Optimal Control of Discrete Hybrid Stochastic Automata

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Abstract. This paper focuses on hybrid systems whose discrete state transitions depend on both deterministic and stochastic events. For such systems, after introducing a suitable hybrid model called Discrete Hybrid Stochastic Automaton (DHSA), different finite-time optimal control approaches are examined: (1) Stochastic Hybrid Optimal Control (SHOC), that "optimistically" determines the trajectory providing the best trade off between the tracking performance and the probability that stochastic events realize as expected, under specified chance constraints; (2) Robust Hybrid Optimal Control (RHOC) that, in addition, less optimistically, ensures that the system remains within a specified safety region for all possible realizations of stochastic events. Sufficient conditions for the asymptotic convergence of the state vector are given for receding-horizon implementations of the above schemes. The proposed approaches are exemplified on a simple benchmark problem in production system management.

1 Introduction

Hybrid systems were proved to be a powerful framework for the analysis and synthesis of embedded systems, as they provide a model in which continuous and discrete behaviors coexist [1]. Several mathematical models were proposed in the last years for *deterministic* hybrid systems, for the analysis of their structural properties, and for controller synthesis. However, there are relatively few studies regarding *stochastic* hybrid systems, except the remarkable results presented in [2, 3] regarding modeling aspects, the ones in [4, 5, 6] regarding structural properties, and important results in applications, such as air traffic control [7], manufacturing systems [8], and communication networks [9].

In this paper we introduce a discrete-time model and suitable control algorithms based on optimization techniques for a class of stochastic hybrid systems, denoted as Discrete Hybrid Stochastic Automata (DHSA), in which the uncertainty appears on the discrete component of the hybrid dynamics, in the form of stochastic events that, together with their deterministic counterparts, determine the transition of the discrete state. As a consequence, mode switches of the continuous dynamics become non-deterministic and uncertainty propagates also to the continuous states.

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Unpredictable behaviors such as delays or faults in digital components and discrete approximations of continuous input disturbances can be both modeled by DHSA. The main advantage of DHSA is that the number of possible values that the overall system state can have at each time instant is finite (although it may be large), so that the problem of controlling DHSA can be conveniently treated by numerical optimization.

The paper is organized as follows. Section 2 is concerned with modeling aspects. In Section 3 we present a control approach that uses stochastic information about the uncertainty to obtain an optimal trajectory whose probability of realization is known and in Section 4 we extend the approach to ensure also robust safety properties. Finally, after presenting an application example in Section 5, in Section 6 we provide sufficient conditions for the asymptotic convergence of the state in case of receding-horizon implementations of the proposed optimal control schemes.

2 Discrete Hybrid Stochastic Automaton

A model for deterministic hybrid systems called Discrete Hybrid Automaton (DHA) was introduced in [10]. We introduce here the Discrete Hybrid Stochastic Automaton (DHSA), that in addition takes into account possible stochastic discrete state transitions.

2.1 Model Formulation

A DHSA is composed by four components: a Switched Affine System (SAS), an Event Generator (EG), a stochastic (non-deterministic) Finite State Machine (sFSM) and a Mode Selector (MS). The switched affine system satisfies the equations

$$\begin{aligned} x_r(k+1) &= A_{i(k)}x_r(k) + B_{i(k)}u_r(k) + f_{i(k)}, \\ y_r(k) &= C_{i(k)}x_r(k) + D_{i(k)}u_r(k) + g_{i(k)}, \end{aligned} \tag{1}$$

in which $k \in \mathbb{K} = \{0, 1, \ldots, \}$ is the time index, $i \in \mathcal{I} = \{1, 2, \ldots, s\}$ is the current mode of the system, $x_r \in \mathcal{X}_r \subseteq \mathbb{R}^n$ is the continuous component of the state, $u_r \in \mathcal{U}_r \subseteq \mathbb{R}^m$ is the continuous input vector, $y_r \in \mathcal{Y}_r \subseteq \mathbb{R}^p$ is the output vector and $\{A_i, B_i, f_i, C_i, D_i, g_i\}_{i\in\mathcal{I}}$, are matrices of suitable dimensions. The EG produces event signals $\delta_e(k) \in \{0, 1\}^{n_e}$, that we consider as the *endogenous* discrete input signals, defined as

$$\delta_e(k) = f_{\mathrm{H}}(x_r(k), u_r(k), k), \tag{2}$$

where $f_{\mathrm{H}} : \mathcal{X}_r \times \mathcal{U}_r \times \mathbb{K} \to \{0, 1\}^{n_e}$ is the event generator function [10]. The mode selector is defined by a discrete function $f_{\mathrm{M}} : \{0, 1\}^{n_b} \times \{0, 1\}^{m_b} \times \{0, 1\}^{n_e} \to \mathcal{I}$

$$i(k) = f_{\mathrm{M}}(x_b(k), u_b(k), \delta_e(k)), \tag{3}$$

where $x_b \in \{0, 1\}^{n_b}$ is the discrete state and $u_b \in \{0, 1\}^{m_b}$ is the discrete *exogenous* input.

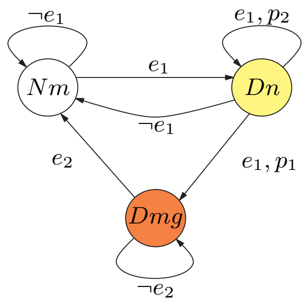

Fig. 1. Stochastic Finite State Machine: 3 states, 2 events and 2 stochastic transitions

The above three building elements are the same as presented in [10] for DHA. The difference between DHA and DHSA[1] is in the element defining the discrete state dynamics: a Finite State Machine (FSM) in DHA, a stochastic FSM (sFSM) in DHSA. While a FSM is defined by the purely discrete difference equation

$$x_b(k+1) = f_{\mathrm{B}}(x_b(k), u_b(k), \delta_e(k)), \tag{4}$$

where $f_{\mathrm{B}} : \{0,1\}^{n_b} \times \{0,1\}^{m_b} \times \{0,1\}^{n_e} \to \{0,1\}^{n_b}$, a sFSM is defined by the probability that the discrete state will take a given value at the next step, given the actual state and inputs:

$$P[x_b(k+1) = \bar{x}_b] = f_{\mathrm{b}}(x_b(k), u_b(k), \delta_e(k), \bar{x}_b), \tag{5}$$

where $f_{\mathrm{b}} : \{0,1\}^{n_b} \times \{0,1\}^{m_b} \times \{0,1\}^{n_e} \times \{0,1\}^{n_b} \to [0,1]$. The information contained in the stochastic finite state machine is the following: Given the state value at step k and the inputs $\delta_e(k), u_b(k)$, the probability that the next discrete state takes a certain value is known. An example of sFSM is reported in Figure 1.

