R}_{[i]} \underline{b} - \bar{b}^\top \bar{R}_{[i]} \bar{b} & V_i^u K & V_i^u K \\ * & \underline{P}^\top \underline{R}_{[i]} \underline{P} & \mathbf{0} \\ * & * & \bar{P}^\top \bar{R}_{[i]} \bar{P} \end{bmatrix} \succ 0. \tag{11}$$

We now formulate a criterion to select the variables formulating (8)-(11) along with $(A,B,d) \in \hat{\Sigma}_T$, leading to an optimization problem. In this formulation, we assume that the matrices $(\underline{P}, \bar{P}, F)$ are known a priori. While this assumption increases conservativeness in our approach, it simplifies the solution procedure. We note that a good set of hyperplanes $(\underline{P}, \bar{P})$ can *guessed* for some initial (A,B,F,d) using [9], and kept constant for our approach. Moreover, our approach can be extended to optimize over $(\underline{P}, \bar{P})$ using the results in [25]. We skip further details here due to space limitations.

3) ***Identification criterion:*** For RMPC synthesis, it is desirable to compute a small RPI set $\Delta\mathcal{X}$ to reduce constraint tightening, and to regulate the system to a small neighborhood of the origin [1]. Hence, we minimize $\|\underline{b}\|_1$, since it corresponds to computing the smallest (in an inclusion sense) RPI set represented by fixed hyperplanes $\underline{P}$ [4, Corollary 1]. Moreover, we know from [1, Proposition 2] that a large terminal set $\mathcal{X}_{\mathrm{t}}$ maximizes the region of attraction Ω_N. Hence, we maximize the size of $\mathcal{X}_{\mathrm{t}}$ by minimizing a distance metric between $\mathcal{X}_{\mathrm{t}}$ and the state constraint set $\mathcal{X}$ as follows: let $\mathcal{B}(\bar{\epsilon}) := \mathcal{P}(\bar{E}, \bar{\epsilon}) \subset \mathbb{R}^{n_x}$ with $\bar{\epsilon} \in \mathbb{R}_+^{m_\epsilon}$ and $\bar{E}$ fixed a priori; then, we minimize $\|\bar{\epsilon}\|_1$ subject to the inclusion $\mathcal{X} \subseteq \mathcal{P}(\bar{P}, \bar{b}) \oplus \mathcal{B}(\bar{\epsilon})$. Assuming to know the vertices $\{x_{[i]}, i \in \mathbb{I}_1^{m_x^v}\}$ of $\mathcal{X}$, $(\bar{b}, \bar{\epsilon}) \in \bar{\mathcal{S}}$ implies $\mathcal{X} \subseteq \mathcal{P}(\bar{P}, \bar{b}) \oplus \mathcal{B}(\bar{\epsilon})$ where $\bar{\mathcal{S}} := \{(\bar{b}, \bar{\epsilon}) :\ x_{[i]} \in \mathcal{P}(\bar{P}, \bar{b}) \oplus \mathcal{P}(\bar{E}, \bar{\epsilon}), \forall\ i \in \mathbb{I}_1^{m_x^v}\}$.

Finally, since the performance matrices $(H_{\mathrm{Q}}, P_{\mathrm{Q}})$ formulating the RMPC controller in (3) are fixed by (A,B,K) as noted in Remark 1, we introduce a way to tune the closed-loop performance: We evaluate the performance using the system $\hat{x}(t+1) = A\hat{x}(t) + B\hat{u}(t)$ inside the terminal set as

$$\hat{x}(0) \in \mathcal{X}_{\mathrm{t}},\ \hat{u}(t) = K\hat{x}(t),\ \sum_{t=0}^{\infty} \|\hat{x}(t)\|_{\tilde{Q}}^2 + \|\hat{u}(t)\|_{\tilde{R}}^2 \leq \tilde{r}, \tag{12}$$

where $\tilde{Q} \in \mathbb{S}_+^{n_x}$ and $\tilde{R} \in \mathbb{S}_+^{n_u}$ are user-defined performance matrices, and we minimize $\tilde{r}$. Then, if $\tilde{\Theta} \in \mathbb{S}_+^{n_x}$ satisfies

$$(A+BK)^\top \tilde{\Theta}(A+BK) - \tilde{\Theta} + \tilde{Q} + K^\top \tilde{R} K \prec 0, \tag{13}$$

