NONLINEAR PREDICTIVE REFERENCE FILTERING FOR CONSTRAINED TRACKING

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Abstract

This paper presents a new methodology for solving control problems where hard contraints on the state and/or the inputs of the system are present. This is achieved by adding to the control architecture a command governor which prefilters the reference to be tracked, taking into account the current value of the state and aiming at optimizing a tracking performance index. The overall system is proved to be asymptotically stable, and feasibility is ensured by a weak condition on the initial state. Though the method can be applied in principle to both nonlinear and linear loops, a complete solution is developed for the latter. The resulting on-line computational burden turns out to be moderate and the related operations executable with current low-priced computing hardware.

1 Introduction

In recent years there have been substantial theoretical advancements in the area of feedback control of dynamic systems with input and/or state-related constraints. For an account of related results see [1] which also includes relevant references. Amongst the various approaches, the developments of this paper are more akin to the receding-horizon or predictive control methodology [2]-[9]. Predictive control, wherein the receding horizon philosophy is used, selects the control action by also taking into account the future evolution of the reference. Such an evolution can be: known in advance as in applications where repetitive tasks are executed, e.g. industrial robots; predicted if a dynamic model for the reference is given; or designed in real time. As a matter of fact, the last instance is a peculiar and important potential feature of predictive control. In fact, taking into account the current value of both the state vector and the reference, a *virtual* reference evolution can be designed on line so as to ensure that the corresponding input and state responses be admissible. We point out the relevance of such an approach, being the feasibility issue one of the most important problems in predictive control. In most cases, predictive control computations require the numerical solution of a convex quadratic programming (QP) problem, which is computationally formidable if, as in predictive control, on-line solutions are required. In order to lighten computations, it would be thus important to know whether and when it is possible to borrow from predictive control the concept of on-line reference management so as to tackle constrained control problems without the computational burden intrinsic to predictive control. The main goal of the present paper is to address this issue. As anticipated, in this direction there are no contributions with the only exception of [7]-[9]. However, the problem of on-line modifying the reference in such a way that a compensated control system can operate within its linear dynamic range with no constraint violation has been recently addressed outside the predictive control realm [10]-[12]. The relationships between these and the approach of the present paper, which improves on [7]-[8], will be pointed out in due course.

2 Problem Formulation

Consider the discrete-time linear time-invariant system

$$\begin{cases} x(t+1) &= \Phi x(t) + Gv(t) \\ y(t) &= Hx(t) \\ c(t) &= H_c x(t) + Dv(t) \end{cases} \qquad (1)$$

where: $t \in \mathbb{Z}_+ := \{0, 1, \ldots\}$; $x(t) \in \mathbb{R}^n$ is the state-vector; $y(t) \in \mathbb{R}^p$ the output that is desired to be close to the *set-point* trajectory $w(t) \in \mathbb{R}^p$; $v(t) \in \mathbb{R}^p$ the *command input*; and $c(t) \in \mathbb{R}^q$ a vector which is required to belong to a specified *constraint* set $\mathcal{C} \subset \mathbb{R}^q$

$$c(t) \in \mathcal{C},\ \forall t \in \mathbb{Z}_+ \qquad (2)$$

The problem that we wish to study is how to choose the command sequence $v(\cdot) := \{v(t)\}_{t=0}^{\infty}$ with

$$v(t) = v(x(t), w(t)) \qquad (3)$$

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so that $y(\cdot)$ can be possibly close to the set-point sequence $w(\cdot)$ while $c(\cdot)$ fulfills the constraints (2). The transformation (3) is referred to as the *command governor* (CG). Eq. (1) can represent a linear time-invariant system under state-feedback. We shall assume that (1) is asymptotically stable and that there exists a non empty set $W \subset \mathbb{R}^p$ such that, $\forall w \in W$, there is an equilibrium state $x_w := (I - \Phi)^{-1} G w$ which fulfills the constraints $c_w := H_c x_w + D w \in \mathcal{C}$, and yields zero offset, $y_w := H x_w = w$.

