MODEL PREDICTIVE CONTROL: A MULTI-PARAMETRIC PROGRAMMING APPROACH

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In this paper, linear model predictive control problems are formulated as multi-parametric quadratic programs, where the control variables are treated as optimization variables and the state variables as parameters. It is shown that the control variables are affine functions of the state variables and each of these affine functions is valid in a certain polyhedral region in the space of state variables. An approach for deriving the explicit expressions of all the affine functions and their corresponding polyhedral regions is presented. The key advantage of this approach is that the control actions are computed off-line: the on-line computation simply reduces to a function evaluation problem.

1. INTRODUCTION

On-line optimization is a commonly used tool in the chemical process industry for operating plants at their maximum performance. Typically, this issue is addressed via a Model Predictive Control (MPC) framework where at regular time intervals the measurements from the plant are obtained and an optimization problem is solved to predict the optimal control actions - for a recent survey on MPC, see [1]. In this work, we propose an alternative approach for the on-line calculation of control actions which requires a very small computational effort as an optimizer is never called on-line. This approach is based upon the fundamentals of parametric programming. In an optimization framework, where the objective is to minimize or maximize a performance criterion subject to a given set of constraints and where some of the parameters in the optimization problem are uncertain, parametric programming is a technique for obtaining the objective function and the optimization variables as a function of the uncertain parameters [2,3]. Here, we present a parametric quadratic programming approach to address linear MPC problems, where the state variables are treated as parameters and the control actions are computed as a function of the state variables. The rest of the paper is organized as follows. First a brief outline of MPC problems is presented and these problems are formulated as multi-parametric quadratic programs (mp-QP). Next a solution approach for mp-QPs is presented, followed by an illustrative example.

2. MODEL PREDICTIVE CONTROL

Model Predictive Control (MPC) has been widely adopted by industry to solve control problems of systems subject to input and output constraints. MPC is based on the so called *receding horizon* philosophy: a sequence of future control actions is chosen according to a prediction of the future evolution of the system and applied to the plant until new measurements are available. Then, a new sequence is determined which replaces the previous one. Each sequence is evaluated by means of an optimization procedure which takes into account two objectives: optimize the tracking performance, and protect the system from possible constraint violations. In a mathematical framework, MPC problems can be formulated as follows.

Consider the following state-space representation of a given process model:

$$\left\{ \begin{array}{rcl} x(t+1) & = & Ax(t) + Bu(t) \\ y(t) & = & Cx(t), \end{array} \right. \tag{1}$$

subject to the following constraints: $y_{\min} \leq y(t) \leq y_{\max}$, $u_{\min} \leq u(t) \leq u_{\max}$, where $x(t) \in \Re^n$, $u(t) \in \Re^m$, and $y(t) \in \Re^p$ are the state, input, and output vectors respectively, subscripts *min* and *max* denote lower and upper bounds respectively and (A, B) is stabilizable. Model Predictive Control (MPC) problems for regulating to the origin can then be posed as the following optimization problems:

$$\begin{array}{ll} \min\limits_{U} & J(U, x(t)) = x'_{t+N_y|t} P x_{t+N_y|t} + \sum\limits_{k=0}^{N_y-1} x'_{t+k|t} Q x_{t+k|t} + u'_{t+k} R u_{t+k} \\ \text{s.t.} & y_{\min} \leq y_{t+k|t} \leq y_{\max},\ k = 1, \ldots, N_c \\ & u_{\min} \leq u_{t+k} \leq u_{\max},\ k = 0, 1, \ldots, N_c \\ & x_{t|t} = x(t) \\ & x_{t+k+1|t} = Ax_{t+k|t} + Bu_{t+k},\ k \geq 0 \\ & u_{t+k} = Kx_{t+k|t},\ N_u \leq k \leq N_y \end{array} \tag{2}$$