the left-hand-side of the inequality in (12) is upper bounded by $\|\hat{x}(0)\|_{\tilde{\Theta}}^2$ [10]. Hence, (12) is satisfied if the inclusion $\mathcal{P}(\bar{P},\bar{b}) \subseteq \mathcal{E}(\tilde{\Theta},\tilde{r})$ holds, thus imposing an upper bound on the size of the terminal set. Following the S-procedure [26, Section 2.6.3], the inclusion $\mathcal{P}(\bar{P},\bar{b}) \subseteq \mathcal{E}(\tilde{\Theta},\tilde{r})$ holds if

$$\exists \tilde{M} \in \mathbb{D}_{+}^{\bar{m}} \quad \text{s.t.} \quad \bar{P}^\top \tilde{M} \bar{P} - \tilde{\Theta} \succ 0, \quad \tilde{r} - \bar{b}^\top \tilde{M} \bar{b} > 0. \tag{14}$$

Based on these considerations, we formulate the identification problem as the following NLPMI

$$\begin{aligned} \min_{Z_{\mathrm{NL}}} \ & \alpha \|\underline{b}\|_1 + \beta \|\bar{\epsilon}\|_1 + \gamma \tilde{r} \\ \text{s.t. } & (A,B,d) \in \hat{\Sigma}_T, \ (\bar{b},\bar{\epsilon}) \in \bar{\mathcal{S}}, \ (8)-(11),(13)-(14) \end{aligned} \tag{15}$$

where $\alpha, \beta, \gamma \geq 0$ are user-defined weights, and

$$Z_{\mathrm{NL}} := \begin{pmatrix} A, B, d, \mathcal{Z}, \lambda, K, \underline{b}, \bar{b}, \tilde{\Theta}, \tilde{r}, \tilde{M}, \bar{\epsilon}, \{\underline{D}_{[i]}, W_{[i]}, i \in \mathbb{I}_1^m\}, \\ \{\bar{D}_{[i]}, i \in \mathbb{I}_1^{\bar{m}}\}, \{\underline{S}_{[i]}, \bar{S}_{[i]}, i \in \mathbb{I}_1^{m_x}\}, \{\underline{R}_{[i]}, \bar{R}_{[i]}, i \in \mathbb{I}_1^{m_u}\} \end{pmatrix}.$$

4) ***Feasible SCP for Problem*** (15): In order to solve Problem (15), a standard SCP approach can be adopted, in which a sequence of SDPs approximating (15) are solved. However, to guarantee feasibility of the iterates, we adopt the following SCP procedure. Starting from an initial feasible iterate Z_{NL}, we solve a sequence of SDPs formulated using sufficient LMI conditions for the constraints of Problem (15), such that the method produces feasible iterates. The sufficient LMI conditions are formulated using convex underestimates [27] of the NLMI constraints at the current iterate. Moreover, the objective value of (15) is non-increasing over the iterates, such that globalization is unnecessary and we terminate when the objective value does not reduce further.

(a) **Convex SDP approximation**: Given a feasible iterate Z_{NL} for Problem (15), we formulate sufficient LMI conditions for (8)-(11), (13), (14) using the following result.

Proposition 1: [9, Lemma 2.1] Let matrices $\boldsymbol{L}, L \in \mathbb{R}^{m \times n}$ and $\boldsymbol{D}, D \in \mathbb{S}_{+}^{m}$, and define the matrix functions $\mathcal{L}_{\boldsymbol{L},\boldsymbol{D}}^{L,D} := \boldsymbol{L}^\top D^{-1} L + L^\top D^{-1} \boldsymbol{L} - L^\top D^{-1} \boldsymbol{D} D^{-1} L$, and $\mathcal{N}_{\boldsymbol{L},\boldsymbol{D}} := \boldsymbol{L}^\top \boldsymbol{D}^{-1} \boldsymbol{L}$. Then, $\mathcal{N}_{\mathbf{L},\mathbf{D}} \succeq \mathcal{L}_{\mathbf{L},\mathbf{D}}^{L,D}$ and $\mathcal{N}_{L,D} = \mathcal{L}_{L,D}^{L,D}$. Hence, if $\exists (L,D)$ such that $\mathcal{N}_{L,D} \succ 0$, then $\exists (\boldsymbol{L},\boldsymbol{D})$ such that $\mathcal{N}_{\boldsymbol{L},\boldsymbol{D}} \succeq \mathcal{L}_{\boldsymbol{L},\boldsymbol{D}}^{L,D} \succ 0$. □

This result implies that if $\mathcal{N}_{L,D} \succ 0$, then the LMI $\mathcal{L}_{\boldsymbol{L},\boldsymbol{D}}^{L,D} \succ 0$ is a convex underestimate and a sufficient condition for $\mathcal{N}_{\boldsymbol{L},\boldsymbol{D}} \succ 0$. We will now use this property to formulate sufficient LMIs for (8)-(11), (13), (14). The claimed SCP feasibility and cost decrease are then obtained as a corollary.