3 Command Governor

Consider the pair state/set point $(x(t), w(t))$ at time t. Introduce a *virtual command* trajectory $v(\cdot|t, \mu) := \{v(t+k|t, \mu)\}_{k=0}^{\infty}$ where

$$v(t+k|t, \mu) := \Gamma^k \mu + w(t), \ k \in \mathbb{Z}_+, \tag{4}$$

$$\Gamma = \text{Diag}\{\gamma_1, \gamma_2, \dots, \gamma_p\}, \quad \gamma_i \in (0,1), \ i = 1, \dots, p, \tag{5}$$

and $\mu \in \mathbb{R}^p$ is a vector to be suitably chosen in order to possibly drive the system state to $x_{w(t)}$ with no constraint violation as $k \to \infty$. The idea is that if $x(t) = x_{\bar{w}}$, $\bar{w}$, $w(t) \in W$, $\mu = \bar{w} - w(t)$, and $\Gamma \approx I$, (4) defines a monotonically slowly-varying sequence and exponentially approaching $w(t)$ from $\bar{w}$. In this way, one can compress the dynamic range of $c(t)$ in order to possibly satisfy the prescribed constraints. In fact, taking an arbitrarily small $\delta > 0$ and defining

$$W_\delta := \{w \in W \mid c_w + \tilde{c} \in \mathcal{C}, \ \forall \|\tilde{c}\| \le \delta\} \tag{6}$$

which is supposed to be non empty, the following result, whose proof is given in [9], can be stated:

Lemma 1 *Consider the system (1) with $\mathcal{C}$ convex and W bounded. Then, given any pair of set points $\bar{w}$ and $w(t)$, $\bar{w}, w(t) \in W_\delta$, (4) drives (1) from an equilibrium state $x(t) = x_{\bar{w}}$ to the equilibrium state $x_{w(t)}$ with no constraint violation by setting $\mu = \bar{w} - w(t)$ and $\gamma_i \in (\gamma_\delta, 1)$, $\forall i = 0, \dots, p$, where $\gamma_\delta > 0$ depends on W, δ, and the system (1).*

From now on, we will take

$$\gamma_i \in (\gamma_\delta, 1), \ i = 1, \dots, p. \tag{7}$$

Define now the quadratic *command selection index*

$$J(x(t), w(t), v(\cdot|t, \mu)) := \|v(t|t, \mu) - w(t)\|^2_{\psi_v} + \sum_{k=0}^{\infty} \|y(t+k|t, \mu) - w(t)\|^2_{\psi_y} \tag{8}$$

with $\|x\|^2_\psi := x'\psi x$, $\psi_v > 0$ diagonal, $\psi_y = \psi_y' > 0$, and $y(t+k|t, \mu)$ the output response of (1) at $t+k$ from the state $x(t)$ at t to the virtual command (4). The sum in (8) accounts for the tracking performance, the first term guarantees internal stability, as will be proved later. Assuming that the minimum exists, set

$$\mu_t := \arg\min_{\mu \in \mathbb{R}^p} \{J(x(t), w(t), v(\cdot|t, \mu)) | \ c(\cdot|t, \mu) \subset \mathcal{C}\} \tag{9}$$

If such a μ_t exists, we say that $(x(t), w(t))$ is *admissible*, and the problem *feasible*. We next study the consequences of (9) under the assumption that the set point is kept constant, $w(t) \equiv w$ and the problem is initially feasible. To this end we resort to a Lyapunov function argument. Look first at

$$V(x(t)) := \min_{\mu \in \mathbb{R}^p} \{J(x(t), w, v(\cdot|t, \mu)) | \ c(\cdot|t, \mu) \subset \mathcal{C}\} \tag{10}$$

