PREDICTIVE CONTROL VIA SET-MEMBERSHIP STATE ESTIMATION FOR CONSTRAINED LINEAR SYSTEMS WITH DISTURBANCES

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Abstract

This paper combines predictive control and set-membership state estimation techniques, for input/state hard constraints fulfilment. Linear systems with unknown but bounded disturbances and partial state information are considered. The adopted worst-case approach guarantees that the constraints are satisfied for all the states which are compatible with the available information and for all the disturbances within the given bounds. A stability result and simulative studies are reported.

1 Introduction

Two features frequently arise in many practical control problems: the necessity of satisfying input/state constraints and the presence of disturbances. In recent years, several control techniques have been developed which are able to handle hard constraints, see e.g. [1]. In particular, in the last decades industry has been attracted by predictive controllers [2]-[5]. These approaches are based on the so called *receding horizon* strategy. This consists in determining a *virtual control input sequence* that optimizes an open-loop *performance function*, according to a prediction of the system evolution over a semi-infinite prediction horizon. Then, the sequence is actually applied to the system, until another sequence based on more recent data is newly computed. The involved prediction depends on the current state, the future state disturbances, and the selected control input. Several strategies, which have been developed for deterministic frameworks [6]-[9] can be applied by neglecting the presence of the state disturbances over the prediction horizon. However, this does not guarantee that state related constraints are actually satisfied. More recently, [10]-[12] and [13] have indipendently developed computationally efficient techniques for solving constrained problems, by manipulating the reference trajectory. In particular, in [13] constraint fulfilment is also guaranteed in the presence of state disturbances. However, these techniques require full state measurements. When these are not available, it is common practice to provide the predictor with an estimate generated by a state observer, *e.g.* a Kalman filter, but again, no guarantee of constraint fulfilment holds.

This paper copes with both the presence of state disturbances and full state information unavailability. We assume that the uncertainties acting on the system (state disturbances and output noises) are unknown but bounded. We adopt a worst-case approach, which entails in: *i)* considering the effect of the worst state disturbance sequence over the prediction horizon; *ii)* handling state unavailability by using the so-called *set-membership* state estimation [14]-[15]. This considers the *state uncertainty set*, i.e. the set of all the state vectors compatible with the model equations, the initial uncertainty, the disturbance bounds and the available output measurements. Due to the tremendous amount of calculations required by the updating of the state uncertainty sets, many recursive approximation algorithms, based on simple regions in the state space, like ellipsoids [14]-[17], or limited complexity polytopes [18], have been proposed in the literature. In this paper, we adopt the minimum volume parallelotopic approximation developed in [19]-[20]. The resulting set-membership estimation algorithm is particularly appealing for predictive control, as it presents both good approximation capabilities and reasonable computational complexity.

2 Problem Formulation and Assumptions

Consider the following linear discrete-time time-invariant system

$$\left\{\begin{array}{rcl} x(t+1) & = & Ax(t)+Bu(t)+\xi(t) \\ y(t) & = & Cx(t)+Du(t)+\zeta(t) \\ c(t) & = & Gx(t)+Hu(t) \end{array}\right. \quad (1)$$

along with a desired output reference $r(t) \in \mathbb{R}^p$, where the state $x(t) \in \mathbb{R}^n$ is supposed to be not directly measurable, $u(t) \in \mathbb{R}^m$ is the control input, $y(t) \in \mathbb{R}^p$ the measured output, $\xi(t) \in \mathbb{R}^n$ the state disturbance, $\zeta(t) \in \mathbb{R}^p$ the output noise, and $t \in \mathbb{Z}_+ = \{0, 1, \ldots\}$. We assume that both $\xi(t)$ and $\zeta(t)$ are unknown but bounded

$$\xi_i^- \leq \xi_i(t) \leq \xi_i^+, \qquad i = 1, \ldots, n \tag{2}$$

$$\zeta_i^- \leq \zeta_i(t) \leq \zeta_i^+, \qquad i = 1, \ldots, p \tag{3}$$

or, in a more compact form, $\xi(t) \in \Xi$, $\zeta(t) \in \mathcal{Z}$, $\forall t \in \mathbb{Z}_+$, and Ξ and $\mathcal{Z}$ are given . We assume that system (1) satisfies the following

Assumption 1 *A is asymptotically stable.*

This assumption is not restrictive, since frequently (1) represents a precompensated feedback system.

