On the Optimal Control Law for Linear Discrete Time Hybrid Systems

Alberto Bemporad[1,2], Francesco Borrelli[1], and Manfred Morari[1]

[1] Automatic Control Laboratory, ETH,
CH-8092 Zurich, Switzerland
Phone: +41 1 632-4158
Fax: +41 1 632-1211
bemporad,borrelli,morari@aut.ee.ethz.ch
[2] Dip. Ingegneria dell'Informazione
Università di Siena
Phone: +39 0577 234-631
Fax: +39 0577 234-632
bemporad@dii.unisi.it

Abstract. In this paper we study the solution to optimal control problems for discrete time linear hybrid systems. First, we prove that the closed form of the state-feedback solution to finite time optimal control based on quadratic or linear norms performance criteria is a time-varying piecewise affine feedback control law. Then, we give an insight into the structure of the optimal state-feedback solution and of the value function. Finally, we briefly describe how the optimal control law can be computed by means of multiparametric programming.

1 Introduction

Different methods for the analysis and design of controllers for hybrid systems have emerged over the last few years [31, 33, 11, 19, 26, 5]. Among them, the class of optimal controllers is one of the most studied. Most of the literature deals with optimal control of continuous-time hybrid systems and is focused on the study of necessary conditions for a trajectory to be optimal [32, 29], and on the computation of optimal or *sub-optimal* solutions by means of Dynamic Programming or the Maximum Principle [18, 20, 10, 30, 12]. Although some techniques for determining feedback control laws seem to be very promising, many of them suffer from the "curse of dimensionality" arising from the *discretization* of the state space necessary in order to solve the corresponding Hamilton-Jacobi-Bellman or Euler-Lagrange differential equations.

In this paper we study the solution to optimal control problems for linear discrete time hybrid systems. Our hybrid modeling framework is extremely general, in particular the control switches can be both internal, i.e., caused by the state reaching a particular boundary, and controllable (i.e., one can decide when to switch to some other operating mode). Even though interesting mathematical phenomena occurring in hybrid systems such as Zeno behaviors [25] do not exist

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in discrete time, we have shown that for such a class of systems we can *characterize* and *compute* the optimal control law *without gridding* the state space. In [3] we proposed a procedure for synthesizing piecewise affine optimal controllers for discrete time linear hybrid systems. The procedure, based on multiparametric programming, consists of finding the state-feedback solution to finite-time optimal control problems with performance criteria based on linear norms.

Sometimes the use of linear norms has practical disadvantages: A satisfactory performance may be only achieved with long time-horizons, with a consequent increase of complexity, and closed-loop performance may not depend smoothly on the weights used in the performance index, i.e., slight changes of the weights could lead to very different closed-loop trajectories, so that the tuning of the controller becomes difficult. This work is a step towards the characterization of the closed form of the state-feedback solution to optimal control problems for linear hybrid systems with performance criteria based on quadratic norms. First, we prove that the state-feedback solution to the finite time optimal control problem is a time-varying piecewise affine feedback control law (possibly defined over nonconvex regions). Then, we give an insight on the structure of the optimal state-feedback solution and of the value function. Finally, we briefly describe how the optimal control law can be computed by means of multiparametric programming.

The infinite horizon optimal controller can be approximated by implementing in a receding horizon fashion a finite-time optimal control law. The resulting state-feedback controller is stabilizing and respects all input and output constraints. The implementation, as a consequence of the results presented here on finite-time optimal control, requires only the evaluation of a piecewise affine function. This opens up the route to use receding horizon techniques to control hybrid systems characterized by fast sampling and relatively small size. In collaboration with different companies we have applied this type of optimal control design to a range of hybrid control problems, for instance in traction control [8].

2 Hybrid Systems

Several modeling frameworks have been introduced for discrete time hybrid systems. Among them, *piecewise affine* (PWA) systems [31] are defined by partitioning the state space into polyhedral regions, and associating with each region a different linear state-update equation

$$\begin{array}{l} x(t+1) = A_i x(t) + B_i u(t) + f_i \\ \qquad \text{if } \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} \in \mathcal{X}_i \triangleq \{ \left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right] : \; H_i x + J_i u \leq K_i \} \end{array} \tag{1}$$

where $x \in \mathbb{R}^{n_c} \times \{0,1\}^{n_\ell}$, $u \in \mathbb{R}^{m_c} \times \{0,1\}^{m_\ell}$, $\{\mathcal{X}_i\}_{i=0}^{s-1}$ is a polyhedral partition of the sets of state+input space $\mathbb{R}^{n+m}$, $n \triangleq n_c + n_\ell$, $m \triangleq m_c + m_\ell$. In the special case $x \in \mathbb{R}^{n_c}$, $u \in \mathbb{R}^{m_c}$ (no binary states and inputs), we say that the PWA system (1) is continuous if the mapping $(x(t), u(t)) \mapsto x(t+1)$ is continuous. The double definition of the state-update function over common boundaries of sets $\mathcal{X}_i$ (the boundaries will also be referred to as *guardlines*) is a technical issue that arises only when the PWA mapping is discontinuous, and can be solved by allowing strict inequalities in the definition of the polyhedral cells in (1). PWA systems can model a large number of physical processes, such as systems with static nonlinearities, and can approximate nonlinear dynamics via multiple linearizations at different operating points.