Theorem 3: Suppose that Z_{NL} is feasible for (15). Then:

(i) *RPI condition* (8): For each $i \in \mathbb{I}_1^m$, there exists $(\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{d}, \boldsymbol{K}, \underline{\boldsymbol{b}}, \hat{\underline{\boldsymbol{D}}}_{[i]}, \hat{\boldsymbol{W}}_{[i]})$ satisfying the LMI

$$\begin{bmatrix} \mathbf{I} & \mathbf{0} & \mathbf{0} & -\boldsymbol{B}^\top \underline{P}_i^\top & \mathbf{0} & \boldsymbol{K} \\ * & \hat{\underline{\boldsymbol{D}}}_{[i]} & \mathbf{0} & \underline{\boldsymbol{b}} & \mathbf{0} & \mathbf{0} \\ * & * & \hat{\boldsymbol{W}}_{[i]} & \boldsymbol{d} & \mathbf{0} & \mathbf{0} \\ * & * & * & 2\underline{\boldsymbol{b}}_i + \mathcal{L}_{\boldsymbol{B}^\top \underline{P}_i^\top, \mathbf{I}}^{B^\top \underline{P}_i^\top, \mathbf{I}} & \underline{P}_i & \underline{P}_i \boldsymbol{A} \\ * & * & * & * & \mathcal{L}_{F, \hat{\boldsymbol{W}}_{[i]}}^{F, W_{[i]}^{-1}} & \mathbf{0} \\ * & * & * & * & * & \mathcal{L}_{\underline{P} \hat{\underline{\boldsymbol{D}}}_{[i]}}^{\underline{P}, \underline{D}_{[i]}^{-1}} + \mathcal{L}_{\boldsymbol{K},\mathbf{I}}^{K,\mathbf{I}} \end{bmatrix} \succ 0, \tag{16}$$

and $(\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{d}, \boldsymbol{K}, \underline{\boldsymbol{b}}, \hat{\underline{\boldsymbol{D}}}_{[i]}^{-1}, \hat{\boldsymbol{W}}_{[i]}^{-1})$ satisfies (8).

(ii) *PI condition* (9): The following LMI is satisfied by some $(\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{K}, \bar{\boldsymbol{b}}, \hat{\bar{\boldsymbol{D}}}_{[i]})$ for each $i \in \mathbb{I}_1^{\bar{m}}$:

$$\begin{bmatrix} \mathbf{I} & \mathbf{0} & -\boldsymbol{B}^\top \bar{P}_i^\top & \boldsymbol{K} \\ * & \hat{\bar{\boldsymbol{D}}}_{[i]} & \bar{\boldsymbol{b}} & \mathbf{0} \\ * & * & 2\bar{\boldsymbol{b}}_i + \mathcal{L}_{\boldsymbol{B}^\top \bar{P}_i^\top, \mathbf{I}}^{B^\top \bar{P}_i^\top, \mathbf{I}} & \bar{P}_i \boldsymbol{A} \\ * & * & * & \mathcal{L}_{\bar{P}, \hat{\bar{\boldsymbol{D}}}_{[i]}}^{\bar{P}, \bar{D}_{[i]}^{-1}} + \mathcal{L}_{\boldsymbol{K},\mathbf{I}}^{K,\mathbf{I}} \end{bmatrix} \succ 0, \tag{17}$$

and $(\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{K}, \bar{\boldsymbol{b}}, \hat{\bar{\boldsymbol{D}}}_{[i]}^{-1})$ satisfies (9).