The problem is to generate the control input $u(t)$ so as to constrain the vector $c(t) \in \mathbb{R}^l$

$$c(t) \in \mathcal{C}, \tag{4}$$

where $\mathcal{C}$ is the convex polyhedron

$$\mathcal{C} = \left\{c \in \mathbb{R}^l : \; A_c c \leq B_c\right\}, \; B_c \in \mathbb{R}^q, \tag{5}$$

without affecting the original tracking properties of the system. W.l.o.g., we consider

$$c(t) = \begin{bmatrix} x(t) \\ u(t) \end{bmatrix},$$

and rewrite (5) as

$$\mathcal{C} = \left\{[x' \; u']' \in \mathbb{R}^{n+m} : \; A_c^x x + A_c^u u \leq B_c\right\} \tag{6}$$

where $'$ denotes transposition. Hereafter, we shall assume that

Assumption 2 *$\mathcal{C}$ is bounded.*

According to the above setting, at a generic time t the available information on the state vector $x(t)$ is given by the model equation (1), the bounds on the state disturbances (2) and the output noise (3), and the observed measurements $y(k)$, $k = 0, 1, \ldots, t$. Let us denote by $\mathcal{X}^*(t_1|t_2)$ the *state uncertainty set* of all state vectors at time t_1, compatible with the information available at time t_2. If the *a priori information set* $\mathcal{X}^*(0|-1)$ is bounded and contains the initial state $x(0)$, then the state uncertainty sets are provided by the following recursion

$$\begin{aligned} \mathcal{X}^*(t|t) &= \mathcal{X}^*(t|t-1) \bigcap \mathcal{X}_y^*(t) \\ \mathcal{X}^*(t+1|t) &= A\mathcal{X}^*(t|t) \oplus \{Bu(t)\} \oplus \Xi \end{aligned}$$

where

$$\mathcal{X}_y^*(t) = \{x \in \mathbb{R}^n : \; y(t) - Cx - Du(t) \in \mathcal{Z}\}$$

is the set of states compatible with the single measurement $y(t)$, and $\oplus$ denotes the vector sum of sets. It is clear that the complexity of $\mathcal{X}^*(t|t)$ and $\mathcal{X}^*(t+1|t)$ grows with t, and therefore it is common practice to approximate these sets by simpler regions, the so-called *set-valued estimates* $\mathcal{X}(t|t)$ and $\mathcal{X}(t|t+1)$ respectively. In the following, the set-valued estimate involved in the optimization procedure at time t will be $\mathcal{X}(t|t-1)$. This means that the input at time t will be computed on the basis of the available information up to time $t-1$, so that the required computations can be performed over one full sample interval.

In order to use efficient optimization procedures, we adopt the strategy proposed in [9] by limiting to N_v the number of control degrees of freedom. This entails in coping with a finite dimensional optimization vector

$$\mathcal{V} := \begin{bmatrix} v(N_v - 1) \\ \vdots \\ v(0) \end{bmatrix} \in \mathbb{R}^{N_v m}$$

and then considering, as virtual control input sequence, its *constant extension* over a semi-infinite horizon

$$S_{\mathcal{V}} := \{v(k)\}_{k=0}^{\infty},$$

which is obtained by setting

$$v(k) \equiv v(N_v - 1), \forall k \geq N_v - 1, \; k \in \mathbb{Z}_+ \, . \tag{7}$$

In the following, we shall indicate with $c(t+k, x, \mathcal{V}, \mathcal{K}_k)$ the constrained vector at time $t+k$, predicted at time t, according to model (1), initial state $x \in \mathcal{X}(t|t-1)$, future state disturbances $\mathcal{K}_k \in \Xi_k$,

$$\begin{aligned} \mathcal{K}_k &:= \begin{bmatrix} \xi(t+k-1) \\ \vdots \\ \xi(t) \end{bmatrix} \\ \Xi_k &:= \left\{\mathcal{K}_k \in \mathbb{R}^{kn} : \; \mathcal{K}_{k,i} \in [\xi_i^-, \xi_i^+]\right\}, \end{aligned}$$

and by setting $u(t+k) = v(k)$, $\forall k \in \mathbb{Z}_+$.