Definition 1. *Given a binary state $x_b(k) = \bar{x}_b$, an exogenous binary input $u_b(k) = \bar{u}_b$, an endogenous vector of events $\delta_e(k) = \bar{\delta}_e$, we say that a discrete transition $\bar{x}_b \to \hat{x}_b$ to the successor state $x_b(k+1) = \hat{x}_b$ is* enabled *for $(\bar{x}_b, \bar{u}_b, \bar{\delta}_e)$, if the probability $P_{\bar{x}_b \to \hat{x}_b} = f_b(\bar{x}_b, \bar{u}_b, \bar{\delta}_e, \hat{x}_b) > 0$. An enabled transition is said* stochastic *if $P_{\bar{x}_b \to \hat{x}_b} < 1$.*

Definition 2. *Given a triple $(\bar{x}_b, \bar{u}_b, \bar{\delta}_e)$, two or more enabled transitions are called* conflicting *on $(\bar{x}_b, \bar{u}_b, \bar{\delta}_e)$.*

A more formal definition of conflicting transitions is given in [11], we just note here that for a correctly defined sFSM the sum of the probabilities of conflicting transitions at every given $(\bar{x}_b, \bar{u}_b, \bar{\delta}_e)$ must be 1.

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[1] The resets maps introduced in [10] can be straightforwardly included also in DHSA, so they are not explicitly considered in this paper.

2.2 Computational Model

The DHSA formulation (1), (2), (3), (5) is good for modeling stochastic discrete effects (such as stochastic delays, failures, unpredictable or external decisions), but not conveniently exploitable for control design, as we will more clearly justify in the beginning of Section 3. For this reason, we need rephrase the DHSA into an equivalent model that is easier to manage in computations.

The key idea is that a sFSM having stochastic conflicting transitions can be equivalently represented by a deterministic FSM having additional exogenous random binary inputs w_1 ,w_2, ..., w_l, that we call *uncontrollable events*, where if $w_i = 1$ the corresponding stochastic transition, if enabled, is taken. Given a system with l stochastic transitions, we denote by $W \subseteq \{0,1\}^l$ the set of vectors $w = [w_1(k)\ \dots\ w_l(k)]^T$ that satisfy the conditions

$$\begin{array}{l}\left[(x_b = \bar{x}_b) \wedge (u_b = \bar{u}_b) \wedge (\delta_e = \bar{\delta}_e)\right] \rightarrow \left[\sum_{i \in I(\bar{x}_b, \bar{u}_b, \bar{\delta}_e)} w_i = 1\right], \\ \forall (\bar{x}_b, \bar{u}_b, \bar{\delta}_e) \in \{0,1\}^{n_b} \times \{0,1\}^{m_b} \times \{0,1\}^{n_e} : |I(\bar{x}_b, \bar{u}_b, \bar{\delta}_e)| > 1\end{array} \tag{6}$$

where $I(\bar{x}_b, \bar{u}_b, \bar{\delta}_e) \subseteq \{1, \dots, l\}$ is the subset of indices of the uncontrollable events associated with the conflicting transitions on $(\bar{x}_b, \bar{u}_b, \bar{\delta}_e)$ and $|\cdot|$ denotes cardinality.

As an example, the sFSM represented in Figure 1 can be associated with a FSM having additional uncontrollable events $w_1, w_2 \in \{0,1\}$ that affect the stochastic transitions: transition $Dn \rightarrow Dn$ happens when $e_1 \wedge w_2$[2] is true, while transition $Dn \rightarrow Dmg$ when $e_1 \wedge w_1$ is true, w_1 and w_2 are mutually exclusive, and $\mathbf{P}[w_1 = 1] = p_1$ and $\mathbf{P}[w_2 = 1] = p_2$. More generally, a sFSM having l stochastic transitions can be transformed into a deterministic automaton, denoted as uncontrollable-events FSM (ueFSM), defined by the state-update function:

$$x_b(k+1) = f_{\mathrm{B}}(x_b(k), u_b(k), \delta_e(k), w(k)), \tag{7}$$

where $w(k) = [w_1(k)\ \dots\ w_l(k)]^T \in W$ is the random vector of uncontrollable events at time k and $f_{\mathrm{B}} : \{0,1\}^{n_b} \times \{0,1\}^{m_b} \times \{0,1\}^{n_e} \times W \rightarrow \{0,1\}^{n_b}$.

An uncontrollable-events Discrete Hybrid Automaton (ueDHA) is obtained from a DHSA by substituting the sFSM with its corresponding ueFSM (7), leaving the switched affine system, the mode selector and the event generator unchanged.

An ueDHA obtained from a DHSA is equivalent to the DHSA itself when the additional exogenous variables w are produced by a random binary number generator under the conditions

$$\mathbf{P}[w_i = 1] = p_i, \quad i = 1, \dots, l, \qquad w \in W, \tag{8}$$

that ensure that the uncontrollable events take value 1 with probability equal to the one associated with the corresponding stochastic transition.

[2] "$\wedge$" denote logic "and".

The advantages of transforming a DHSA into the related ueDHA are three:

1. Uncertainty is now associated with binary signals $w(k)$.
2. The ueDHA is an extended DHA model, thus it can be converted into equivalent hybrid models, and in particular into Mixed Logical Dynamical (MLD) systems [12] for solving optimization problems.
3. The probability of a given discrete state trajectory can be obtained as a function of the uncontrollable event vector $\{w(k)\}_{k=0}^{N-1}$, as explained in the following paragraphs.