Furthermore, we mention here linear complementarity (LC) systems [21, 35, 22] and extended linear complementarity (ELC) systems [13], max-min-plus-scaling (MMPS) systems [14], and mixed logical dynamical (MLD) systems [5]. Recently, the equivalence of PWA, LC, ELC, MMPS, and MLD hybrid dynamical systems was proven constructively in [23, 4]. Thus, the theoretical properties and tools can be easily transferred from one class to another. Each modeling framework has its advantages. For instance, stability criteria were formulated for PWA systems [24, 27] and control and verification techniques were proposed for MLD discrete time hybrid models [5, 7]. In particular, MLD models have proven successful for recasting hybrid dynamical optimization problems into mixed-integer linear and quadratic programs, solvable via branch and bound techniques [28].

MLD systems [5] allow specifying the evolution of continuous variables through linear dynamic equations, of discrete variables through propositional logic statements and automata, and the mutual interaction between the two. Linear dynamics are represented as difference equations $x(t+1) = Ax(t)+Bu(t)$, $x \in \mathbb{R}^n$. Boolean variables are defined from linear-threshold conditions over the continuous variables. The key idea of the approach consists of embedding the logic part in the state equations by transforming Boolean variables into 0-1 integers, and by expressing the relations as mixed-integer linear inequalities [5, 36].

By collecting the equalities and inequalities derived from the representation of the hybrid system we obtain the Mixed Logical Dynamical (MLD) system [5]

$$x(t+1) = Ax(t) + B_1u(t) + B_2\delta(t) + B_3z(t) \tag{2a}$$

$$E_2\delta(t)+E_3z(t) \leq E_1u(t) + E_4x(t) + E_5 \tag{2b}$$

where $x \in \mathbb{R}^{n_c} \times \{0,1\}^{n_\ell}$ is a vector of continuous and binary states, $u \in \mathbb{R}^{m_c} \times \{0,1\}^{m_\ell}$ are the inputs, $\delta \in \{0,1\}^{r_\ell}$, $z \in \mathbb{R}^{r_c}$ represent auxiliary binary and continuous variables respectively, which are introduced when transforming logic relations into mixed-integer linear inequalities, and A, B_{1-3}, E_{1-5} are matrices of suitable dimensions. We assume that system (2) is *completely well-posed* [5], which means that for all x, u within a bounded set the variables δ, z are uniquely determined, i.e., there exist functions F, G such that, at each time t, $\delta(t) = F(x(t), u(t))$, $z(t) = G(x(t), u(t))$. This allows one to assume that $x(t+1)$ is uniquely defined once $x(t)$, $u(t)$ are given, and therefore that x-trajectories exist and are uniquely determined by the initial state $x(0)$ and input signal $u(t)$. It is clear that the well-posedness assumption stated above is usually guaranteed by the procedure used to generate the linear inequalities (2b), and therefore this hypothesis is typically fulfilled by MLD relations derived from modeling real-world plants through the tool HYSDEL [34].

In the next section we use the PWA modeling framework to derive the main properties of the state-feedback solution to finite time optimal control problem for hybrid systems. Thanks to the aforementioned equivalence between PWA and MLD systems, the latter will be used in Section 4 to compute the optimal control law.

3 Finite-Time Constrained Optimal Control

Consider the PWA system (1) subject to hard input and state constraints[1]

$$\begin{aligned} u_{\min} \leq u(t) \leq u_{\max}, \\ x_{\min} \leq x(t) \leq x_{\max} \end{aligned} \tag{3}$$

for $t \geq 0$, and denote by constrained PWA system (CPWA) the restriction of the PWA system (1) over the set of states and inputs defined by (3),

$$\begin{aligned} x(t+1) &= A_i x(t) + B_i u(t) + f_i \\ &\text{if } \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} \in \tilde{\mathcal{X}}_i \triangleq \{ \left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right] : \ \tilde{H}_i x + \tilde{J}_i u \leq \tilde{K}_i \} \end{aligned} \tag{4}$$

where $\{\tilde{\mathcal{X}}_i\}_{i=0}^{s-1}$ is the new polyhedral partition of the sets of state+input space $\mathbb{R}^{n+m}$ obtained by intersecting the polyhedrons $\mathcal{X}_i$ in (1) with the polyhedron described by (3).