Let $v(t) := v(t|t, \mu_t) = \mu_t + w$ be the *actual* command used in (1). Noting that by (4) $v(t+1+k|t, \mu) = v(t+1+k|t+1, \Gamma\mu)$, $\forall k \in \mathbb{Z}_+$, and assuming that the minimizer exists and equals μ_t, (10) becomes

$$V(x(t)) = \|\mu_t\|^2_{\bar{\psi}_v} + \|y(t) - w\|^2_{\psi_y} + J(x(t+1), w, v(\cdot|t+1, \Gamma\mu_t)) \tag{11}$$

where $\bar{\psi}_v := \psi_v - \Gamma'\psi_v\Gamma$. Taking into account that

$$\begin{aligned} J(x(t+1), w, v(\cdot|t+1, \Gamma\mu_t)) &\ge \\ \min_{\mu \in \mathbb{R}^p} \{J(x(t+1), w, v(\cdot|t+1, \mu)) &| c(\cdot|t+1, \mu) \subset \mathcal{C}\} \\ = V(x(t+1)) \end{aligned}$$

we find that along the trajectories of the system

$$V(x(t)) - V(x(t+1)) \ge \|\mu_t\|^2_{\bar{\psi}_v} + \|y(t) - w\|^2_{\psi_y} \tag{12}$$

Hence $V(x(t))$, being positive and monotonically nonincreasing, converges as $t \to \infty$. Summing both sides of (12) from t to ∞ we find

$$\sum_{i=t}^{\infty} \left[\|\mu_i\|^2_{\bar{\psi}_v} + \|y(i) - w\|^2_{\psi_y} \right] < \infty.$$

Proposition 1 *Consider the system (1) along with the command governor (4), (5), (7), (9). Let the set point at and after time t be constant and equal to $w \in W$, and (1) be fed by $v(i) = \mu_i + w$, $\forall i \ge t$. Suppose that the pair $(x(t), w)$ be admissible, the minimizer μ_i, $i \ge t$, exist, and $\bar{\psi}_v$ be positive definite. Then the overall system results in an asymptotically stable behaviour in that*

$$c(i) \in \mathcal{C}, \ \forall i \ge t \tag{13}$$

and

$$\lim_{i \to \infty} y(i) = w \tag{14}$$

$$\lim_{i \to \infty} v(i) = w \tag{15}$$

at a rate faster than $1/i^{\frac{1}{2}}$.

Notice that (15) is the same as $\lim_{i \to \infty} \mu_i = 0$, and implies $\lim_{i \to \infty} x(i) = x_w$.

Remark 1 Because the argument used to prove Prop. 1 does not involve the linearity of (1), stabilizing properties could be achieved as well when the CG is applied to nonlinear systems.

In order to proceed further, we introduce some extra notation. We denote by $c(\cdot, x, \mu, w)$ the c-variable response from state x and command $v(k) = \Gamma^k \mu + w$. Then

$$\mathcal{M}(t) := \{\mu \in \mathbb{R}^p : \ c(\cdot, x(t), \mu, w(t)) \subset \mathcal{C}\} \tag{16}$$

will be referred to as the *admissible set*. We next specify the command governor that will be considered from now on:

Let Γ be as in (5), (7). Then at each $t \in \mathbb{Z}_+$ define a *virtual command* of the form

$$v(t+k|t) = \begin{cases} \Gamma^k \mu_t + w(t), & \mathcal{M}(t) \text{ is non empty} \\ v(t+k|t-1, \mu_{t-1}), & \text{otherwise} \end{cases} \tag{17a}$$