The uncontrollable events contain the whole information about stochastic transitions, thus, when vectors $\{w(k)\}_{k=0}^{N-1}$ are known, the probability of the state trajectory $\{x(k)\}_{k=0}^{N}$ can be computed once $\{u(k)\}_{k=0}^{N-1}$ and $x(0)$ are also given. Consider a system with l uncontrollable events and let $w(k) = [w_1(k) \dots w_l(k)]^T$ be the uncontrollable event vector at step k. Consider an additional $w_{l+1}(k)$ taking value 1 when the transition taken by the DHSA at step k is deterministic and extend conditions (6) with this. Consider the vector $p = [p_1 \dots p_l \, 1]^T$ containing the probability coefficients of the stochastic transitions. Then, consider the products

$$\begin{bmatrix} \pi(0) \\ \vdots \\ \pi(N-1) \end{bmatrix} = \begin{bmatrix} w^T(0) \\ \vdots \\ w^T(N-1) \end{bmatrix} \cdot p, \quad \pi = \pi(w(0), \dots, w(N-1)) = \prod_{k=0}^{N-1} \pi(k). \tag{9}$$

The coefficient $\pi(k)$ contains the probability of transition at step k, π the probability of the complete trajectory. In this way it is possible to know the probability to have a certain trajectory given the initial condition and input signals.

Finally we mention that the well posedness of a DHSA is ensured if its related ueDHA is a well posed DHA [10], if conditions (6) hold, and if the probability coefficients of stochastic transitions are correctly defined as proven in [11], where it is also shown the existing relations between DHSA, Markov Chains and Piecewise Deterministic Markov Processes [2, 13].

Thanks to the uncontrollable events, the whole statistical information about transitions is removed from the system structure and associated to the stochastic properties of the binary signals. In the following sections we will show how the ueDHA can be used to formulate optimization problems that consider the information regarding trajectory probability in the objective function and in the constraints.

3 Stochastic Hybrid Optimal Control

In [11] we showed that it is not possible to obtain average state optimal control of DHSA by exploiting similarities with Markov Chains average state optimal control [14], as some of the control signals of the discrete dynamics are not exogenous and they depend on the continuous dynamics. The only way to optimally control the average state is to use a “scenario enumeration” approach [15], which however generates a numerically intractable problem as the optimal control horizon N gets large. In this paper we take a different approach and consider the problem of choosing the input profile that optimizes the most favorable situation, under penalties and hard constraints on the probability of the disturbance realization that determines such a situation. Given a DHSA, by exploiting the equivalent ueDHA and the probability computed in (2), we can formulate such an optimal control problem as an MIP.

3.1 Problem Setup

Consider the convex *performance index*

$$\mathcal{C}_d = \sum_{k=0}^{N-1} \ell_k(x(k+1) - r_x(k+1), u(k) - r_u(k)), \tag{10}$$

which is a function of $x(k)$, $u(k)$, $k = 0, \ldots, N-1$. Typically $\ell_k(x,u) = \|Q(x - r_x)\|_\infty + \|R(u - r_u)\|_\infty$ where Q, R full rank or $\ell_k(x,u) = (x - r_x)^T Q(x - r_x) + (u - r_u)^T R(u - r_u)$ where $Q \geq 0$, $R > 0$, in which r_x and r_u are given references on the state and on the input, respectively.

Next, consider the *probability cost*

$$\mathcal{C}_p = \ln \frac{1}{\pi(w(0), \ldots, w(N-1))} = -\ln\left(\pi(w(0), \ldots, w(N-1))\right), \tag{11}$$

which is a function of $w(k)$, $k = 0, \ldots, N-1$. The smaller is the probability of a trajectory, the larger is the probability cost, so that the trajectories that realize rarely are penalized. The most desirable situation is to obtain a trajectory with good performance and high probability. For this reason, we define as the *objective function* the cost

$$\mathcal{C} = \mathcal{C}_d + q_p \mathcal{C}_p\,, \tag{12}$$

in which $q_p \in (0, +\infty)$ is a trade off coefficient called *probability coefficient.*

In order to hardly eliminate trajectories that realize rarely, we also wish to impose the *chance constraint*

$$\pi(w(0), \ldots, w(N-1)) \geq \tilde{p}\,, \tag{13}$$

where the coefficient $\tilde{p} \in (0, 1]$ is called *probability limit.*

The chance constraint (13) ensures that when the chosen input profile $\{u(k)\}_{k=0}^{N-1}$ is applied to the system, the corresponding trajectory $\{x(k)\}_{k=0}^{N}$ realizes with probability greater or equal to $\tilde{p}$. Other constraints on probabilities may be imposed, such as constraints defining the minimum allowed probability at every single step.

The problem of optimally control a DHSA in respect to the cost function (12), considering (13) as additional constraint is then formulated as:

Problem 1 (Stochastic Hybrid Optimal Control, SHOC).

$$\min_{\{w(k),u(k)\}_{k=0}^{N-1}} \quad \mathcal{C}_d + q_p\mathcal{C}_p \tag{14a}$$

$$\text{s.t.} \quad \text{DHSA dynamics } (1),(2),(3),(7),(6) \tag{14b}$$

$$\text{chance constraint } (13)\,. \tag{14c}$$

3.2 Optimization Problem

In order to cast problem (14) as a mixed-integer linear or quadratic problem, we need to transform (11) and (13) into linear functions of the uncontrollable event values w. The performance index in (10) can be dealt with as described in [16] for deterministic hybrid systems.