(iii) *Constraint inclusions* (10), (11): For each $i \in \mathbb{I}_1^{m_x}$, there exists $(\underline{\boldsymbol{b}}, \bar{\boldsymbol{b}}, \hat{\underline{\boldsymbol{S}}}_{[i]}, \hat{\bar{\boldsymbol{S}}}_{[i]})$, and for each $i \in \mathbb{I}_1^{m_u}$, there exists $(\boldsymbol{K}, \underline{\boldsymbol{b}}, \bar{\boldsymbol{b}}, \hat{\underline{\boldsymbol{R}}}_{[i]}, \hat{\bar{\boldsymbol{R}}}_{[i]})$ satisfying the LMIs

$$\begin{bmatrix} \hat{\underline{\boldsymbol{S}}}_{[i]} & \mathbf{0} & \underline{\boldsymbol{b}} & \mathbf{0} & \mathbf{0} \\ * & \hat{\bar{\boldsymbol{S}}}_{[i]} & \bar{\boldsymbol{b}} & \mathbf{0} & \mathbf{0} \\ * & * & 2v_i^x & V_i^x & V_i^x \\ * & * & * & \mathcal{L}_{\underline{P}, \hat{\underline{\boldsymbol{S}}}_{[i]}}^{\underline{P}, \underline{S}_{[i]}^{-1}} & \mathbf{0} \\ * & * & * & * & \mathcal{L}_{\bar{P}, \hat{\bar{\boldsymbol{S}}}_{[i]}}^{\bar{P}, \bar{S}_{[i]}^{-1}} \end{bmatrix} \succ 0, \tag{18}$$

$$\begin{bmatrix} \hat{\underline{\boldsymbol{R}}}_{[i]} & \mathbf{0} & \underline{\boldsymbol{b}} & \mathbf{0} & \mathbf{0} \\ * & \hat{\bar{\boldsymbol{R}}}_{[i]} & \bar{\boldsymbol{b}} & \mathbf{0} & \mathbf{0} \\ * & * & 2v_i^u & V_i^u \boldsymbol{K} & V_i^u \boldsymbol{K} \\ * & * & * & \mathcal{L}_{\underline{P}, \hat{\underline{\boldsymbol{R}}}_{[i]}}^{\underline{P}, \underline{R}_{[i]}^{-1}} & \mathbf{0} \\ * & * & * & * & \mathcal{L}_{\bar{P}, \hat{\bar{\boldsymbol{R}}}_{[i]}}^{\bar{P}, \bar{R}_{[i]}^{-1}} \end{bmatrix} \succ 0, \tag{19}$$

and $(\underline{\boldsymbol{b}}, \bar{\boldsymbol{b}}, \hat{\underline{\boldsymbol{S}}}_{[i]}^{-1}, \hat{\bar{\boldsymbol{S}}}_{[i]}^{-1})$, $(\boldsymbol{K}, \underline{\boldsymbol{b}}, \bar{\boldsymbol{b}}, \hat{\underline{\boldsymbol{R}}}_{[i]}^{-1}, \hat{\bar{\boldsymbol{R}}}_{[i]}^{-1})$ satisfy (10), (11).

(iv) *Dissipativity condition* (13): There exists $(\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{K}, \tilde{\boldsymbol{\Theta}})$ satisfying the LMI

$$\begin{bmatrix} \mathbf{I} & \mathbf{0} & \mathbf{0} & -\boldsymbol{B}^\top & \boldsymbol{K} \\ * & \tilde{Q}^{-1} & \mathbf{0} & \mathbf{0} & \mathbf{I} \\ * & * & \tilde{R}^{-1} & \mathbf{0} & \boldsymbol{K} \\ * & * & * & \mathcal{L}_{\mathbf{I},\tilde{\boldsymbol{\Theta}}}^{\mathbf{I},\tilde{\Theta}} + \mathcal{L}_{\boldsymbol{B}^\top,\mathbf{I}}^{B^\top,\mathbf{I}} & \boldsymbol{A} \\ * & * & * & * & \tilde{\boldsymbol{\Theta}} + \mathcal{L}_{\boldsymbol{K},\mathbf{I}}^{K,\mathbf{I}} \end{bmatrix} \succ 0, \tag{20}$$

and $(\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{K}, \tilde{\boldsymbol{\Theta}})$ satisfies (13).

(v) *Performance ellipsoid inclusion condition* (14): There exists $(\bar{\boldsymbol{b}}, \tilde{\boldsymbol{\Theta}}, \tilde{\boldsymbol{r}}, \hat{\tilde{\boldsymbol{M}}})$ satisfying the LMI

$$\mathcal{L}_{\bar{P}, \hat{\tilde{\boldsymbol{M}}}}^{\bar{P}, \tilde{M}^{-1}} - \tilde{\boldsymbol{\Theta}} \succ 0, \quad \begin{bmatrix} \hat{\tilde{\boldsymbol{M}}} & \bar{\boldsymbol{b}} \\ * & \tilde{\boldsymbol{r}} \end{bmatrix} \succ 0, \tag{21}$$

and $(\bar{\boldsymbol{b}}, \tilde{\boldsymbol{\Theta}}, \tilde{\boldsymbol{r}}, \hat{\tilde{\boldsymbol{M}}}^{-1})$ satisfies (14).