We wish to select a performance function $J(t, \mathcal{V})$ in such a way that the minimization of $J(t, \mathcal{V})$ w.r.t. $\mathcal{V}$ ensures good tracking properties. In the predictive control literature, it is common use to weight the sum of the predicted tracking error squares over a semi-infinite horizon, which makes the performance function depend on the current state $x(t)$. On the other hand, we have assumed that $x(t)$ is not available. In the present worst-case formulation, a possible solution consists in defining

$$\begin{aligned} J(t, \mathcal{V}) = & \max_{\substack{x \,\in\, \mathcal{X}(t|t-1) \\ \mathcal{K}_\infty \,\in\, \Xi_\infty}} \left\{ \sum_{k=0}^{\infty} \|y(t+k, x, \mathcal{V}, \mathcal{K}_\infty) \right. \\ & \left. -r(t)\|_{\Upsilon_y}^2 + \|\mathcal{V}\|_{\Upsilon_v}^2 \right\}, \end{aligned}$$

where $\Upsilon_y, \; \Upsilon_v > 0$, and $\|y\|_{\Upsilon_y}^2 = y'\Upsilon_y y$. However, a control law based on such a cost function would require the solution of a very complex constrained min-max optimization problem at each time step t. In order to sidestep these difficulties, we adopt the following function

$$J(t,\mathcal{V}) = \sum_{k=0}^{N_v-2} \|v(k)-v(N_v-1)\|^2_{\Upsilon_u} + \|C(I-A)^{-1}Bv(N_v-1)-r(t)\|^2_{\Upsilon_y}, \quad (8)$$

Notice that no feedback term is present in (8). Since as explained later the constraints involved in the minimization of (8) depend on the current set-valued state estimate $\mathcal{X}(t|t-1)$, feedback will be present only when the constraints are active. This should not be considered as a drawback, since noise and unmodeled dynamics effects rejection can be achieved by designing a precompensator and then labeling as (1) the resulting closed-loop system.

At each time t, the selection of the optimal vector $\mathcal{V}_t$ proceeds as follows. Denote by $\Omega(t)$ the set of all vectors $\mathcal{V}$ leading to feasible evolutions of the constrained vector,

$$\Omega(t) = \{\mathcal{V}\in\mathbb{R}^{N_v m}: \ c(t+k,x,\mathcal{V},\mathcal{K}_k)\in\mathcal{C}, \ \forall x\in\mathcal{X}(t|t-1), \ \forall\mathcal{K}_k\in\Xi_k, \ \forall k\in\mathbb{Z}_+\}. \quad (9)$$

If $\Omega(t)$ is nonempty, define

$$\mathcal{V}_t^* = \arg\min_{\mathcal{V}\in\Omega(t)} J(t,\mathcal{V}) \quad (10)$$

Then, denoting by $\mathcal{V}_t^1$ the extension of the previous optimal vector $\mathcal{V}_{t-1}$, i.e.

$$\mathcal{V}_{t-1} = \begin{bmatrix} v_{t-1}(N_v-1) \\ v_{t-1}(N_v-2) \\ \vdots \\ v_{t-1}(1) \\ v_{t-1}(0) \end{bmatrix}, \ \mathcal{V}_t^1 = \begin{bmatrix} v_{t-1}(N_v-1) \\ v_{t-1}(N_v-1) \\ v_{t-1}(N_v-2) \\ \vdots \\ v_{t-1}(1) \end{bmatrix},$$

we set

$$\mathcal{V}_t = \begin{cases} \mathcal{V}_t^* & \text{if } \Omega(t)\neq\emptyset \text{ and } J(t,\mathcal{V}_t^*) < J(t,\mathcal{V}_t^1)-\epsilon \\ \mathcal{V}_t^1 & \text{otherwise} \end{cases} \quad (11)$$

where $\epsilon = \min\{\rho_1 J(t,\mathcal{V}_t^1),\rho_2)$, and ρ_1, ρ_2 are fixed arbitrarily small scalars. Then, according to the receding horizon strategy described above, we set

$$u(t) = v_t(0). \quad (12)$$

The entire procedure is then repeated at time $t+1$.