Consider a DHSA with l stochastic transitions whose probabilities are collected in vector $p = [p_1 \dots p_l]^T$, and consider the equivalent ueDHA with uncontrollable events $w = [w_1 \dots w_l]^T$[3]. The probability of a trajectory depends only on the transitions, thus it can be computed as a function of the uncontrollable events as $\pi(w(0), \dots, w(N-1)) = \prod_{k=0}^{N-1} \prod_{i=1}^{l} \pi_i(k)$ where $\pi_i(k)$ represents the contribution of the stochastic transition i at step k on the trajectory probability,

$$\pi_i(k) = \begin{cases} 1 & \text{if } w_i(k) = 0 \\ p_i & \text{if } w_i(k) = 1. \end{cases} \tag{15}$$

Equivalently, $\pi_i(k) = 1 + (p_i - 1)\, w_i(k)$, $w_i(k) \in \{0, 1\}$. The probability cost (11) is equal to

$$-\sum_{k=0}^{N-1} \sum_{i=1}^{l} \ln \pi_i(k). \tag{16}$$

With an exp-log transformation, provided that $\pi(w(0), \dots, w(N-1)) > 0$, $\pi(w(0), \dots, w(N-1)) = \exp(\ln \prod_{i,k} \pi_i(k))$, thus $\ln \pi(w(0), \dots, w(N-1)) = \ln \prod_{i,k} \pi_i(k) = \sum_{i,k} \ln \pi_i(k) = \sum_{i,k} \ln(1 + (p_i - 1)\, w_i(k))$. Although this expression is still nonlinear in $w_i(k)$ because of the logarithms, we note that $\ln \pi_i(k) = w_i(k)\, \ln(p_i)$ for $w_i(k) \in \{0, 1\}$. Hence, the logarithm of the trajectory probability $\ln \pi(w(0), \dots, w(N-1)) = \sum_{k=0}^{N-1} \sum_{i=1}^{l} w_i(k)\, \ln(p_i)$, and therefore the probability cost (16) can be expressed as a linear function of the uncontrollable events $w_i(k) \in \{0, 1\}$, so that the chance constraint (13) becomes a linear constraint on $w_i(k) \in \{0, 1\}$.

By converting the ueDHA into MLD form [10], the optimal control problem (14) can be solved by standard mixed integer programming solvers [17].

The solution of (14) is a couple (u^*, w^*), where u^* is the optimal control sequence and w^* is the desired uncontrollable events sequence. Only u^* is actuated, thus the actual trajectory may be different from the expected one. However, if the realization of the stochastic events is equal to w^* the actual trajectory is

[3] This can be extended by considering the fictitious event for deterministic transitions having $p_d = 1$. As explained below its contribution will disappear because $\log p_d = 0$.

equal to the one obtained from (14). The largest q_p is, the most likely the actual w will coincide with w^*, and the most cautious will be the control action.

In [11] it is shown that several DHA can be extracted from a single DHSA by fixing a nominal behavior for the uncertain transitions, that is, by fixing $w = \bar{w} \in W$ in the equivalent ueDHA. The Stochastic Hybrid Optimal Control problem solved on the DHSA will always give a better solution than the optimal control problem formulated on an extracted DHA having $\mathcal{C}_d$ as cost function: the solution of the SHOC has either higher probability or better performance.

4 Robust Hybrid Optimal Control of DHSA

The approach of Section 3 does not ensure that the behavior of the system is correct when the actual w is different from w^*, as some constraints may be violated for particular realizations. Therefore, this approach can be used only if possible deviations from the desired trajectory are not critical. However, considering those situations in which constraint violation is critical, we define another control approach that considers not only the desired trajectory, but also the possible deviations from it, due to unexpected stochastic transitions.

Definition 3. *Given a stochastic system $x(k+1) = f(x(k), u(k), \phi(k))$ in which $\phi(k) \in \Phi$ is a stochastic disturbance, the constraint $h(x(k), u(k), \phi(k)) < 0$ is* robustly satisfied *at time k if $h(x(k), u(k), \phi(k)) < 0$, $\forall \phi(k) \in \Phi$.*

Problem 2 (Robust Hybrid Optimal Control, RHOC).

$$\min_{\{w(k),u(k)\}_{k=0}^{N-1}} \quad \mathcal{C}_d + q_p \mathcal{C}_p \tag{17a}$$

$$\text{s.t.} \quad \text{DHSA dynamics (1), (2), (3), (7), (6)} \tag{17b}$$

$$\text{chance constraint (13)} \tag{17c}$$

$$\text{constraint } h(\cdot) \leq 0 \text{ is robustly satisfied, } \forall k \in [0, N-1]\,. \tag{17d}$$

Compared to Problem 1, Problem 2 (RHOC) also requires that the optimal input u^* is such that a set of constraints $h(\cdot) \leq 0$ is always satisfied for all the admissible values of stochastic events w that may realize.

By exploiting the techniques developed in Section 3 and in [12], problem (17) can be rephrased as:

$$\min_{u,w,\xi} \quad f(u, w, \xi) \tag{18a}$$

$$\text{s.t.} \quad A_u\, u + A_w\, w + A_\xi\, \xi \leq b \tag{18b}$$

$$\mathbf{P}[w] \geq \tilde{p} \tag{18c}$$

$$h(u, w, \xi) \leq 0,\ \forall w \in W \text{ such that } \mathbf{P}[w] > p_s, \tag{18d}$$

where u is the vector of deterministic decision variables, w is the vector of uncontrollable events, ξ is the vector of auxiliary variables (z, δ) obtained by translating the ueDHA into MLD form[4] and $\mathbf{P}[\cdot]$ denotes the probability of its argument. Cost function (18a), system dynamics/operation constraints (18b), and chance constraint (18c) are the same of the SHOC problem (14). The quantified constraints (18d) are safety constraints that must be robustly enforced with respect to stochastic events having at least probability $p_s \geq 0$: (18d) is the implicit expression extended along the whole control horizon $k \in [0, N-1]$ on ueDHA of $h(x, u, \phi) \leq 0$ in Definition 3, where the role of ϕ is taken by w. If $p_s = 0$, safety with respect to all trajectories having finite probability is ensured, hence obtaining a complete robustness. Robustness in probability is otherwise enforced.

4.1 Robust Optimal Control Algorithm

Because of the quantified constraints (18d), problem (18) cannot be directly formulated as an MIP. As the feasible values of (w, ξ) are finite, in principle it is possible to explode the quantified constraints in several groups of constraints, one for each realization of stochastic events, according to the so called "scenario enumeration" approach of stochastic optimization [15]. However, the number of scenarios is combinatorial with respect to the number of stochastic events and control horizon, so that the numerical problem is intractable in most practical cases.

On the other hand, one only needs to ensure robust safety of the *optimal* sequence u^*, thus only the stochastic event sequences potentially unsafe and enabled by u^* must be considered. Following this consideration we can apply a strategy based on the interaction between a "partially" robustly safe control problem and a reachability analysis problem, described in Algorithm 4.1.