Proof: The proof follows by using the Schur complement and Proposition 1 on (8)-(11). We detail the proof of Part (i), since Parts (ii)-(v) follow with similar arguments. Part (i) : As $(A, B, K, \underline{b}, d, \underline{D}_{[i]}, W_{[i]})$ in Z_{NL} satisfy (8), we take a Schur complement of the $(1,1)$ block to obtain

$$\begin{bmatrix} \underline{D}_{[i]}^{-1} & \mathbf{0} & \underline{b} & \mathbf{0} & \mathbf{0} \\ * & W_{[i]}^{-1} & d & \mathbf{0} & \mathbf{0} \\ * & * & 2\underline{b}_i & \underline{P}_i & \underline{P}_i A + \underline{P}_i B K \\ * & * & * & F^\top W_{[i]} F & \mathbf{0} \\ * & * & * & * & \underline{P}^\top \underline{D}_{[i]} \underline{P} \end{bmatrix} \succ 0. \tag{22}$$

Defining $\hat{W}_{[i]} := W_{[i]}^{-1}$ and $\hat{\underline{D}}_{[i]} := \underline{D}_{[i]}^{-1}$, Eq. (22) is nonlinear in variables $(B, K, \hat{\underline{D}}_{[i]}, \hat{W}_{[i]})$ in the blocks $(4,4)$,

$(5,5)$, $(3,5)$ and $(5,3)$. Then, we write (22) as

$$\begin{bmatrix} \hat{\underline{D}}_{[i]} & \mathbf{0} & \underline{b} & \mathbf{0} & \mathbf{0} \\ * & \hat{W}_{[i]} & d & \mathbf{0} & \mathbf{0} \\ * & * & \mathcal{N}_{i,33} & \underline{P}_i & \underline{P}_i A \\ * & * & * & \mathcal{N}_{F,\hat{W}_{[i]}} & \mathbf{0} \\ * & * & * & * & \mathcal{N}_{i,55} \end{bmatrix} - \mathcal{K}_i^\top \mathcal{K}_i \succ 0, \tag{23}$$

where $\mathcal{N}_{i,33} := 2\underline{b}_i + \mathcal{N}_{B^\top \underline{P}_i^\top, \mathbf{I}}$, $\mathcal{N}_{i,55} := \mathcal{N}_{\underline{P}, \hat{\underline{D}}_{[i]}} + \mathcal{N}_{K,\mathbf{I}}$, and $\mathcal{K}_i := [\mathbf{0} \;\; \mathbf{0} \;\; -B^\top \underline{P}_i^\top \;\; \mathbf{0} \;\; K]$ (with the function $\mathcal{N}_{\cdot,\cdot}$ defined in Proposition 1). Taking Schur complement of (23),

$$\begin{bmatrix} \mathbf{I} & \mathbf{0} & \mathbf{0} & -B^\top \underline{P}_i^\top & \mathbf{0} & K \\ * & \hat{\underline{D}}_{[i]} & \mathbf{0} & \underline{b} & \mathbf{0} & \mathbf{0} \\ * & * & \hat{W}_{[i]} & d & \mathbf{0} & \mathbf{0} \\ * & * & * & 2\underline{b}_i + \mathcal{N}_{B^\top \underline{P}_i^\top, \mathbf{I}} & \underline{P}_i & \underline{P}_i A \\ * & * & * & * & \mathcal{N}_{F,\hat{W}_{[i]}} & \mathbf{0} \\ * & * & * & * & * & \mathcal{N}_{\underline{P}, \hat{\underline{D}}_{[i]}} + \mathcal{N}_{K,\mathbf{I}} \end{bmatrix} \succ 0 \tag{24}$$

results, with all nonlinear components collected in the diagonal blocks. Using Proposition 1 on these components, we conclude that (16) is a sufficient LMI condition for (24). ■