with μ_t chosen in accordance with (9), and set

$$v(t) = v(x(t), w(t)) = v(t|t) \tag{17b}$$

The rationale for using the command governor logic (17) stems from Proposition 1 along with the following considerations. Suppose that $(x(0), w(0))$ be admissible. Hence, $\mathcal{M}(0)$ is non empty, and $v(k|0, \mu_0) = \Gamma^k \mu_0 + w(0)$ for some $\mu_0 \in \mathcal{M}(0)$. Then, $v(0) = v(0|0, \mu_0) = \mu_0 + w(0)$ is applied to system (1). At $t = 1$, if it happens that $\mathcal{M}(1)$ is non empty, we compute and apply $v(1) = v(1|1, \mu_1) = \mu_1 + w(1)$. If on the contrary $(x(1), w(1))$ is not admissible, $v(1) = v(1|0, \mu_0)$ by definition of $v(k|0, \mu_0)$ results in an admissible command input in that the constraint $c(1) \in \mathcal{C}$ is not violated. Moreover, $v(1|0, \mu_0)$ brings the state to $x(2)$ for which $v(2) = v(2|0, \mu_0)$ is an admissible command input. Thus, if we adopt the CG logic (17), the condition $(x(0), w(0))$ admissible ensures constraint satisfaction at all future times. The other important issue is the tracking performance achievable by (17). Next theorem, whose proof exploits Lemma 1 and Proposition 1, shows that (17) yields desirable asymptotic performance properties provided that the set-points be restricted to W_δ.

Theorem 1 (Conditional stability and offset-free behaviour) *Consider the system (1) along with the command governor (17) with γ_δ replaced by $\gamma^- := \gamma_{\delta-\eta}$, η being a arbitrarily small positive number. Let: the initial state $x(0)$ at time 0 be admissible for some virtual command sequence $v(t|-1, \mu_{-1}) = \Gamma^k \mu_{-1} + w(-1)$, $w(-1) \in W_\delta$; the set-point sequence be such that $w(t) \in W_\delta$, $\forall t \in \mathbb{Z}_+$; and $w(t) = w$, $\forall t \geq \bar{t} \geq 0$. Then, the overall system results in an offset-free asymptotically stable behaviour in that*

$$c(i) \in \mathcal{C}, \ \forall i \geq t \tag{18}$$

and

$$\lim_{t\to\infty} y(t) = w \tag{19}$$

$$\lim_{t\to\infty} v(t) = w \tag{20}$$

at a rate faster that $1/t^{\frac{1}{2}}$.

Proof. We show that there exists a finite time $\hat{t}$, $\hat{t} \geq \bar{t}$, at which $\mathcal{M}(\hat{t})$ is non empty, and, hence, (18)-(20) follow from Proposition 1. Suppose, by contradiction, that such a $\hat{t}$ does not exist. Then, $\forall t \geq \bar{t}$ $\mathcal{M}(t)$ is empty and $v(t) = v(t|\tau, \mu_\tau)$, where τ, $-1 \leq \tau < \bar{t}$, is the greatest integer at which an admissible virtual reference was determined. Now, $v(t|\tau, \mu_\tau) = \Gamma^{t-\tau} \mu_\tau + w(\tau)$, and consequently, as t increases, $v(t) \to w(\tau)$. Thus $\lim_{t\to\infty} x(t) = x_{w(\tau)}$. Consider the pair $(x_{w(\tau)}, w)$. By Lemma 1, there exists a virtual command which asymptotically drives the system state from $x_{w(\tau)}$ to x_w. This reference gives rise to an evolution for the c-variable of the form $c(k) = \bar{c}(k) + \tilde{c}_1(k)$ with $\bar{c}(k)$ corresponding to the steady-state c-response to the set point $\Gamma^k w(\tau) + (I - \Gamma^k) w \in W_\delta$, $\gamma^- < \Gamma_{ii} < 1$, $\forall i = 1, 2, \ldots, p$, and $\|\tilde{c}_1(k)\| \leq \delta - \eta$. Look now at the perturbed pair $(x_{w(\tau)} + \tilde{x}, w)$. The evolution of the c-variable from the perturbed state due to the previous virtual reference is given by $c(k) = \bar{c}(k) + \tilde{c}_1(k) + \tilde{c}_2(k)$, where $\tilde{c}_2(t)$ depends linearly on $\tilde{x}$. Then, there exists a positive ϵ such that $\|\tilde{c}_2(k)\| \leq \eta$ for all $\|\tilde{x}\| < \epsilon$, and hence $\|\tilde{c}_1(k) + \tilde{c}_2(k)\| \leq \delta$. Consequently, $(x_{w(\tau)} + \tilde{x}, w)$ is admissible for some $\tilde{x}$. Therefore, there exists a finite time at which $(x(t), w)$ is admissible, and this contradicts the assumption.