The algorithm is based on the iterative solution of an optimal control problem, whose dimension increases at each iteration of step 3.3.1., and that looks for a candidate solution $\widetilde{u}_i$, and a verification problem, whose dimension remains constant, and that looks for an unsafe[5] stochastic event sequence for $u = \widetilde{u}_i$. Both problems can be solved via MIP. The dimension of the control problem keeps increasing as long as an unsafe stochastic sequence $\widetilde{w}_i$ is found. The ξ variables and the constraints are duplicated to explicitly enforce safety with respect to the trajectory generated by $\widetilde{w}_i$ while optimizing a different trajectory: in this way we request that the trajectory generated by $\widetilde{w}_i$ satisfies $h(x(k), u(k), \widetilde{w}_i(k)) \leq 0$, $\forall k \in [0, N-1]$. Algorithm 4.1 terminates in finite time because the number of admissible stochastic event sequences w is finite.

Let $\mathcal{V}$ be the set of input sequences that fulfils constraints (18b), (18c) and (18d) without quantification, and $\mathcal{S}$ be the set of input sequences that fulfils (18b), (18c), (18d). $\mathcal{V}$ is the feasible input set for the SHOC problem, $\mathcal{S}$ is the feasible input set for the RHOC problem, and $\mathcal{S} \subseteq \mathcal{V}$. The behavior of Algorithm 4.1 is the following. At the beginning $\mathcal{V}$ is known, since it is defined by the constraints of the optimal control problem, while $\mathcal{S}$ is not. The information

[4] Possibly ξ also includes slack variables required to optimize infinity norms, unless 2-norms are used.

[5] In case a set of robust constraints is considered, it is sufficient that one is violated.

1. Let the control problem be (18) after removing quantification from (18d);
2. Set $i = 0$;
3. do
   3.1. $i \leftarrow i + 1$
   3.2. Solve the control problem and get a candidate solution $\widetilde{u}_i$;
   3.3. if $\widetilde{u}_i \neq \varnothing$
   Solve a reachability problem for $\widetilde{u}_i$ and find $\widetilde{w}_i : \exists k : h(x(k), \widetilde{u}_i(k), \widetilde{w}_i(k)) > 0$
      3.3.1. if $\widetilde{w}_i \neq \varnothing$
      Add to the control problem variables ξ_i and constraints $A_u\, u + A_w\, \widetilde{w}_i + A_\xi\, \xi_i \leq b$ and $h(u, \widetilde{w}_i, \xi_i) \leq 0$ that enforce safety with respect to $\widetilde{w}_i$;
   while $\widetilde{u}_i \neq \varnothing$ and $\widetilde{w}_i \neq \varnothing$
4. if $\widetilde{u}_i = \varnothing$
   4.1. Problem (18) is unfeasible.
   else
   4.2. Set $u^* = \widetilde{u}_i$.

Algorithm 4.1: Robust hybrid optimal control algorithm

obtained from the verification problem is used to cut a part of $\mathcal{V}$ while maintaining $\mathcal{S} \subseteq \mathcal{V}$. This procedure continues until the optimal point computed at step 3.2 belongs to $\mathcal{S}$, and therefore the RHOC problem is solved, without in most cases explicitly characterizing $\mathcal{S}$.

Usually, only a small fraction of stochastic events affects the evolution of the system when a particular control sequence is chosen, and an even smaller fraction brings the system to the unsafe region. The iterative approach aims at considering only these stochastic event sequences among all the possible ones, thus solving many smaller problems rather than one large MIP in which all possible realizations of stochastic events are enumerated. Nevertheless, it must be noted that in the worst case Algorithm 4.1 still has a combinatorial complexity with respect to the control horizon and the number of uncontrollable events.

Remark 1. The SHOC and RHOC approaches are different from the more common control approach for stochastic systems, where the *average* state is controlled. In our setting, the uncertainty affecting DHSA has a discrete nature, so that taking averages may lead to unsatisfactory solutions. Consider the following problem: control to the origin the state of the system having three modes with dynamics $x(k+1) = x(k)$, $x(k+1) = x(k) + u(k) - 1$, $x(k+1) = x(k) + u(k) + 1$, respectively mode $1, 2, 3$. Consider the system starting in $x(0) = 0$ in mode 1 and assume at time $\bar{k}$ the mode switches to state 2 or 3, both with probability 0.5. An average state control policy would choose $u(k) = 0$, $\forall k$, with the consequence that the trajectories of the system will always diverge from the desired state. On the other hand, SHOC and RHOC would choose one of the two possible behaviors and optimize the system for that situation, e.g. by setting $u(k) = 1$ if the system is predicted to switch to mode 2. In 50% of the cases the state would be brought to the origin (clearly, in the remaining 50% the error would be larger than in the case of the average control policy).

5 Application Example

As a benchmark test we consider a problem in production systems where the goal is to control a small factory production facility subject to random failures depending on wear.

5.1 Modeling

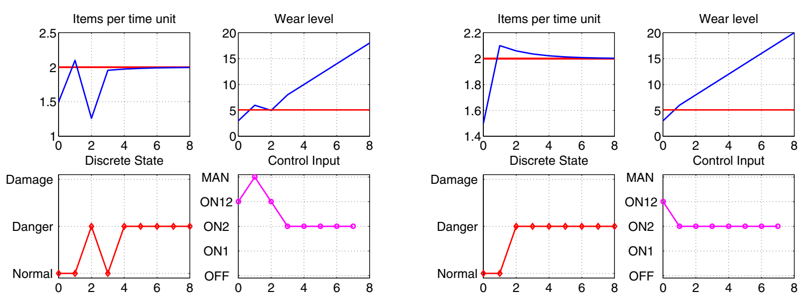
The considered production facility is constituted by two lines having different fixed production rates. The factory production rate must track a given reference forecasted demand.

The production system accumulates wear. When the wear is above a certain level, there is a probability p_{break} that the system breaks. Maintenance can be decided and executed to reduce wear, at the price of stopping the production. Production is interrupted when the system is damaged and the system must be repaired before production starts again.