Finally, in order to complete the above scheme, we make the following hypothesis on $\mathcal{X}(0|-1)$.

Assumption 3 *For the a priori information set $\mathcal{X}(0|-1)$ there exists a finite input sequence $\mathcal{V}_{-1}$ such that $\mathcal{V}_{-1}^1\in\Omega(0)$.*

3 Main Results

The optimization problem (10) involves an *infinite* number of linear constraints. However, in order to be able to computationally solve (10) via standard quadratic programming tools, a *finite* number of constraints is desiderable. Next Theorem 1 shows that this can be achieved by adding an extra linear constraint on $\mathcal{V}$.

Theorem 1 *There exist an index $k_o\geq N_v$ and $\delta>0$ such that, if $\mathcal{V}$ satisfies*

$$\left[A_c^x(I-A)^{-1}B + A_c^u\right]v(N_v-1) \leq B_c - \delta\,\underline{1} \quad (13)$$

where $\underline{1}=[1,\ldots,1]'$, then $\mathcal{V}\in\Omega(t)$ iff

$$c(t+k,x,\mathcal{V},\mathcal{K}_k)\in\mathcal{C}, \ \forall x\in\mathcal{X}(t|t-1), \ \forall\mathcal{K}_k\in\Xi_k, \ \forall k=0,\ldots,k_o, \quad (14)$$

Proof. W.l.o.g. set $t=0$. Let $k\geq N_v$, $x\in\mathcal{X}(0|-1)$, and consider $\mathcal{V}$ such that (13) is satisfied, and $c(h,x,\mathcal{V},\mathcal{K}_h)\in\mathcal{C}$, $\forall h=0,\ldots,N_v$. Consider the prediction of the state at time k

$$x(k,x,\mathcal{V},\mathcal{K}_k) = A^{k-N_v}x(N_v,x,\mathcal{V},\mathcal{K}_{N_v}) + \sum_{i=0}^{k-N_v-1}A^iBv(N_v-1) + \sum_{i=0}^{k-N_v-1}A^i\xi(k-1-i)$$

where

$$A_c^x x(N_v,x,\mathcal{V},\mathcal{K}_{N_v}) + A_c^u v(N_v-1) \leq B_c.$$

By Assumption 2, there exist constants $\Delta_{\mathcal{C}}^x$ and $\Delta_{\mathcal{C}}^u$ such that

$$\|x\|_\infty\leq\Delta_{\mathcal{C}}^x, \|u\|_\infty\leq\Delta_{\mathcal{C}}^u, \ \forall\begin{bmatrix} x \\ u\end{bmatrix}\in\mathcal{C}.$$

By letting M and λ such that

$$\|A^k\|_\infty \leq M\lambda^k,$$

where $\|A\|_\infty$ denotes the ∞-induced matrix norm, one gets

$$\begin{aligned}\|\sum_{i=0}^{k-N_v-1}A^i\xi(i)\|_\infty &\leq \sum_{i=0}^{k-N_v-1}M\lambda^i\|\xi(i)\|_\infty \\ &\leq \sum_{i=0}^{k-N_v-1}M\lambda^i\bar{\xi} \leq \frac{M}{1-\lambda}\bar{\xi}\end{aligned}$$

where $\bar{\xi}=\max_{i=1,\ldots,n}\{\max\{|\xi_i^-|,|\xi_i^+|\}\}$. Being

$$(I-A)^{-1}Bv(N_v-1) = \sum_{i=0}^{\infty}A^iBv(N_v-1)$$

it results, after some calculations

$$\begin{aligned}&\left\|A_c^x\left[x(k,x,\mathcal{V},\mathcal{K}_k)-(I-A)^{-1}Bv(N_v-1)\right]\right\|_\infty \\ &\leq M\|A_c^x\|_\infty\lambda^{k-N_v}(\Delta_{\mathcal{C}}^x + \|(I-A)^{-1}B\|_\infty\Delta_{\mathcal{C}}^u) \\ &+\frac{M\|A_c^x\|_\infty}{1-\lambda}\bar{\xi}\leq\delta,\end{aligned} \quad (15)$$