where $U \triangleq \{u_t, \ldots, u_{t+N_u-1}\}$, $Q = Q' \succeq 0$, $R = R' \succ 0$, $P \succeq 0$, $(Q^{\frac{1}{2}}, A)$ detectable, $N_y \geq N_u$, and K is some feedback gain. The problem (2) is solved repetitively at each time t for the current measurement $x(t)$ and a vector of predicted state variables, $x_{t+1|t}, \ldots, x_{t+k|t}$ at time $t+1, \ldots, t+k$ respectively and corresponding control actions $u_t, \ldots, u_{t+k-1}$ is obtained. In the next section, we present a parametric programming approach where the repetitive solution of (2) at each time interval is avoided and instead an optimization problem is solved only once.

3. MULTI-PARAMETRIC QUADRATIC PROGRAMMING

Parametric programming has largely been used for incorporating the uncertainties in the model, where (i) the objective function and the optimization variables are obtained as a function of uncertain parameters and (ii) the regions in the space of the uncertain parameters where these functions are valid are also obtained [2–5]. The main advantage of using the parametric programming techniques to address the issue of uncertainty is that for problems pertaining to plant operations, such as for process planning [6] and scheduling, one obtains a complete map of all the optimal solutions and as the operating conditions fluctuate, one does not have to re-optimize for the new set of conditions since the optimal solution as a function of uncertain parameters (or the new set of conditions) is already available. In the following paragraphs, we present a parametric programming approach which avoids a repetitive solution of (2). First, we do some algebraic manipulations to recast (2) in a form suitable for using and developing some new parametric programming concepts. By making the following substitution in (2):

$$x_{t+k|t} = A^k x(t) + \sum_{j=0}^{k-1} A^j B u_{k-1-j} \tag{3}$$

the objective $J(U, x(t))$ can be written as the following Quadratic Programming (QP) problem:

$$\begin{array}{ll} \min\limits_{U} & \frac{1}{2}U'HU + x'(t)FU + x'(t)Yx(t) \\ \text{s.t.} & GU \leq W + Kx(t) \end{array} \tag{4}$$

where $U \triangleq [u'_t, \ldots, u'_{t+N_u-1}]' \in \Re^s$, $s \triangleq mN_u$, is the vector of optimization variables, $H = H' \succ 0$, and H, F, Y, G, W, K are obtained from Q, R, and (2)–(3). With the transformation, $z \triangleq U + H^{-1}F'x(t)$, where $z \in \Re^s$, (4) can be written as the following Multi-parametric Quadratic Program (mp-QP):

$$\begin{array}{lll} \mu(x) = & \min\limits_{z} & \frac{1}{2}z'Hz \\ & \text{s.t.} & Gz \leq W + Sx(t), \end{array} \tag{5}$$

where $S \triangleq K + GH^{-1}F'$, z represents the vector of optimization variables and x represents the vector of parameters. The main advantage of writing (2) in the form given in (5) is that z (and therefore U) can be obtained as an affine function of x for the complete feasible space of x. To derive these results, we first state the following theorem (see also [7]).

Theorem 1 *For the problem in (5) let x_0 be a vector of parameter values and (z_0, λ_0) a KKT pair, where $\lambda_0 = \lambda(x_0)$ is a vector of nonnegative Lagrange multipliers, λ, and $z_0 = z(x_0)$ is feasible in (5). Also assume that (i) linear independence constraint qualification and (ii) strict complementary slackness conditions hold. Then,*

$$\begin{bmatrix} z(x) \\ \lambda(x) \end{bmatrix} = -(M_0)^{-1}N_0(x - x_0) + \begin{bmatrix} z_0 \\ \lambda_0 \end{bmatrix} \tag{6}$$

where,

$$M_0 = \begin{pmatrix} H & G_1^T & \cdots & G_q^T \\ -\lambda_1 G_1 & -V_1 & & \\ \vdots & & \ddots & \\ -\lambda_p G_q & & & -V_q \end{pmatrix}, \quad N_0 = (Y, \lambda_1 S_1, \ldots, \lambda_p S_p)^T$$

where G_i denotes the i^{th} row of G, S_i denotes the i^{th} row of S, $V_i = G_i z_0 - W_i - S_i x_0$, W_i denotes the i^{th} row of W and Y is a null matrix of dimension $(s \times n)$.