The production rate dynamics $\psi(k)$ is modeled as a first order asymptotically stable linear system, the wear dynamics $\nu(k)$ as an integrator. The production facility can be in three discrete states: *Normal*, *Danger* (=risk of damage) and *Damaged*. The sFSM describing the possible discrete state transitions is presented in Figure 1, where the events e_1, e_2 represents the *risky* threshold crossing ($\nu(k) \geq 5.1$) and the completion of the repairing ($\nu(k) \leq 0.1$), respectively. There are three binary control commands, two for activating independently the production lines and the third, mutually exclusive with the others, activating maintenance. A more detailed description of the system can be found in [11].

5.2 Control Design and Results

All the tests presented here have been performed on an Intel Pentium Xeon 2.8 MHz running Matlab 6.5 and Cplex 9.0. We set $N = 8$, $\tilde{p} = 0.4$ and the



(a) Stochastic control, $q_p = 10$.

(b) Stochastic control, $q_p = 10^{-3}$.

Fig. 2. Stochastic control of a production system

objective function $\sum_{i=0}^{N}|(\psi(k)-r(k))|$, in which r is the forecasted demand, $r(k)=2, \forall k\in[0,8]$. The constraints involve the discrete and continuous dynamics of the system and the additional mutual exclusivity constraint among production line activation signals and maintenance execution signal. The trade off coefficient q_p is used as a tuning parameter and set either to 10 or to 10^{-3}, while $p_{break}=0.1$. The initial state is $\psi(0)=1.5$, $\nu(0)=3$ and the discrete state is *Normal*. The optimal control sequence found is applied in open loop.

The expected trajectory for $q_p=10$ has probability 0.66 and it is shown in Figure 2(a). Note that the probability is higher than the limit $\tilde{p}$ because of the probability cost $\mathcal{C}_p$. In Figure 2(b) the expected trajectory for $q_p=10^{-3}$ is reported: it has higher performance but the probability of the optimal trajectory decreases to 0.53. In both cases the computation time to solve the associated MIP is less than 0.1 seconds.

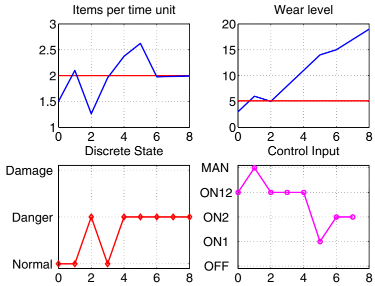
The stochastic control does not ensure that constraints will be met when u^* is implemented. If we require that the production rate remains above a certain threshold $\bar{\psi}_m=0.92$ items per time unit in all possible situations during the whole horizon, a RHOC approach must be used. For $q_p=10^{-3}$, the robust algorithm requires two additional iterations to solve the problem and a computation time of 0.68 seconds. The predicted trajectory is reported in Figure 3(a).

In Figure 3(b) the robust control solution is reported for $q_p=10$. In this case only one additional iteration is required with respect to the stochastic control under the same conditions and the computation time is 0.49 seconds.

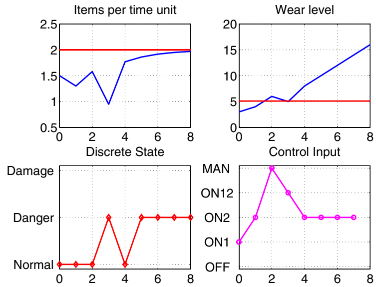
Figures 3(c), 3(d) depict the worst case situation in which the system suddenly breaks down when it is in danger, in order to compare SHOC and RHOC. Probability coefficients $q_p=10^{-3}$ (Figure 3(c)) and $q_p=10$ (Figure 3(d)) are tested in both approaches. The trajectory obtained by stochastic control (dashed line) is initially closer to the desired production rate, but it crosses the line of minimum desired rate. Instead, when the input profile obtained by robust control algorithm (solid line) is applied, the production rate remains in the desired region during the whole control horizon.

6 Actuation Policies and Convergence Results

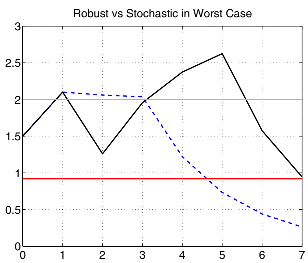
So far we have considered open loop optimal control problems. Feedback control can be achieved through repeated optimization schema, such as Model Predictive Control strategies. In this section we provide preliminary results on sufficient conditions for asymptotic convergence of the state vector when SHOC/RHOC is applied repeatedly. In order to prove convergence of the SHOC/RHOC control of DHSA we separated the problem of obtaining convergence of a deterministic system and the problem of obtaining convergence of a system affected by stochastic disturbances. The first is solved using well known results of receding horizon asymptotic convergence [12, 18], the second using techniques of Markov Chain convergence theory [14].



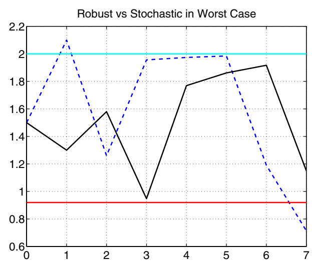
(a) Robust control ($q_p = 10^{-3}$).



(b) Robust control ($q_p = 10$).



(c) RHOC vs. SHOC ($q_p = 10^{-3}$).



(d) RHOC vs. SHOC ($q_p = 10$).

Fig. 3. Robust control (solid) and comparison with stochastic control (dashed)

6.1 Repeated Open-Loop Optimal Control

The simpler policy is Repeated Open-Loop Optimal Control (ROLOC): from a given state $x(0)$ an SHOC/RHOC problem is solved and the whole input sequence $u^* = \{u^*(i)\}_{i=0}^{N-1}$ is applied. Then, a new problem is solved from $x(N)$, and so on.

Since the system is stochastic, asymptotic convergence in probability is considered here. A sequence of random variables $\{\phi(i)\}_{i=0}^{\infty}$ converges in probability to a random variable $\bar{\phi}$ if $\forall \varepsilon > 0, \lim_{i\to\infty} \mathbf{P}[|\phi(i) - \bar{\phi}| > \varepsilon] = 0$ (see [19]).