Define the following cost function

$$J(U_0^{T-1}, x(0)) \triangleq \sum_{k=0}^{T-1} \|Q(x(k) - x_e)\|_p + \|R(u(k) - u_e)\|_p + \|P(x(T) - x_e)\|_p \tag{5}$$

and consider the finite-time optimal control problem (FTCOC)

$$J^*(x(0)) \triangleq \min_{\{U_0^{T-1}\}} J(U_0^{T-1}, x(0)) \tag{6}$$

$$\text{s.t. } \begin{cases} x(t+1) = A_i x(t) + B_i u(t) + f_i \\ \quad \text{if } \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} \in \tilde{\mathcal{X}}_i \end{cases} \tag{7}$$

where the column vector $U_0^{T-1} \triangleq [u'(0), \ldots, u(T-1)']' \in \mathbb{R}^{mT}$, is the optimization vector and T is the time horizon. In (5), $\|Qx\|_p = x'Qx$ for $p = 2$ and $\|Qx\|_p = \|Qx\|_{1,\infty}$ for $p = 1, \infty$, where $R = R' \succ 0$, $Q = Q', P = P' \succeq 0$ if $p = 2$ and $Q,\ R,\ P$ nonsingular if $p = \infty$ or $p = 1$.

[1] Although the form (3) is very common in standard formulation of constrained optimal control problems, the results of this paper also hold for the more general mixed constraints $Ex(t) + Lu(t) \leq M$ arising, for example, from constraints on the input rate $\Delta u(t) \triangleq u(t) - u(t-1)$.

We also need to recall the following definitions:

Definition 1. *A collection of sets $R_1, \dots, R_N$ is a* partition *of a set Θ if* (i) $\bigcup_{i=1}^{N} R_i = \Theta$, (ii) $(R_i \backslash \partial R_i) \cap (R_j \backslash \partial R_j) = \emptyset$, $\forall i \neq j$, *where ∂ denotes the boundary. Moreover $R_1, \dots, R_N$ is a* polyhedral partition *of a polyhedral set Θ if $R_1, \dots, R_N$ is a partition of Θ and R_i's are polyhedral sets.*

Definition 2. *A function $h(\theta) : \Theta \mapsto \mathbb{R}^k$, where $\Theta \subseteq \mathbb{R}^s$, is* piecewise affine (PWA) *if there exists a partition $R_1, \dots, R_N$ of Θ and $h(\theta) = H^i\theta + k^i$, $\forall \theta \in R_i$, $i = 1, \dots, N$.*

Definition 3. *A function $h(\theta) : \Theta \mapsto \mathbb{R}^k$, where $\Theta \subseteq \mathbb{R}^s$, is* PWA on polyhedrons *(PPWA) if there exists a polyhedral partition $R_1, \dots, R_N$ of Θ and $h(\theta) = H^i\theta + k^i$, $\forall \theta \in R_i$, $i = 1, \dots, N$.*

In the following we need to distinguish between optimal control based on the 2-norm and optimal control based on the 1-norm or ∞-norm.

3.1 FTCOC - $p = 2$

Theorem 1. *The solution to the optimal control problem (5)-(7) is a PWA state feedback control law of the form*

$$u(x(k)) = F_i^k x(k) + G_i^k \text{ if } x(k) \in \mathcal{P}_i^k \triangleq \{x : \; x(k)' L_i^k x(k) + M_i^k x(k) \leq N_i^k\}, \\ k = 0, \dots, T-1 \tag{8}$$

where $\mathcal{P}_i^k$, $i = 1, \dots, N_i$ is a partition of the set D^k of feasible states $x(k)$.

Proof: We will give the proof for $u(x(0))$, the same arguments can be repeated for $u(x(1)), \dots, u(x(T-1))$.
Depending on the initial state $x(0)$ and on the input sequence $U = [\ u(0)', \dots, u(k-1)']$ the state $x(k)$ is either infeasible or it belongs to a certain polyhedron $\tilde{\mathcal{X}}_i$. Suppose for the moment that there are no binary inputs, $m_\ell = 0$. The number of all possible locations of the state sequence $x(0), \dots, x(T)$ is equal to s^{T+1}. Denote by v_i, $i = 1, \dots, s^{T+1}$ the list of all possible switching sequences over the horizon T, and by v_i^k the k-th element of the sequence v_i, i.e., $v_i^k = j$ if $x(k) \in \tilde{\mathcal{X}}_j$.