for

$$\begin{aligned} \delta &\geq \frac{2M\|A_c^x\|_\infty \bar{\xi}}{1-\lambda} && (16)\\ k &\geq k^* = N_v + \\ & \log_\lambda \frac{\delta}{2M\|A_c^x\|_\infty(\Delta_{\mathcal{C}}^x + \|(I-A)^{-1}B\|_\infty \Delta_{\mathcal{C}}^u)} && (17)\end{aligned}$$

Hence, by (13) and (15),

$$\begin{aligned} & A_c^x x(k,x,\mathcal{V},\mathcal{K}_k) + A_c^u v(k) \\ & \leq A_c^x \left[x(k,x,\mathcal{V},\mathcal{K}_k) - (I-A)^{-1}Bv(N_v-1) \right] \\ & + B_c - \delta \underline{1} \leq B_c \end{aligned}$$

or equivalently

$$\left[\begin{array}{c} x(k,x,\mathcal{V},\mathcal{K}_k) \\ v(k) \end{array} \right] \in \mathcal{C}, \ \forall k \geq k^*$$

Then, there exist integers $k_o \leq k^*$ such that (14) guarantees $\mathcal{V} \in \Omega(0)$. □

Notice that (13) imposes that the predicted steady-state constrained vector, corresponding to the constant input level $v(N_v-1)$, lies inside $\mathcal{C}$ by at least a fixed distance away from the border. Then, by virtue of Assumption 1, this implies that, after a finite transient, all the trajectories of the constrained vector will be inside $\mathcal{C}$. In the following we will assume that the set defined by the inequality (13), with δ as in (16), is nonempty, and hence the existence of feasible solutions $\mathcal{V}$ is allowed.

Next Theorem 2 describes the asymptotical behaviour of the overall control scheme.

Theorem 2 *Consider system (1) and a sequence of approximated state uncertainty sets* $\{X(t|t-1)\}_{t=0}^{\infty}$. *Let* $r(t) \equiv r$, $\forall t > t_r \in \mathbb{Z}_+$. *Then, the control strategy (10)-(12), based on the optimization of the performance function (8) in the presence of constraints (9) and (13), guarantees stability of the overall control loop.*

Proof. Omitted due to lack of space. □

4 Constrained Optimization Algorithm

In this section, we derive the solution of the constrained optimization problem posed in Section 2. The control algorithm must perform two main tasks:

i) updating the approximated state uncertainty set $\mathcal{X}(t|t-1)$;

ii) performing the constrained optimization (10) with the additional constraint (13).

In this paper, we will consider parallelotopes [19] as approximating regions for the state uncertainty sets.

Definition 1 *Let a nonsingular matrix* $T \in \mathbb{R}^{n\times n}$ *and a vector* $\hat{x} \in \mathbb{R}^n$ *be given. Then*

$$\mathcal{P}(T,\hat{x}) = \{x: \ x = \hat{x} + T\alpha, \|\alpha\|_\infty \leq 1\}$$

defines a parallelotope *in* $\mathbb{R}^n$, *with center* $\hat{x}$ *and edges parallel to the column vectors of* T.

Recently, a recursive algorithm for the outer approximation of the uncertainty state set of a linear system through parallelotopic regions has been proposed in [20]. The recursive approximation is computed according to a minimum volume criterion. At a generic time t, the following two steps are performed

- *measurement update*: given the parallelotope $\mathcal{X}(t|t-1) = \mathcal{P}(t-1)$, compute the minimum volume parallelotope $\overline{\mathcal{P}}$ outbounding $\mathcal{P}(t-1) \bigcap \mathcal{X}_y^*(t)$;

- *time update*: compute the minimum volume parallelotope $\mathcal{P}(t)$ outbounding $A\overline{\mathcal{P}} \oplus \{Bu(t)\} \oplus \Xi$ and set $\mathcal{X}(t+1|t) = \mathcal{P}(t)$.