Consider a DHSA in initial state x_0, stochastic hybrid optimal control with ROLOC policy and a target state $\bar{x}$. Let $\mathcal{X} = \mathcal{X}_r \times \{0,1\}^{n_b}$ be the full (continuous and discrete) state set and let $\mathcal{R}(\bar{x}, N) \subseteq \mathcal{X}$ be the set of states from which the state $\bar{x}$ is reachable within N steps. Define $\tilde{p}$ such that $0 < \tilde{p} \leq \min_{x\in\mathcal{R}(\bar{x},N)}\{P[\mathcal{T}(x,\bar{x})]\} < 1$, where $\mathcal{T}(x,\bar{x})$ is the trajectory with maximum probability from state x to $\bar{x}$. Let $\mathcal{X}_s \subseteq \mathcal{X}$ be the set of states x_0 for which problem (14) is feasible from $x(0) = x_0$ and let the initial state be $x_0 \in \mathcal{R}(\bar{x}, N) \cap \mathcal{X}_s$.

Proposition 1. *Consider the stochastic hybrid optimal control* (14) *applied in ROLOC policy with horizon N from initial state $x_0 \neq \bar{x}$. If:*

1. *the terminal state constraint $x(N) = \bar{x}$ is used as an additional constraint in the optimization,*
2. *the probability limit is fixed to $\tilde{p}$ and $0 < \tilde{p} \leq \min_{x \in \mathcal{R}(\bar{x},N)}\{P[\mathcal{T}(x,\bar{x})]\} < 1$,*
3. *$\forall x \in \mathcal{R}(\bar{x},N) \cap \mathcal{X}_s, \forall w \in W^N,\ \tilde{x} = F(x,u^*,w) \in \mathcal{R}(\bar{x},N) \cap \mathcal{X}_s$, where u^* is the deterministic component of the optimal solution and F is the function that maps the initial state x, the input sequence u^* and the stochastic event sequence w in the final state $\tilde{x}$,*
4. *the objective state $\bar{x}$ is an equilibrium point of the system, it is not affected by stochastic events and the optimal performance index is zero for $x = \bar{x}$,*

then the state x converges asymptotically in probability to $\bar{x}$.

Proof. Consider a generic instant kN, $k \in \mathbb{K}$, $k > 0$. We are interested in computing $\mathbf{P}[x(kN) = \bar{x}]$.

By applying the total probability theorem we get

$$\mathbf{P}[x(kN) = \bar{x}] = \mathbf{P}[x(kN) = \bar{x} | x((k-1)N) = \bar{x}]\ \mathbf{P}[x((k-1)N) = \bar{x}] + \mathbf{P}[x(kN) = \bar{x} | x((k-1)N) \neq \bar{x}]\ \mathbf{P}[x((k-1)N) \neq \bar{x}]\ . \quad (19)$$

Because of hypothesis 4, $\mathbf{P}[x(kN) = \bar{x} | x((k-1)N) = \bar{x}] = 1$, and $\mathbf{P}[x(kN) = \bar{x} | x((k-1)N) \neq \bar{x}] = \hat{p}_{k-1} \geq \tilde{p}$ because of hypothesis 2. Denoting by $P_k = \mathbf{P}[x(kN) = \bar{x}]$, we can write (19) as $P_k = P_{k-1} + \hat{p}_{k-1}(1 - P_{k-1})$.
We prove convergence by induction. For $k = 1$ we have $P_1 = P_0 + \hat{p}_0(1 - P_0) = \hat{p}_0 \geq \tilde{p} = \sum_{i=0}^{0} \tilde{p}(1-\tilde{p})^i$ where $P_0 = 0$ because $x_0 \neq \bar{x}$. Assume that

$$P_{k-1} \geq \sum_{i=0}^{k-2} \tilde{p}(1-\tilde{p})^i. \quad (20)$$

Then $P_k = P_{k-1} + \hat{p}_{k-1}(1 - P_{k-1}) \geq P_{k-1} + \tilde{p}(1 - P_{k-1}) = P_{k-1}(1-\tilde{p}) + \tilde{p}$. By the induction hypothesis (20), $P_k \geq \sum_{i=0}^{k-2} \tilde{p}(1-\tilde{p})^{i+1} + \tilde{p} = \sum_{i=1}^{k-1} \tilde{p}(1-\tilde{p})^i + \tilde{p}(1-\tilde{p})^0 = \sum_{i=0}^{k-1} \tilde{p}(1-\tilde{p})^i$, and thus we have $\sum_{i=0}^{k-1} \tilde{p}(1-\tilde{p})^i \leq P_k \leq 1$. Since $\lim_{k\to\infty} \sum_{i=0}^{k-1} \tilde{p}(1-\tilde{p})^i = 1$, we conclude that $\lim_{k\to\infty} \mathbf{P}[x(kN) = \bar{x}] = 1$. □

Note that hypothesis 1 forces convergence to $\bar{x}$, hypothesis 2 ensures feasibility of (13) in optimization and hypothesis 3 ensures not to lose feasibility because of an unexpected stochastic event; this hypothesis might be difficult to verify, thus it can be convenient to verify a condition including it (e.g. that the condition is feasible for each valid input sequence and not only for the optimal one), and it can be removed if RHOC is used. Hypothesis 4 ensures that the objective state will never be left, once it is reached. We can note that the larger is $\tilde{p}$, the faster the probability of reaching the target state converges to one.

6.2 Model Predictive Control

We now consider a Model Predictive Control (MPC) policy, where an optimal control problem is repeated at each step k and only $u^*(0)$ is applied as the input $u(k)$, while $\{u^*(1), \ldots u^*(N-1)\}$ are discarded. In order to obtain convergence of such an MPC policy, we make the probability limit $\tilde{p}$ time varying.