Remark 2 The hypothesis of Theorem 1 are fulfilled when $x(0) = x_{w(-1)}$ is an equilibrium state and $\mu(-1) = 0$.

4 Reduction to a finite constraint number and computations

We now concentrate on finding the analytical form of the command selection index (8) in terms of the vector μ for the system (1). Let $\xi_v(k)$ be the state of a linear system which generates $v(k)$, $\xi_v(k+1) = \Gamma \xi_v(k)$, $\xi_v(0) = \mu$, $v(k) = \xi_v(k) + w(t)$. Define $\tilde{x}(k) := x(t+k|t, \mu) - x_{w(t)}$. Recalling that $x_{w(t)} = (I - \Phi)^{-1} G w(t)$, then $\tilde{x}(k+1) = \Phi \tilde{x}(k) + G \xi_v(k)$, and $\epsilon(t+k|t, \mu) := y(t+k|t, \mu) - w(t) = H \tilde{x}(k)$. Defining $\xi(k) := [\xi_v'(k) \ \tilde{x}'(k)]'$ and

$$A := \begin{bmatrix} \Gamma & 0 \\ G & \Phi \end{bmatrix}, \ C := \begin{bmatrix} 0_{p\times p} & H \end{bmatrix}$$

one has $\xi(k+1) = A\xi(k)$, $\epsilon(t+k|t, \mu) = C\xi(k)$, $\xi(0) = [\mu' \ \tilde{x}(0)']'$. Then, setting $J(\mu) := J(x(t), w(t), v(\cdot|t, \mu))$,

$$\begin{aligned} J(\mu) &= \|\mu\|^2_{\psi_v} + \sum_{k=0}^{\infty} \|CA^k \xi(0)\|^2_{\psi_y} \\ &= \mu' \psi_v \mu + \xi(0)' \mathcal{L} \xi(0) \end{aligned}$$

where $\mathcal{L} = \mathcal{L}'$ solves the Lyapunov equation

$$\mathcal{L} = A' \mathcal{L} A + C' \psi_y C \tag{21}$$

Then, denoting by $\mathcal{L}_{(i_1:i_2, j_1:j_2)}$ the submatrix of $\mathcal{L}$ obtained by extracting the entries $\mathcal{L}_{i,j}$ for $i_1 \leq i \leq i_2$ and $j_1 \leq j \leq j_2$, and setting $n := \dim x(t)$, we have

$$J(\mu) = \mu' A_J \mu + 2B_J' \mu + C_J$$

where

$$A_J \ = \ \psi_v + \mathcal{L}_{(1:p,1:p)} \geq \psi_v > 0$$

$$
\begin{aligned}
B_J &= \mathcal{L}_{(1:p,p+1:p+n)}[x(t) - x_{w(t)}] \\
C_J &= [x(t) - x_{w(t)}]'\mathcal{L}_{(p+1:p+n,p+1:p+n)}[x(t) - x_{w(t)}]
\end{aligned}
$$

Notice that if the constraints are non active, the minimizer equals $\mu_t = -A_J^{-1}B_J$. In this case the CG builds $v(t)$ as a linear combination of the desired trajectory $w(t)$ and the state.

Because we require that $c(t+k|t,\mu) \in \mathcal{C}$, $\forall k \in \mathbb{Z}_+$, one has to minimize the quadratic functional $J(\mu)$ with an infinite number of constraints. We shall transform this into a finite constraint problem by adopting the approach in [11] by proving that $\mathcal{M}(t)$ can be determined by a finite number of constraints.