Corollary 1: Suppose that Z_{NL} is feasible for Problem (15). Then, the solution of the SDP

$$\begin{aligned} &\min_{\boldsymbol{Z}} \;\; \alpha \left\|\underline{\boldsymbol{b}}\right\|_1 + \beta \left\|\bar{\boldsymbol{\epsilon}}\right\|_1 + \gamma \tilde{\boldsymbol{r}} \\ &\text{s.t. } (\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{d}) \in \hat{\Sigma}_T, \quad (\bar{\boldsymbol{b}}, \bar{\boldsymbol{\epsilon}}) \in \bar{\mathcal{S}}, \quad (16)-(21), \end{aligned} \tag{25}$$

$$\boldsymbol{Z} := \begin{pmatrix} \boldsymbol{A}, \boldsymbol{B}, \boldsymbol{d}, \boldsymbol{\mathcal{Z}}, \boldsymbol{\lambda}, \boldsymbol{K}, \underline{\boldsymbol{b}}, \bar{\boldsymbol{b}}, \tilde{\boldsymbol{\Theta}}, \tilde{\boldsymbol{r}}, \hat{\tilde{\boldsymbol{M}}}, \bar{\epsilon}, \{\hat{\underline{\boldsymbol{D}}}_{[i]}, \hat{\boldsymbol{W}}_{[i]}, i \in \mathbb{I}_1^m\}, \\ \{\hat{\bar{\boldsymbol{D}}}_{[i]}, i \in \mathbb{I}_1^{\bar{m}}\}, \{\hat{\underline{\boldsymbol{S}}}_{[i]}, \hat{\bar{\boldsymbol{S}}}_{[i]}, i \in \mathbb{I}_1^{m_x}\}, \{\hat{\underline{\boldsymbol{R}}}_{[i]}, \hat{\bar{\boldsymbol{R}}}_{[i]}, i \in \mathbb{I}_1^{m_u}\} \end{pmatrix},$$

is feasible for Problem (15), and satisfies the cost decrease condition $\alpha \left\|\underline{\boldsymbol{b}}\right\|_1 + \beta \left\|\bar{\boldsymbol{\epsilon}}\right\|_1 + \gamma \tilde{\boldsymbol{r}} \le \alpha \left\|\underline{b}\right\|_1 + \beta \left\|\bar{\epsilon}\right\|_1 + \gamma \tilde{r}$. □

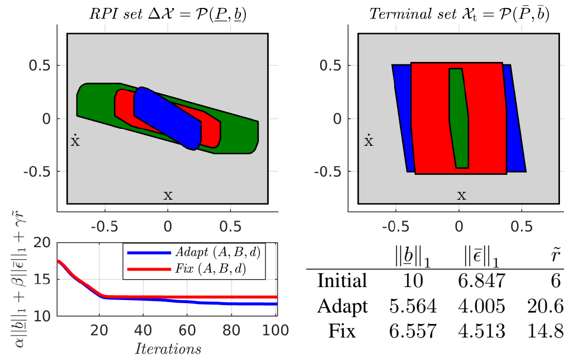
Proof: The feasibility of $\boldsymbol{Z}$ for Problem (15) follows from Theorem 3, and the cost decrease condition holds since Z_{NL} is feasible for Problem (25). ■

We propose the procedure in Algorithm 1 to solve Problem (15). The convergence of this algorithm can be studied using the results in [27, Chapter 4], and is left for future research.

Algorithm 1 Update solution of Problem (15)

1: Obtain an initial feasible solution Z_{NL} for (15).
2: Solve SDP (25) for $\boldsymbol{Z}$, recover Z_{NL} from the solution.
3: Evaluate the objective value $\alpha \left\|\underline{b}\right\|_1 + \beta \left\|\bar{\epsilon}\right\|_1 + \gamma \tilde{r}$.
4: If there is a reduction from previous iteration, repeat Step 2 using Z_{NL} for linearization. Else, terminate.

(b) **Initialization procedure**: We propose the following procedure to compute an initial feasible solution Z_{NL}.
(i) Select some $\hat{\theta}_T > 0$ through a guess to characterize $\hat{\Sigma}_T$.
(ii) Solve the LP $\arg\min_{A,B,d}\{\|d\|_1 \text{ s.t. } (A,B,d) \in \hat{\Sigma}_T\}$ for an initial feasible model (A,B,d).
(iii) Use the method in [9] to compute an initial RPI set $\Delta\mathcal{X} = \mathcal{P}(\underline{P}, \underline{b})$ satisfying (7a) along with a feedback gain K, while enforcing $\mathcal{P}(\underline{P}, \underline{b}) \subset \mathcal{X}$ and $K\mathcal{P}(\underline{P}, \underline{b}) \subset \mathcal{U}$.
(iv) Compute the tightened constraint set $\mathcal{O}_0 := \{x : x \in \mathcal{X} \ominus \Delta\mathcal{X}, Kx \in \mathcal{U} \ominus K\Delta\mathcal{X}\}$, and then compute a PI set $\mathcal{X}_{\mathrm{t}} = \mathcal{P}(\bar{P}, \bar{b})$ using the method in [9] for $x(t+1) = (A+BK)x(t)$.