Consider solving problem (14) from the initial state $x(0) = x_0$, with probability limit $\tilde{p}(0) = \tilde{p}$ as defined in hypothesis 2. Let (u^*, w^*) be the optimizer, and let the predicted next state be $\hat{x}(1) = f(x(0), u^*(0), w^*(0))$. After applying the first input $u^*(0)$ we get a new state $x(1)$, from which a new optimization problem is solved with probability limit $\tilde{p}(1)$ defined by

$$\tilde{p}(k+1) = \begin{cases} \frac{\tilde{p}(k)}{\mathbf{P}[w^*(k)]} & \text{if} \quad x_b(k+1) = \hat{x}_b(k+1) \\ \tilde{p}(0) & \text{if} \quad x_b(k+1) \neq \hat{x}_b(k+1). \end{cases} \tag{21}$$

The value $\mathbf{P}[w^*(k)]$ represents the probability of the transition predicted at step k and it is known from the result of the MIP, while x_b is the discrete component of the state. The purpose of updating the probability limit is to force the probability of a path between two unexpected transitions to be greater or equal than $\tilde{p}$, therefore avoiding the generation of trajectories having "almost-0" probability.

Assumption 1. The "deterministic behavior" of the MPC closed-loop system, where both u and w are manipulated variables, is asymptotically stable.

Assumption 1 can be satisfied by using final state constraints and defining cost weight matrices in the objective function as reported in [12, 18], since the problem is that of stabilizing a deterministic ueDHA by manipulating the inputs u and w in a receding horizon fashion. When the above strategy is applied, we can prove convergence using the same arguments of Proposition 1. A path that reaches the objective without unexpected transitions in the worst case has probability $\tilde{p}$, thus the probability of having one or more of them is $1 - \tilde{p}$.

Proposition 2. *The stochastic hybrid optimal control* (14) *applied to the DHSA with MPC policy and probability limit update* (21)*, under the same hypotheses of Proposition 1 and Assumption 1, converges asymptotically in probability to the objective state $\bar{x}$.*

Proof. The final state constraint and preliminary assumption on ueDHA ensure that, if there are no unexpected transitions in an interval "long enough", the system state converges to the objective, as shown in [18]. The probability of having no unexpected transitions in the worst case is $\tilde{p}$, and the probability of having h of them is $\tilde{p}(1-\tilde{p})^h$. The probability of converging with not more than m unexpected transition is $\sum_{h=0}^{m} \tilde{p}(1-\tilde{p})^h$. As $k \to \infty$, there might be $m \to \infty$ unexpected transitions, but the probability of converging is $\sum_{h=0}^{\infty} \tilde{p}(1-\tilde{p})^h$. This series has been shown to converge at value 1, thus $\lim_{k\to\infty} \mathbf{P}[x(k) = \bar{x}] = 1$. □

Even in this case we can relax hypothesis 3 if the RHOC approach is used.

7 Conclusions

In this paper we have shown that by modeling hybrid systems affected by stochastic uncertainty as DHSA several classes of optimal control problems can be solved. We have shown how to trade off between performance and probability, how to impose the chance constraints and how to satisfy constraints robustly. The approach was exemplified on an application study and a set of sufficient conditions, under which asymptotic convergence of repeated optimization schemes can be proved, has been given.

References

1. Antsaklis, P.: A brief introduction to the theory and applications of hybrid systems. Proc. IEEE, Special Issue on Hybrid Systems: Theory and Applications **88** (2000) 879–886
2. Pola, G., Bujorianu, M., Lygeros, J., Di Benedetto, M.: Stochastic hybrid models: an overview with application to air traffic management. In: IFAC–ADHS03,IFAC conference on analysis and design of hybrid systems. (2003)
3. Hu, J., Lygeros, J., Sastry, S.: Towards a theory of stochastic hybrid systems. In Krogh, B., Lynch, N., eds.: Hybrid Systems: Computation and Control. Volume 1790 of Lecture Notes in Computer Science. Springer-Verlag (2000) 160–173
4. Bujorianu, M., Lygeros, J.: Reachability questions in piecewise deterministic markov processes. In Maler, O., Pnueli, A., eds.: Hybrid Systems: Computation and Control. Number 2623 in Lecture Notes in Computer Science, Springer-Verlag (2003) 126–140
5. Liberzon, D., Chatterjee, D.: On stability of stochastic switched systems. In: Proc. 43th IEEE Conf. on Decision and Control, Paradise Island, Bahamas (2004)
6. Strubbe, S., Julius, A., van der Schaft, A.: Communicating piecewise deterministic markov processes. In: in Proc. IFAC Conf. Analysis and Design of Hybrid Systems. (2003) 349–354
7. Prandini, M., Hu, J., Lygeros, J., Sastry, S.: A probabilistic approach to aircraft conflict detection. IEEE Transactions on Intelligent Transportation Systems **1** (2000) 199–220
8. Cassandras, C., Mookherje, R.: Receding horizon optimal control for some stochastic hybrid systems. In: Proc. 41th IEEE Conf. on Decision and Control. (2003) 2162–2167
9. Hespanha, J.: Stochastic hybrid systems: application to communication networks. In Alur, R., Pappas, G., eds.: Hybrid Systems: Computation and Control. Volume 2993 of Lecture Notes in Computer Science. Springer-Verlag (2004) 387–401
10. Torrisi, F., Bemporad, A.: HYSDEL — A tool for generating computational hybrid models. IEEE Trans. Contr. Systems Technology **12** (2004) 235–249
11. Bemporad, A., Di Cairano, S.: Modelling and optimal control of hybrid systems with event uncertainty. Technical report, University of Siena (02/04, 2004) Available at www.dii.unisi.it/∼dicairano/papers/tr0204.pdf.
12. Bemporad, A., Morari, M.: Control of systems integrating logic, dynamics, and constraints. Automatica **35** (1999) 407–427
13. Davis, M.: Markov models and optimization. Chapman-Hall, London (1993)
14. Cassandras, C.: Discrete event systems. Aksen associates (1993)

15. Birge, J., Louveaux, F.: Introduction to Stochastic Programming. Springer, New York (1997)
16. Bemporad, A.: Hybrid Toolbox – User's Guide. (2003) `http://www.dii.unisi.it/hybrid/toolbox`.
17. ILOG, Inc.: CPLEX 8.1 User Manual, Gentilly Cedex, France. (2003)
18. Lazar, M., Heemels, W., Weiland, S., Bemporad, A.: Stability of hybrid model predictive control. In: Proc. 43th IEEE Conf. on Decision and Control, Paradise Island, Bahamas (2004)
19. Papoulis, A.: Probability, random variables and stochastic processes. McGraw-Hill (1991)