The space of x where this solution, (6), remains optimal is defined as the Critical Region (CR^0) and can be obtained as follows. Let CR^R represent the set of inequalities obtained (i) by substituting $z(x)$ into the inequalities in (5) and (ii) from the positivity of the Lagrange multipliers, as follows:

$$CR^R = \{Gz(x) \leq W + Sx(t), \lambda(x) \geq 0\}, \tag{7}$$

then CR^0 is obtained by removing the redundant constraints from CR^R as follows:

$$CR^0 = \Delta\{CR^R\}, \tag{8}$$

where Δ is an operator which removes the redundant constraints. Since for a given space of state-variables, X, so far we have characterized only a sub-space of X i.e. $CR^0 \subseteq X$, in the next step the rest of the region CR^{rest}, is defined as follows [3]:

$$CR^{rest} = X - CR^0. \tag{9}$$

The above steps, (6–9) are repeated and a set of $z(x), \lambda(x)$ and corresponding CR^0s are obtained. The solution procedure terminates when no more regions can be obtained, i.e. when $CR^{rest} = \emptyset$. For the regions which have the same solution and can be unified to give a convex region, such a unification is performed and a compact representation is obtained. The continuity and convexity properties of the optimal solution are summarized in the next theorem.

Theorem 2 *For the mp-QP problem, (5), the set of feasible parameters $X_f \subseteq X$ is convex, the optimal solution, $z(x) : X_f \mapsto \Re^s$ is continuous and piecewise affine, and the optimal objective function $\mu(x) : X_f \mapsto \Re$ is continuous, convex and piecewise quadratic.*

Based upon the above theoretical developments the solution of an mp-QP of the form given in (5), to calculate U as an affine function of x and characterize X by a set of polyhedral regions, CRs, can be obtained. This approach provides a significant advancement in the solution and on-line implementation of MPC problems. Since its application results in a complete set of control actions as a function of state-variables (from (6)) and the corresponding regions of validity (from (8)), which are computed off-line. Therefore during on-line optimization, no optimizer needs to be called and instead for the current set of measurements the region, CR^0, where these measurements are valid, can be identified by substituting the value of these measurements into the inequalities which define the regions. Then, the corresponding control actions can be computed by using a function evaluation of the corresponding affine function. In the next section, we present an example to illustrate these concepts.

4. NUMERICAL EXAMPLE

Consider the following state-space model representation:

$$\left\{ \begin{array}{rcl} x(t+1) & = & \begin{bmatrix} 0.7326 & -0.0861 \\ 0.1722 & 0.9909 \end{bmatrix} x(t) + \begin{bmatrix} 0.0609 \\ 0.0064 \end{bmatrix} u(t) \\ y(t) & = & \begin{bmatrix} 0 & 1.4142 \end{bmatrix} x(t) \end{array} \right. \tag{10}$$

together with the following constraints: $-2 \leq u(t) \leq 2$. The corresponding optimization problem of the form (2) for regulating to the origin is given as follows:

$$\begin{array}{ll} \min\limits_{u_t, u_{t+1}} & x'_{t+2|t} P x_{t+2|t} + \sum\limits_{k=0}^{1} x'_{t+k|t} x_{t+k|t} + .01 u_{t+k}^2 \\ \text{s.t.} & -2 \leq u_{t+k} \leq 2, \; k = 0, 1 \\ & x_{t|t} = x(t) \end{array} \tag{11}$$

where P solves the Lyapunov equation $P = A'PA + Q$, $Q = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right]$, $R = 0.01$, $N_u = N_y = N_c = 2$. The corresponding mp-QP problem of the form (5) has the following constant vectors and matrices.