Fix a certain v_i and constrain the state to switch according to the sequence v_i. Problem (5)-(7) becomes

$$J_{v_i}^*(x(0)) \triangleq \min_{\{U_0^{T-1}\}} J(U_0^{T-1}, x(0)) \tag{9}$$

$$\text{s.t.} \begin{cases} x(t+1) = A_i x(t) + B_i u(t) + f_i \\ \qquad \text{if } \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} \in \tilde{\mathcal{X}}_i \\ x(k) \in \tilde{\mathcal{X}}_{v_i^k} \quad k = 0, \dots, T \end{cases} \tag{10}$$

Problem (9)-(10) is equivalent to a finite time optimal control problem for a linear time-varying system with time-varying constraints and can be solved by using the approach of [6]. It's solution is the PPWA feedback control law

$$\tilde{u}^i(x(0)) = \tilde{F}^i_j x(0) + \tilde{G}^i_j, \quad \forall x(0) \in \mathcal{T}^i_j, \quad j = 1, \dots, N_{ri} \tag{11}$$

where $\mathcal{D}^i = \bigcup_{j=1}^{N_{ri}} \mathcal{T}^i_j$ is a polyhedral partition of the convex set $\mathcal{D}^i$ of feasible states $x(0)$ for problem (9)-(10). N_{ri} is the number of regions of the polyhedral partition of the solution and is a function of the number of constraints in problem (9)-(10). The upper-index i in (11) denotes that the input $\tilde{u}^i(x(0))$ is optimal when the switching sequence v_i is fixed.

The optimal solution $u(x(0))$ to the original problem (5)-(7) can be found by solving problem (9)-(10) for all v_i. The set D^0 of all feasible states at time 0 is $D^0 = \bigcup_{i=1}^{s^{T+1}} \mathcal{D}^i$ and, in general, is not convex.

As some initial states can be feasible for different switching sequences, the sets $\mathcal{D}^i$, $i = 1, \dots, s^{T+1}$, in general, can overlap. The solution $u(x(0))$ can be computed in the following way. For every polyhedron $\mathcal{T}^i_j$ in (11),

1. If $\mathcal{T}^i_j \cap \mathcal{T}^l_m = \emptyset$ for all $l \neq i$, $l = 1, \dots, s^{T-1}$, $m = 1, \dots, N_{r_l}$, then the switching sequence v_i is the only feasible one for all the states belonging to $\mathcal{T}^i_j$ and therefore the optimal solution is given by (11), i.e.

$$u(x(0)) = \tilde{F}^i_j x(0) + \tilde{G}^i_j, \quad \forall x \in \mathcal{T}^i_j. \tag{12}$$

2. If $\mathcal{T}^i_j$ intersects one or more polyhedrons $\mathcal{T}^{l_1}_{m_1}$,$\mathcal{T}^{l_2}_{m_2}$,..., the states belonging to the intersection are feasible for more than one switching sequence v_i, v_{l_1}, v_{l_2}, ... and therefore the corresponding value functions $J^*_{v_i}(x(0))$, $J^*_{v_{l_1}}(x(0))$,$J^*_{v_{l_2}}(x(0))$, ... in (9) have to be compared in order to compute the optimal control law.
Consider the simple case when only two polyhedrons overlap, i.e. $\mathcal{T}^i_j \cap \mathcal{T}^l_m \triangleq \tilde{\mathcal{T}}^{i,l}_{j,m} \neq \varnothing$. We will refer to $\tilde{\mathcal{T}}^{i,l}_{j,m}$ as a polyhedron of multiple feasibility. For all states belonging to $\tilde{\mathcal{T}}^{i,l}_{j,m}$ the optimal solution is:

$$u(x(0)) = \begin{cases} \tilde{F}^i_j x(0) + \tilde{G}^i_j, & \forall x(0) \in \tilde{\mathcal{T}}^{i,l}_{j,m} : J^*_{v_i}(x(0)) < J^*_{v_l}(x(0)) \\ \tilde{F}^l_m x(0) + \tilde{G}^l_m, & \forall x(0) \in \tilde{\mathcal{T}}^{i,l}_{j,m} : J^*_{v_i}(x(0)) > J^*_{v_l}(x(0)) \\ \begin{cases} \tilde{F}^i_j x(0) + \tilde{G}^i_j \\ \tilde{F}^l_m x(0) + \tilde{G}^l_m \end{cases} \text{or} & \forall x(0) \in \tilde{\mathcal{T}}^{i,l}_{j,m} : J^*_{v_i}(x(0)) = J^*_{v_l}(x(0)) \end{cases} \tag{13}$$

Because $J^*_{v_i}(x(0))$ and $J^*_{v_l}(x(0))$ are quadratic functions on $\tilde{\mathcal{T}}^i_j$ and $\tilde{\mathcal{T}}^l_m$ respectively, the theorem is proved. In general, a polyhedron of multiple feasibility where n value functions intersect is partitioned into n subsets where in each one of them a certain value function is greater than all the others.
The proof can be repeated in the presence of binary inputs, $m_\ell \neq 0$. In this case the switching sequences v_i are given by all combinations of region indices