The iterations above are initialized by setting $\mathcal{X}(0|-1)$ equal to the given a priori information set $\mathcal{X}^*(0|-1)$. The computational complexity of the algorithm has been proved to be polynomial in the state dimension n.

In order to solve the optimization problem (10), we need to express the set $\Omega(t)$ in (9) in terms of the optimization vector $\mathcal{V}$. By (6), the fulfilment of the constraints $c(t+k,x,\mathcal{V},\mathcal{K}_k) \in \mathcal{C}$, for every $x \in \mathcal{X}(t|t-1)$ and $\mathcal{K}_k \in \Xi_k$, over a finite horizon $k = 0,\ldots,k_o$, can be expressed as

$$\begin{aligned} & A_c^x x(t+k,x,\mathcal{V},\mathcal{K}_k) + A_c^u v(k) \leq B_c, && (18)\\ & \forall x \in \mathcal{X}(t|t-1), \ \forall \mathcal{K}_k \in \Xi_k, \ \forall k = 0,\ldots,k_o \end{aligned}$$

where

$$x(t+k,x,\mathcal{V},\mathcal{K}_k) = A^k x + R_k^v M\mathcal{V} + R_k^\xi \mathcal{K}_k \qquad (19)$$

and

$$\begin{aligned} R_k^v &= \left[\begin{array}{cccc} B & AB & \ldots & A^{k-1}B \end{array} \right], \\ R_k^\xi &= \left[\begin{array}{cccc} I_n & A & \ldots & A^{k-1} \end{array} \right], \\ M &= \left[\begin{array}{c|c} \begin{array}{c} I_m \\ \vdots \\ I_m \end{array} & 0_{m(k-N_v)\times m(N_v-1)} \\ \hline \multicolumn{2}{c}{I_{mN_v}} \end{array} \right]. \end{aligned}$$

According to (19), after some algebraic manipulations, (18) can be rewritten as

$$\begin{aligned} & \mathcal{A}^x x + \mathcal{A}^v \mathcal{V} + \mathcal{A}^\xi \mathcal{K}_{k_o} \leq \mathcal{B}, \ \mathcal{B} \in \mathbb{R}^h, && (20)\\ & \forall x \in \mathcal{X}(t|t-1), \ \forall \mathcal{K}_{k_o} \in \Xi_{k_o}, \end{aligned}$$

where $h = q(k_o+1)$, and $\mathcal{A}^x \in \mathbb{R}^{h\times n}$, $\mathcal{A}^v \in \mathbb{R}^{h\times mN_v}$, $\mathcal{A}^\xi \in \mathbb{R}^{h\times nk_o}$ are suitably defined matrices.

Next Lemma 1 shows how to express (20) as a set of linear inequalities on the optimization vector $\mathcal{V}$.

Lemma 1 *Let*

$$V = \{v \in \mathbb{R}^v : \; P_1 v \le P_2\}, \; P_1 \in \mathbb{R}^{h \times v}, \; P_2 \in \mathbb{R}^h$$

be bounded and nonempty. Denote by $[P]^i$ *the i-th row of* P *and by*

$$\max_{v\in V} Pv := \begin{bmatrix} \max_{v\in V} [P]^1 v \\ \vdots \\ \max_{v\in V} [P]^h v \end{bmatrix}.$$

Then, the following sets

$$\begin{aligned} \mathcal{D} &= \{w \in \mathbb{R}^w : \; P_3 v + P_4 w \le P_5, \; \forall v \in V\} \\ \overline{\mathcal{D}} &= \left\{w \in \mathbb{R}^w : \; P_4 w \le P_5 - \max_{v\in V} P_3 v\right\} \end{aligned}$$

with $P_3 \in \mathbb{R}^{k\times v}$, $P_4 \in \mathbb{R}^{k\times w}$, $P_5 \in \mathbb{R}^k$, *are equal.*

The above lemma proves that $\mathcal{V}$ satisfies the constraints (20) iff

$$\mathcal{A}^v \mathcal{V} \le \mathcal{B} - \max_{x\in \mathcal{X}(t|t-1)} \mathcal{A}^x x - \max_{\mathcal{K}_{k_o} \in \Xi_{k_o}} \mathcal{A}^\xi \mathcal{K}_{k_o}. \tag{21}$$

Notice that the second term in the RHS of (21) depends on the current approximated state uncertainty set $\mathcal{X}(t|t-1)$, and hence it provides feedback from new output measurements. On the other hand, the third term can be computed off-line. Therefore, at each time instant t one has to solve h linear programming problems in order to compute the second term in the RHS of (21). Then, the optimum $\mathcal{V}$ can be obtained by solving a quadratic programming problem with cost (8) and linear constraints (13) and (21).