$$H = \begin{bmatrix} 0.0196 & 0.0063 \\ 0.0063 & 0.0199 \end{bmatrix}, F = \begin{bmatrix} 0.1470 & 0.1123 \\ 0.1058 & -0.0834 \end{bmatrix}, G = \begin{bmatrix} 1 & 0 \\ -1 & 0 \\ 0 & 1 \\ 0 & -1 \end{bmatrix}, W = \begin{bmatrix} 2 \\ 2 \\ 2 \\ 2 \end{bmatrix}, K = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

The solution of the mp-QP problem as computed by using the solution approach described in Section 3 is provided in Table 1. Note that the regions 3,4 and 7,8 in Table 1 are combined together and a compact convex representation is obtained. To illustrate how on-line optimization reduces to a function evaluation problem, consider the starting point $x(0) = [1 \ 1]'$. This point is substituted into the constraints defining the regions in Table 1 and it satisfies only the constraints of the regions 7,8. The control action corresponding to the regions 7,8 from Table 1 is $u = -2$, which is obtained without any further optimization calculations.

5. SUMMARY AND CONCLUDING REMARKS

In this work, linear MPC problems were formulated as mp-QPs. An approach for the solution of mp-QPs was proposed. It was shown that the solution (a set of control actions)

Table 1
Solution of the numerical example

Region#	Region	u
1	$\begin{bmatrix} -6.3202 & -7.5004 \\ 6.3202 & 7.5004 \\ -3.6447 & 6.5748 \\ 3.6447 & -6.5748 \end{bmatrix} x \leq \begin{bmatrix} 2.0000 \\ 2.0000 \\ 2.0000 \\ 2.0000 \end{bmatrix}$	$\begin{bmatrix} -6.3202 & -7.5004 \end{bmatrix} x$
2	$\begin{bmatrix} 0.1123 & -0.0834 \\ 0.1470 & 0.1058 \end{bmatrix} x \leq \begin{bmatrix} -0.0524 \\ -0.0519 \end{bmatrix}$	2.0000
3,4	$\begin{bmatrix} -5.6485 & 4.1968 \\ 5.6485 & -4.1968 \\ 0.1114 & 0.1322 \end{bmatrix} x \leq \begin{bmatrix} 2.6341 \\ 1.3659 \\ -0.0353 \end{bmatrix}$	2.0000
5	$\begin{bmatrix} -7.4906 & -5.3891 \\ -0.0651 & 0.1174 \\ 7.4906 & 5.3891 \end{bmatrix} x \leq \begin{bmatrix} 1.3577 \\ -0.0357 \\ 2.6423 \end{bmatrix}$	$\begin{bmatrix} -7.4906 & -5.3891 \end{bmatrix} x + 0.6423$
6	$\begin{bmatrix} -0.1470 & -0.1058 \\ -0.1123 & 0.0834 \end{bmatrix} x \leq \begin{bmatrix} -0.0519 \\ -0.0524 \end{bmatrix}$	-2.0000
7,8	$\begin{bmatrix} -5.6485 & 4.1968 \\ 5.6485 & -4.1968 \\ -0.1114 & -0.1322 \end{bmatrix} x \leq \begin{bmatrix} 1.3659 \\ 2.6341 \\ -0.0353 \end{bmatrix}$	-2.0000
9	$\begin{bmatrix} 7.4906 & 5.3891 \\ 0.0651 & -0.1174 \\ -7.4906 & -5.3891 \end{bmatrix} x \leq \begin{bmatrix} 1.3577 \\ -0.0357 \\ 2.6423 \end{bmatrix}$	$\begin{bmatrix} -7.4906 & -5.3891 \end{bmatrix} x - 0.6423$

of mp-QPs is an affine function of parameters (state-variables) which is valid in certain regions of optimality which are described by linear inequalities. The main advantage of this approach is that control actions are computed off-line. The on-line computation thus simply reduces to a function evaluation problem. Current work focusses on the extension of the algorithms for multi-parametric mixed-integer programs [3] for hybrid control problems [8].

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