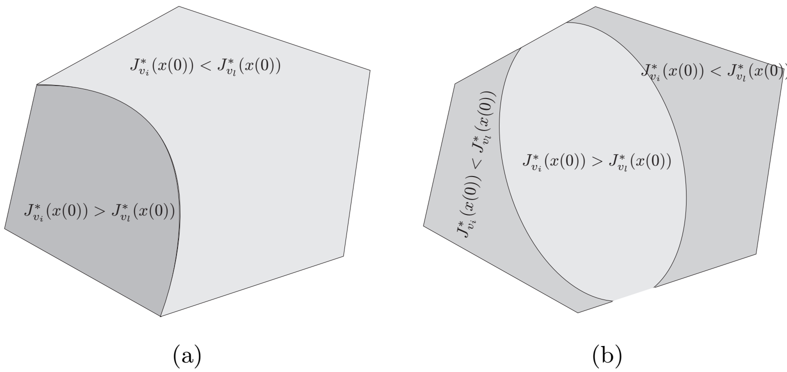


Fig. 1. Possible partitions corresponding to the optimal control law in case 2.d of Remark 1

and *binary inputs*, i.e. $i = 1, \ldots, (s * m_\ell)^{T+1}$. The continuous component of the optimal input is given by (12) or (13). Such an optimal continuous component of the input has an associated optimal sequence v_i which provides the remaining binary components of the optimal input.

$\square$

Remark 1. Let $\tilde{\mathcal{T}}^{i,l}_{j,m}$ be a polyhedron of multiple feasibility and let $\mathcal{F} = \{x \in \tilde{\mathcal{T}}^{i,l}_{j,m} : J^*_{v_i}(x) = J^*_{v_l}(x)\}$ be the set where the quadratic functions $J^*_{v_i}(x)$ and $J^*_{v_l}(x)$ intersect (for the sake of simplicity we consider the case where only two polyhedrons intersect). We distinguish four cases (sub-cases of case 2 in Theorem 1):

2.a $\mathcal{F} = \varnothing$, i.e., $J^*_{v_i}(x)$ and $J^*_{v_l}(x)$ do not intersect over $\tilde{\mathcal{T}}^{i,l}_{j,m}$.
2.b $\mathcal{F} = \{x : Ux = P\}$ and $J^*_{v_i}(x)$ and $J^*_{v_l}(x)$ are tangent on $\mathcal{F}$.
2.c $\mathcal{F} = \{x : Ux = P\}$ and $J^*_{v_i}(x)$ and $J^*_{v_l}(x)$ are not tangent on $\mathcal{F}$.
2.d $\mathcal{F} = \{x : x'Yx + Ux = P\}$ with $Y \neq 0$.

In the first case $\tilde{\mathcal{T}}^{i,l}_{j,m}$ is not further partitioned, the optimal solution in $\tilde{\mathcal{T}}^{i,l}_{j,m}$ is either $\tilde{F}^i_j x(0) + \tilde{G}^i_j$ or $\tilde{F}^l_m x(0) + \tilde{G}^l_m$. In case 2.*b*, $\tilde{\mathcal{T}}^{i,l}_{j,m}$ is not further partitioned but there are multiple optima on the set $Ux = P$. In case 2.c, $\tilde{\mathcal{T}}^{i,l}_{j,m}$ is partitioned into two polyhedrons. In case 2.d $\tilde{\mathcal{T}}^{i,l}_{j,m}$ is partitioned into two sets (not necessarily connected) as shown in Figure 1.

In the special case where case 2.*c* or 2.*d* occur but the control laws are identical, i.e., $F^i_j = F^l_m$ and $\tilde{G}^i_j = \tilde{G}^l_m$, we will assume that the set $\tilde{\mathcal{T}}^{i,l}_{j,m}$ is not further partitioned.

Example 1. Consider the following simple system

$$\begin{cases} x(t+1) = \begin{cases} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \text{ if } x(t) \in \mathcal{X}_1 = \{x : [0\ 1]x \geq 0\} \\ \begin{bmatrix} -1 & -1 \\ 3 & -1 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \text{ if } x(t) \in \mathcal{X}_2 = \{x : [0\ 1]x < 0\} \end{cases} \\ x(t) \in [-5, 5] \times [-5, 5] \\ u(t) \in [-1, 1] \end{cases} \tag{14}$$

and the optimal control problem (5)-(7), with $T = 1$, $Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $R = 1$.

The possible switching sequences are $v_1 = \{1, 1\}$, $v_2 = \{1, 2\}$, $v_3 = \{2, 1\}$, $v_4 = \{2, 2\}$. The solution to problem (9)-(10) is depicted in Figure (2). In Figure 3(a) the four solutions are intersected, the white region corresponds to polyhedrons of multiple feasibility. The state-space partition of the optimal control law is depicted in Figure 3(b) (for lack of space, we do not report here the analytic expressions of the regions and the corresponding affine gains).