	$\|\underline{b}\|_1$	$\|\bar{\epsilon}\|_1$	$\tilde{r}$
Initial	10	6.847	6
Adapt	5.564	4.005	20.613
Fix	6.557	4.513	14.865

Fig. 1: Results of Algorithm 1. Gray sets-$\mathcal{X}$, Green sets-Initialization, Blue sets-(A,B,d) as optimization variables, Red sets - Fix (A,B,d) to initial values.

(v) Compute the remaining variables formulating Problem (15) by solving $\min_{Z_{\mathrm{I}}}\{\tilde{r} : \text{(8)–(11), (13)–(14)}\}$, where

$$Z_{\mathrm{I}} := \begin{pmatrix} \{\underline{D}_{[i]}, W_{[i]}, i \in \mathbb{I}_1^m\}, \{\bar{D}_{[i]}, i \in \mathbb{I}_1^{\bar{m}}\}, \{\underline{S}_{[i]}, \bar{S}_{[i]}, i \in \mathbb{I}_1^{m_x}\}, \\ \{\underline{R}_{[i]}, \bar{R}_{[i]}, i \in \mathbb{I}_1^{m_u}\}, \tilde{\Theta}, \tilde{r}, \tilde{M} \end{pmatrix}.$$

Remark 4: In Steps (ii),(iii), the methods in [9] guarantee the feasibility of the SDP in Step (v), since they are also formulated using Theorem 2. Other methods, e.g. [5], [6], can also be used if the feasibility of Step (v) is ensured. □

IV. Numerical Example

We consider a nonlinear mass-spring-damper system with dynamics $\mathrm{F} = \mathrm{m}\ddot{\mathrm{x}} + (\mathrm{K}\mathrm{x} + \mathrm{K}_{\mathrm{NL}}\mathrm{x}^2) + (\mathrm{c}\dot{\mathrm{x}} + \mathrm{c}_{\mathrm{NL}}\dot{\mathrm{x}}^2) + \mathrm{F}_\delta$, where $u = \mathrm{F}$, $x = [\mathrm{x} \;\; \dot{\mathrm{x}}]^\top$, and constraints $\mathcal{X} = \{x : \|x\|_\infty \le 0.8\}$, $\mathcal{U} = \{u : \|u\|_\infty \le 2.5\}$. We simulate the plant using $\mathrm{ode45}$ integration to build the dataset $\mathcal{D}$ with $T = 1000$ at a $0.1s$ time interval. We set $\mathrm{K}_{\mathrm{NL}}, \mathrm{c}_{\mathrm{NL}} = 0.12$, and uniformly sample the parameters $\mathrm{m}, \mathrm{K}, \mathrm{c}$ in $(0.44, 0.56)$ and F_δ in $(-0.12, 0.12)$ at every timestep $0.1s$. We then use Algorithm 1 to synthesize a model and RPI sets required for RMPC synthesis. To this end, we follow the initialization procedure described in Section III-4(b) to obtain an initial feasible Z_{NL} for Problem (15). We first parametrize the disturbance set $\mathcal{W}$ with $m_w = 10$ hyperplanes. Then, following Step (i), we characterize the set $\hat{\Sigma}_T$ with $\hat{\theta}_T = 1 \cdot 10^{-3}$. Then, we compute the initial model $A = \begin{bmatrix} 0.9967 & 0.0951 \\ -0.0637 & 0.9036 \end{bmatrix}$, $B = \begin{bmatrix} 0.0098 \\ 0.1914 \end{bmatrix}$ and $\|d\|_1 = 0.5816$ following Step (ii). We initialize the feedback gain as $K = [-0.4140 \;\; -2.3734]$ which is the optimal LQR gain corresponding to matrices $\tilde{Q} = \mathrm{diag}(1, 15)$ and $\tilde{R} = 1$. Then, we compute an RPI set $\Delta\mathcal{X} = (\underline{P}, \underline{b})$ following Step (iii) with $\underline{m} = 10$ hyperplanes using [5]. Similarly, we compute the PI terminal set $\mathcal{X}_{\mathrm{t}} = \mathcal{P}(\bar{P}, \bar{b})$ following Step (iv) with $\bar{m} = 15$ hyperplanes using [6]. Finally, with Step (v) we compute the remaining variables formulating Z_{NL}. We parameterize $\mathcal{B}(\bar{\epsilon})$ with $m_\epsilon = 10$ for terminal set maximization. The results obtained with Algorithm 1 with weights $\alpha = 1, \beta = 1, \gamma = 0.1$ using the MOSEK SDP solver [28] in MATLAB are shown in