5 Feasibility and Set-Membership State Estimation

In this section, we study the conditions which have to be fulfilled by the approximated state uncertainty set $\mathcal{X}(t|t-1)$ in order to guarantee *feasibility*. We distinguish between two different definitions.

Definition 2 *A vector* $\mathcal{V}$ *(and its constant extension* $S_\mathcal{V}$*) is said to be* virtually admissible at time t *if it fulfils constraints (13) and (18)* $\forall x \in \mathcal{X}(t|t-1)$, $\forall \mathcal{K}_k \in \Xi_k$, *and* $\forall k \in \mathbb{Z}_+$.

Definition 3 *A vector* $\mathcal{V}$ *(and its constant extension* $S_\mathcal{V}$*) is said to be* actually admissible at time t *if, by applying the command input* $\{u(t+k)\}_{k=0}^\infty = S_\mathcal{V}$ *to system (1), the corresponding evolution of the constrained vector satisfies* $c(t+k) \in \mathcal{C}$, $\forall k \in \mathbb{Z}_+$.

It is worth pointing out the difference between virtual and actual admissibility. Whilst virtual admissibility is an analytical property of vector $\mathcal{V}$, actual admissibility depends a posteriori on the specific state $x(t)$ and disturbance sequence realization $\{\xi(k)\}_{k=t}^\infty$. Intuitively, if the approximated uncertainty state set $\mathcal{X}(t|t-1)$ is too small, it can happen that the actual state vector $x(t) \notin \mathcal{X}(t|t-1)$, and hence an input $\mathcal{V}$ is virtually but not actually admissible. Conversely, when $\mathcal{X}(t|t-1)$ is too large, it may not exist a vector $\mathcal{V}$ that satisfies (18) for every $x \in \mathcal{X}(t|t-1)$. However, for the particular experiment, actually admissible vectors $\mathcal{V}$ may exist. The result presented in this section provide a theoretical ground for the intuitions above.

As the next theorem points out, the relationship between the true state uncertainty set $\mathcal{X}^*(t|t-1)$ and its approximation $\mathcal{X}(t|t-1)$ is a key factor for guaranteeing actual admissibility.

Theorem 3 *Suppose that* $\mathcal{V}$ *is virtually admissible at time* t. *Then*

i) $\mathcal{X}^*(t|t-1) \subseteq \mathcal{X}(t|t-1) \implies S_\mathcal{V}^j = \{v(k+j)\}_{k=0}^\infty$, *is actually admissible at each time* $t+j$, $j \in \mathbb{Z}_+$.

ii) $\mathcal{X}^*(t|t-1) \not\subseteq \mathcal{X}(t|t-1) \implies S_\mathcal{V}$ *is not guaranteed to be actually admissible at time* t.

Proof. Omitted due to lack of space. □

An immediate consequence of the above theorem is that outer approximations of the state uncertainty set must be chosen in order to guarantee actual admissibility. In fact, actual admissibility of $\mathcal{V}(t)$, $\forall t \in \mathbb{Z}_+$, is equivalent to fulfil constraint (4). Then, an important consequence of the previous result is the following.

Corollary 1 *If* $\mathcal{X}(t|t-1) \supseteq \mathcal{X}^*(t|t-1)$, $\forall t \in \mathbb{Z}_+$, *the control strategy (8)-(13) guarantees that* $c(t) \in \mathcal{C}$, $\forall t \in \mathbb{Z}_+$.

Notice that the parallelotopic approach adopted in the present paper provides an outer state uncertainty set approximation, and hence the previous corollary holds.