Theorem 2. *Suppose that the PWA system (1) is continuous, then the value function $J^*(x(0))$ in (7) is continuous.*

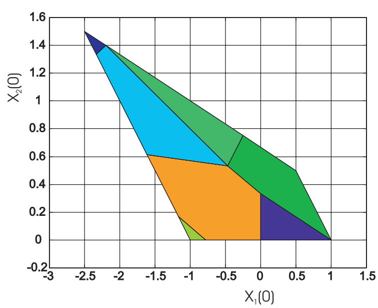
Proof: The continuity of the PWA system (1) implies the continuity of $J(U_0^{T-1}, x(0))$ in (5) as the composition of continuous functions. From the main results on sensitivity analysis [16], $J^*(x(0))$ is also continuous. □

Theorem 3. *Assume $n_\ell = 0$, $m_\ell = 0$ (no discrete states and inputs). Suppose that the cost function $J(U_0^{T-1}, x(0))$ in (5) is continuous and that the optimizer ${U_0^{T-1}}^*(x(0))$ is unique for all $x(0)$. Then the solution to the optimal control problem (5)-(7) is the PPWA state feedback control law*

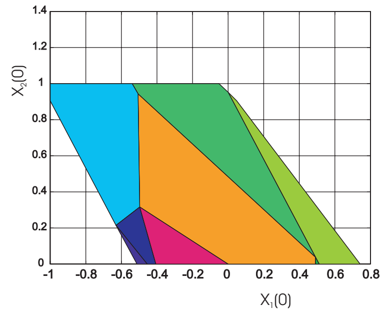
$$\begin{array}{c} u(x(k)) = F_i^k x(k) + G_i^k \text{ if } x(k) \in \mathcal{P}_i^k \triangleq \{x : \ M_i^k x(k) \leq N_i^k\}, \\ k = 0, \ldots, N-1 \end{array} \tag{15}$$

Proof: : We will show that case 2.d in Remark 1 cannot occur by contradiction. Suppose case 2.d occurs. From the hypothesis the optimizer $u(x(0))$ is unique and from Theorem 1 the value function $J^*(x(0))$ is continuous on $\mathcal{F}$, this implies that $\tilde{F}_j^i x(0) + \tilde{G}_j^i = \tilde{F}_m^l x(0) + \tilde{G}_m^l$, $\forall x(0) \in \mathcal{F}$. That contradicts the hypothesis since the set $\mathcal{F}$ is not a hyperplane. The same arguments can be repeated for $u(x(k))$, $k = 1, \ldots, N-1$. □

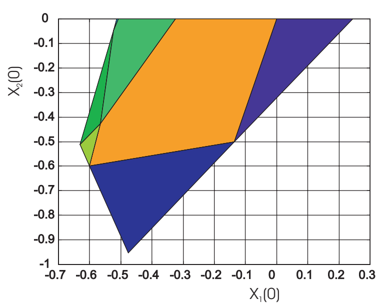
Theorem 4. *Assume $n_\ell = 0$, $m_\ell = 0$ (no discrete states and inputs). Suppose that the cost function $J(U_0^{T-1}, x(0))$ in (5) is strictly convex with respect to U_0^{T-1} and x_0. Then, the solution to the optimal control problem (5)-(7) is a PPWA state feedback control law of the form (15). Moreover the solution $u(x(k))$ is continuous, $J^*(x(0))$ is convex and cases 2.c and 2.d in Remark 1 will never occur.*



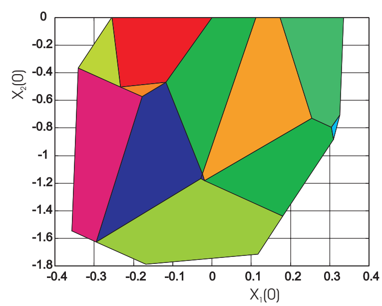
(a) States-space partition corresponding to the solution to problem (5)-(10) for $v_1 = \{1, 1\}$



(b) States-space partition corresponding to the solution to problem (5)-(10) for $v_2 = \{1, 2\}$



(c) States-space partition corresponding to the solution to problem (5)-(10) for $v_3 = \{2, 1\}$

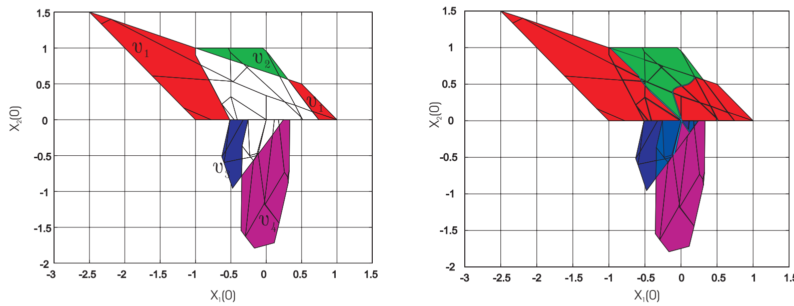


(d) States-space partition corresponding to the solution to problem (5)-(10) for $v_4 = \{2, 2\}$