Figure 1. For the purpose of comparison, we also plot the results when the model (A, B, d) is fixed to the initial value. We observe that by allowing Algorithm 1 to adapt the system model using $\hat{\Sigma}_T$, we obtain a lower objective value with a larger terminal set $\mathcal{X}_{\rm t}$ and a smaller RPI set $\Delta\mathcal{X}$. The model at termination is $A = \begin{bmatrix} 0.9967 & 0.0951 \\ -0.0625 & 0.8990 \end{bmatrix}$, $B = \begin{bmatrix} 0.0098 \\ 0.1958 \end{bmatrix}$, and $\|d\|_1 = 0.5833$, and the computed feedback gain is $K = [-1.7062 \ -2.6306]$: the model consists of a larger disturbance set than the initialized value, with (A, B, d) optimizing (15) instead of best fitting the data. In case the model is fixed to the initial value, the feedback gain at termination is $K = [-1.0033 \ -3.0882]$. In order to study the effect of the parameter $\hat{\theta}_T$ characterizing $\hat{\Sigma}_T$, we run Algorithm 1 for increasing values of $\hat{\theta}_T$. The objective values at termination are 11.631 for $\hat{\theta}_T = 1 \cdot 10^{-3}$, 11.659 for $\hat{\theta}_T = 1.2 \cdot 10^{-3}$, 11.668 for $\hat{\theta}_T = 1.3 \cdot 10^{-3}$, 13.4376 for $\hat{\theta}_T = 1.5 \cdot 10^{-3}$: we observe that conservativeness increases with $\hat{\theta}_T$, while increasing robustness with respect to the underlying plant. Note that this trend is not guaranteed since Problem (15) is an NLPMI.

Computational Complexity: The SDP in (25) consists of an LMI constraint with $\underline{m}(2n_x + m_w + n_u + \underline{m} + 1) + \bar{m}(n_x + n_u + \bar{m} + 1) + (m_x + m_u)(2n_x + \underline{m} + \bar{m} + 1) + (3n_x + 2n_u) + (n_x + \bar{m} + 1) + n_x = 1086$ rows, $n_x m_x^v = 8$ linear equality constraints, and $2m_w T + 2m_w(2n_x + n_u) + (2n_x + n_u) + m_w + 2m_x^v(\bar{m} + m_\epsilon) = 20275$ linear inequality constraints over $2(n_x^2 + n_x n_u + n_x m_x^v + 1) + m_w(2n_x + n_u + 1) + \underline{m}(\underline{m} + m_w + 1 + m_x + m_u) + \bar{m}(\bar{m} + 2 + m_x + m_u) + m_\epsilon = 638$ variables. Over multiple runs, the average solving time for the SDP in (25) on a laptop with an Intel i7-7500U processor and 16GB of RAM running Ubuntu 16.04 is approximately 1.57s when the model is allowed to adapt, and 0.78s when the model is fixed. We note that the number of LMI constraints and variables scale quadratically in $\bar{m}$ and $\underline{m}$. Hence, the approach can be computationally expensive if a large number of hyperplanes are required for RPI set representation. Comparing our approach to [16] using data from a real-world system is a subject of future work.

V. Conclusions

This paper has presented a data-driven method based on RPI sets to synthesize RMPC controllers. To this end, a set of LTI models that can describe the underlying plant behavior is characterized using an input-state dataset. Then, a suitable model along with RPI sets are concurrently computed for RMPC synthesis. This procedure is demonstrated to compute RPI sets with reduced conservativeness when compared to a sequential procedure. Future research will be devoted to estimation techniques for $\hat{\theta}_T$, reducing conservativeness in Theorem 1, using input-output datasets, multiplicative uncertainty models, and combining our approach with [19].

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