6 Simulation Results

The proposed control strategy has been investigated by simulations on the following second order discrete time SISO system

$$\begin{aligned} x(t+1) &= \begin{bmatrix} 1.6463 & -0.7866 \\ 1 & 0 \end{bmatrix} x(t) + \\ & \quad \begin{bmatrix} 1 & 0 \end{bmatrix}' u(t) + \xi(t) \\ y(t) &= \begin{bmatrix} 0.1404 & 0 \end{bmatrix} x(t) + \zeta(t) \\ c(t) &= \begin{bmatrix} -1.9313 & 2.2121 \end{bmatrix} x(t), \end{aligned} \tag{22}$$

whose y- and c-step responses are depicted in Fig. 1 (dashed lines).

The transfer function from the input u to the constrained variable c is underdumped and nonminimum phase. In order to compress the dynamics of c within the range

$$\mathcal{C} = [-1, 3],$$

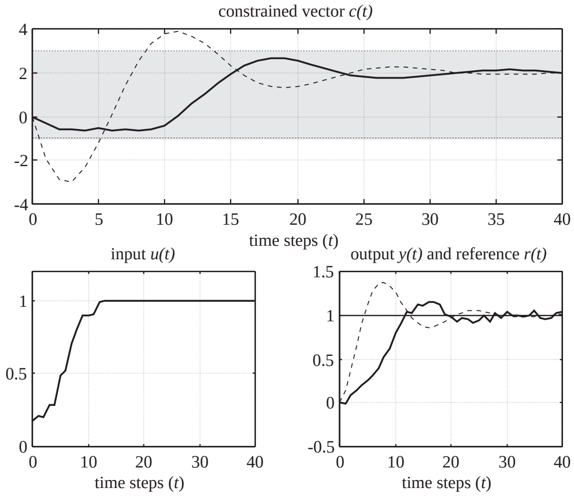

Figure 1: Closed loop behaviour (thick lines) and unconstrained response (dashed lines) for $r(t) \equiv 1$.

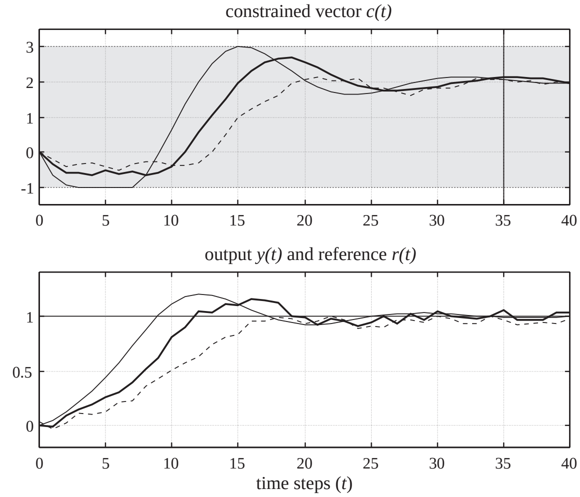

Figure 2: Effect of different disturbance intensities: no disturbance and known initial state $x(0)$ (thin line); $\|\xi(t)\|_\infty \leq 0.01$ (thick line); $\|\xi(t)\|_\infty \leq 0.04$ (dashed line).

and make the output y track the constant reference $r(t) \equiv 1$, we adopt the control law (8)-(13) along with the parameters $\Upsilon_u = 1$, $\Upsilon_y = 0.1$, $N_v = 2$, $\rho_1, \rho_2 \approx 0$. Also, we set $\delta \approx 0$ and $k_o = 16$, which have shown to guarantee a good constraint fulfilment, even if the conservative bounds (16)-(17) may not be satisfied. Fig. 1 shows the resulting trajectories (solid lines) when system (22) is affected by independent randomly generated disturbances $\|\xi(t)\|_\infty \leq 0.01$ and $|\zeta(t)| \leq 0.05$, for the a priori information set $\mathcal{X}(0|-1) = 0.25 \cdot [-1, 1] \times [-1, 1]$. Notice that the constraints are fulfilled at the price of a slower output response.

In Fig. 2 the effect of different state disturbance bounds is investigated. Due to the adopted worst-case approach, it results that as the size of the disturbance increases, the constraints are fulfilled in a more conservative way, and the output dynamics gets slower.

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