Fig. 2. First step for the solution of Example 1. Problem (5)-(10) is solved for different v_i, $i = 1, \dots, 4$

Proof: The convexity of D_k and $J^*(x(0))$ and the continuity of $u(x(0))$ follow from the main theorems on sensitivity analysis [16]. Suppose cases 2.c or cases 2.d occur, then two (or more) value functions $J^*_{v_i}(x)$, $J^*_{v_l}(x)$ intersects over a polyhedron of multiple feasibility $\tilde{\mathcal{T}}^{i,l}_{j,m}$. In $\tilde{\mathcal{T}}^{i,l}_{j,m}$ the value function $J^*(x(0))$ is $\min\{J^*_{v_i}(x), J^*_{v_l}(x)\}$, which is not convex. $\square$

Remark 2. Theorem 2 relies on a rather weak uniqueness assumption. As the proof indicates, the key point is to exclude case 2d in Remark 1. Therefore, it is reasonable to believe that there are other conditions or problem classes which satisfy this structural property without claiming uniqueness. We are also



(a) Feasibility domain corresponding to the solution of Example 1 obtained by joining the solutions plotted in Figure 2. The white region corresponds to polyhedrons of multiple feasibility.

(b) State-space partition corresponding to the optimal control law of Example 1

Fig. 3. State-space partition corresponding to the optimal control law of Example 1

currently trying to identify and classify situations where it is usually the state transition structure that guarantees the absence of disconnected sets as shown in Figure 1(b).

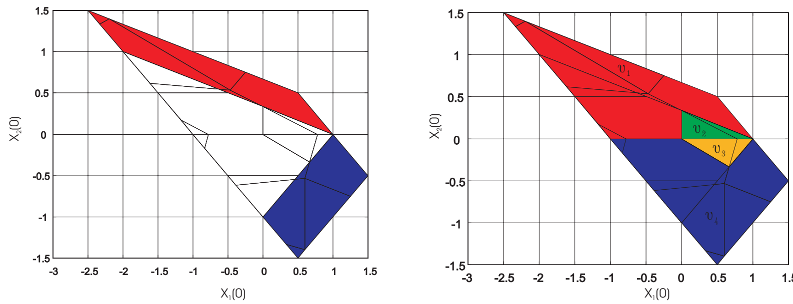
Example 2. Consider the following simple system

$$\begin{cases} x(t+1) = \begin{cases} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \text{ if } x(t) \in \mathcal{X}_1 = \{x : [0\ 1]x \geq 0\} \\ \begin{bmatrix} -1 & -1 \\ 0 & -1 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \text{ if } x(t) \in \mathcal{X}_2 = \{x : [0\ 1]x < 0\} \end{cases} \\ x(t) \in [-5, 5] \times [-5, 5] \\ u(t) \in [-1, 1] \end{cases} \tag{16}$$

and the optimal control problem (5)-(7), with $T = 1$, $Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $R = 1$.

The possible switching sequences are $v_1 = \{1, 1\}$, $v_2 = \{1, 2\}$, $v_3 = \{2, 1\}$, $v_4 = \{2, 2\}$. In Figure 4(a) the white region corresponds to polyhedrons of multiple feasibility. The state-space partition of the optimal control law is depicted in Figure 4(b). Note that the feasible domain is convex and that the partition corresponding to the optimal control law is polyhedral.

The following proposition summarizes the properties enjoyed by the solution to problem (5)-(7) as a direct consequence of Theorems 1-3 and Remark 1



(a) Feasibility domain corresponding to the solution of Example 2. The white region corresponds to polyhedrons of multiple feasibility.

(b) State-space partition corresponding to the optimal control law of Example 2

Fig. 4. State-space partition corresponding to the optimal control law of Example 2

Proposition 1.

1. *$u(x(k))$ and $J^*(x(k))$ are, in general, discontinuous and D^k may be nonconvex.*
2. *$J^*(x(k))$ can be discontinuous only on a facet of a polyhedron of multiple feasibility.*
3. *If there exists a polyhedron of multiple feasibility with $\mathcal{F} = \{x : x'Yx + Ux = P\}$, $Y \neq 0$, then on $\mathcal{F}$ $u(x(k))$ is not unique, except possibly at isolated points.*

3.2 FTCOC - $p = 1, \infty$

The results of the previous section can be extended to piecewise linear cost functions, i.e., cost functions based on the 1-norm or the ∞-norm.

Theorem 5. *The solution to the optimal control problem (5)-(7) with $p = 1, \infty$ is a PPWA state feedback control law of the form (15), where $\mathcal{P}_i^k$, $i = 1, \dots, N_i$ is a partition of the set D^k of feasible states $x(k)$.*

Proof: The proof is similar to the proof of Theorem 1. Fix a certain switching sequence v_i, consider the problem (5)-(7) and constrain the state to switch according to the sequence v_i to obtain problem (9)-(10). Problem (9)-(10) can be viewed as a finite time optimal control problem with performance index based on 1-norm or ∞-norm for a linear time varying system with time varying constraints and can be solved by using the multiparameric linear program as described in [2]. It solution is a PPWA feedback control law

$$\tilde{u}^i(x(0)) = \tilde{F}_j^i x(0) + \tilde{G}_j^i, \quad \forall x \in \mathcal{T}_j^i, \quad j = 1, \dots, N_{ri} \tag{17}$$

and the value function $J^*_{v_i}$ is piecewise affine and convex. The rest of the proof follows the proof of Theorem 1. Note that in this case the value functions to be compared are piecewise affine and not piecewise quadratic. □

Theorem 6. *Suppose that the cost function $J(U_0^{T-1}, x(0))$ in (5) is convex with respect to U_0^{T-1} and x_0. Then the solution to the optimal control problem (5)-(7) is a PPWA and continuous state feedback control law of the form (15), where $D_k = \bigcup_i \mathcal{P}_i^k$ and $J^*(x(0))$ are convex and $u(x(0))$ is continuous.*

Proof: The proof is similar to the proof of Theorem 4. □

4 Efficient Computation of the Solution

In the previous section the properties enjoyed by the solution to hybrid optimal control problems were investigated. Despite the fact that the proof is constructive (as shown in the figures), it is based on the enumeration of all the possible switching sequences of the hybrid system, the number of which grows exponentially with the time horizon. Although the computation is performed off line (the on-line complexity is the one associated with the evaluation of the PWA control law (15)), more efficient methods than enumeration are desirable. Here we show that MLD framework can be used in order to avoid enumeration. Consider the equivalent MLD system (2) of the PWA system (4). Problem (5)-(7) can be rewritten as:

$$\min_{\{U_0^{T-1}\}} J(U_0^{T-1}, x(0)) \triangleq \sum_{k=0}^{T-1} \|Ru(t)\|_p + \|Qx(t)\|_p + \|Px(T|t)\|_p \tag{18}$$

$$\text{subj. to} \begin{cases} x(k+1) = \Phi x(k) + G_1 v(k) + G_2 \delta(k) + G_3 z(k) \\ E_2 \delta(k) + E_3 z(k) \leq E_1 v(k) + E_4 x(k) + E_5 \end{cases} \tag{19}$$

The optimal control problem in (18)-(19) can be formulated as a *Mixed Integer Quadratic Program* (MIQP) when the squared Euclidean norm $p = 2$ is used [5], or as a *Mixed Integer Linear Program* (MILP), when $p = \infty$ or $p = 1$ [3],

$$\begin{aligned} &\min_{\varepsilon} \quad \varepsilon' H_1 \varepsilon + \varepsilon' H_2 x(0) + x(0)' H_3 x(0) + f_1' \varepsilon + f_2' x(0) + c \\ &\text{subj. to } G\varepsilon \leq S + Fx(0) \end{aligned} \tag{20}$$

where H_1, H_2, H_3, f_1, f_2, G, S, F are matrices of suitable dimensions, $\varepsilon = [\varepsilon_c', \varepsilon_d']$ where ε_c, ε_d represent continuous and discrete variables, respectively and H_1, H_2, H_3, are null matrices if problem (20) is an MILP.

Given a value of the initial state $x(0)$, the MIQP (or MILP) (20) can be solved to get the optimal input $\varepsilon^*(x(0))$. Multiparametric programming [17, 15, 6, 9] can be used to efficiently compute the explicit form of the optimal state-feedback control law $u(x(k))$. By generalizing the result of [6] for linear systems to hybrid systems, the state vector $x(0)$, which appears in the objective function and in the linear part of the rhs of the constraints, can be handled as a vector of parameters. Then, for performance indices based on the ∞-norm or 1-norm, the optimization problem can be treated as a *multi-parametric MILP* (mp-MILP), while for performance indices based on the 2-norm, the optimization problem can be treated as a *multi-parametric MIQP* (mp-MIQP).

Solving an mp-MILP (mp-MIQP) amounts to expressing the solution of the MILP (MIQP) (20) as a function of the parameters $x(0)$. Two main approaches have been proposed for solving mp-MILP problems in [1, 15], while, to the authors' knowledge, there does not exist an efficient method for solving mp-MIQPs. An efficient algorithm for the solution of mp-MIQP problems arising from hybrid control problems that uses mp-QP solvers and dynamic programming is currently under development.

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