Let $c_{w(t)} = H_c x_{w(t)} + Dw(t)$ be the steady-state value taken on by the c-vector corresponding to a constant command $v(t+k|t) \equiv w(t)$. The evolution $c(\cdot|t,\mu)$ over the prediction horizon can be written as $\xi(k+1) = A\xi(k)$, $c(t+k|t,\mu) - c_{w(t)} = C_c\xi(k)$, $\xi(0) = \left[\begin{array}{cc} \mu' & x(t)' - x'_{w(t)} \end{array}\right]'$, where $C_c := [D \; H_c]$. Hence

$$
c(t+k|t,\mu) = c_{w(t)} + C_c A^k \left(\left[\begin{array}{c} \mu \\ x(t) \end{array}\right] - \left[\begin{array}{c} 0_p \\ x_{w(t)} \end{array}\right] \right). \tag{22}
$$

Let us introduce the sets

$$
\mathcal{X}_k(w) := \left\{ \left[\begin{array}{c} \mu \\ x \end{array}\right] \in \mathbb{R}^{n+p} : \; c(h,x,w,\mu) \in \mathcal{C}, \forall h = 0 \dots k \right\}
$$

$$
\mathcal{X}_\infty(w) := \lim_{k\to\infty} \mathcal{X}_k.
$$

Note that $\mathcal{X}_\infty(w) \subseteq \mathcal{X}_{k+1}(w) \subseteq \mathcal{X}_k(w)$, and at least $[0 \; x'_w]' \in \mathcal{X}_\infty(w)$. Without loss of generality we assume hereafter that (Φ, H_c) is an observable pair. If this were not the case, the evolution $c(\cdot|t,\mu)$ could be evaluated with the observable quadruple resulting from a canonical observability decomposition. In the following we shall assume that in (5), (7)

$$
\gamma_i \neq \gamma_j, \; \forall i \neq j, \; i,j = 1,\dots,m \tag{23a}
$$
$$
H_c \,\mathrm{Adj}(\gamma_i I - \Phi) G_i \neq 0_q, \; \forall i = 1,\dots,m \tag{23b}
$$

where G_i is the i-th column of G. Then we can estabilish next Theorem 2, whose proof is given in [9].

Theorem 2 *For all $x(t) \in \mathbb{R}^n$ and for all $w(t) \in W_\delta$, $\mathcal{M}(t)$ can be determined by a finite number $\bar{k}$ of constraints.*

We are interested in determining the minimum k such that $\mathcal{X}_\infty(w) = \mathcal{X}_k(w)$. To this end, we can prove the following lemmas [9].

Lemma 2 *For all $w \in W_\delta$, if $\mathcal{X}_{\bar{k}}(w) = \mathcal{X}_{\bar{k}+1}(w)$ then $\mathcal{X}_\infty(w) = \mathcal{X}_{\bar{k}}(w)$*

From now on we shall assume that $\mathcal{C}$ is a set of the form

$$
\mathcal{C} = \{c \in \mathbb{R}^q : g_i(c) \leq 0, \; \forall i = 0,1,\dots,m\} \tag{24}
$$

with

$$
\begin{array}{ll}
\text{(i)} & \mathcal{C} \text{ bounded and convex} \\
\text{(ii)} & g_i : \mathbb{R}^q \to \mathbb{R} \text{ continuous}, \; \forall i = 0,\dots,m.
\end{array} \tag{25}
$$

Lemma 3 *Suppose $\mathcal{C}$ is defined as in (24)-(25). Then*

(i) $\mathcal{C}$ is compact
(ii) $\mathcal{N}_k$ is compact

where

$$
\mathcal{N}_k := \left\{ \left[\begin{array}{c} w \\ \mu \\ x \end{array}\right] \in \mathbb{R}^{n+2p} : \; w \in W_\delta, \left[\begin{array}{c} \mu \\ x \end{array}\right] \in \mathcal{X}_k(w) \right\}. \tag{26}
$$

The following algorithm [11] can be used to find the index

$$
k^o := \min_{k\geq 0}\{k | \; \mathcal{X}_k = \mathcal{X}_\infty\} \tag{27}
$$

Algorithm 1:

1. $k \leftarrow 0$
2. Let
$$
\mathcal{G}_j := \max_{\left[\begin{array}{ccc} w' & \mu' & x' \end{array}\right]'} \{g_j(c(k+1,x,\mu,w))\}
$$
$$
\text{subject to} \left\{ \begin{array}{l} w \in W_\delta \\ g_i(c(h,x,\mu,w)) \leq 0, \\ \;\forall h = 0,\dots,k, \; \forall i = 0,\dots,M \end{array} \right.
$$
3. If $\mathcal{G}_j \leq 0$, $\forall j = 0,\dots,M$, then let $k^o \leftarrow k$ and stop
4. $k \leftarrow k+1$
5. Go to step 2

This algorithm stops when the minimum k such that $\mathcal{X}_k(w) \subset \mathcal{X}_{k+1}(w)$, $\forall w \in W_\delta$, is found. Notice that $\mathcal{G}_j$ is the maximum of $g_j(c(k+1,x,\mu,w))$ over $\mathcal{N}_k$. Then Lemma 3 and the assumption that $g_j(c)$ are continuous prove that $\mathcal{G}_j$ is well defined.

Theorem 3 *Let $w(t) \in W_\delta$ and $x(t) \in \mathbb{R}^n$ be respectively the desired set-point and the system state at time t. Let $\mathcal{C}$ be given as in (24)-(25). Let k^o as in (27) be determined by Algorithm 1. If the vector μ fulfils the k^o+1 constraints*

$$
c(k, x(t), \mu, w(t)) \in \mathcal{C}, \; \forall k = 0,\dots,k_o \tag{28}
$$

then the virtual command $v(t+k|t,\mu) = \Gamma^k\mu + w(t)$ yields a c-evolution $c(\cdot|t,\mu) \subset \mathcal{C}$.

We have reduced a quadratic programming problem with an infinite number of constraints in one with a finite number of constraints. Notice that when $\mathcal{C}$ is a polytope the constraints become linear and k^o can be easily computed by standard optimization routines.

4.1 SISO plants with saturating actuators

Suppose that the plant is SISO ($p = 1$), c is the input of the plant ($q = 1$), and $\mathcal{C} = [C^-, C^+]$. If the dc-gain from the command input v to c $F_{cv}(1) \neq 0$, then W_δ is a closed interval whose extremes are $W^- := (C^- + \delta)/F_{cv}(1)$, $W^+ := (C^+ - \delta)/F_{cv}(1)$. Moreover (22) can be rewritten in the form

$$c(t+k|t,\mu) = c_{1k}(t) + c_{2k}\mu$$

where

$$\begin{aligned} c_{1k}(t) &:= c_{w(t)} + H_c\Phi^k\tilde{x}(t), \\ c_{2k} &:= D\Gamma^k + \sum_{i=0}^{k-1} H_c\Phi^i G\Gamma^{k-1-i} \end{aligned}$$

and Γ is scalar chosen as in (7), (23). The problem is solved in two stages. In the first stage we off-line find k^o via Algorithm 1. The second stage consists of computing on-line, at each time step t, the scalar μ_t yielding the constrained minimum of $J(\mu)$.

Off-line stage. The determination of k^o by Algorithm 1 would require two minimizations of a linear functional with linear constraints at each iteration, being

$$\begin{aligned} g_1(c(k,x,\mu,w)) &= c_w H_c\Phi^k(x - x_w) + c_{2k}\mu - C^+ \\ g_2(c(k,x,\mu,w)) &= -c_w H_c\Phi^k(x - x_w) - c_{2k}\mu + C^-. \end{aligned}$$

The problem can be symmetrized in order to speeding-up computations. Defined $\bar{W} := (W^+ + W^-)/2$, $\Delta W := (W^+ - W^-)/2$, and $\Delta C := (C^+ - C^-)/2$, it is easy to show that, if $F_{cv}(1) \neq 0$, the constraints (28) with $w(t) \in W_\delta$ are equivalent to $H_c\Phi^k(x - x_{\Delta w} - x_{\bar{w}}) + c_{2k}\mu + c_{\Delta w} \in [-\Delta C, \Delta C]$ with $\Delta w \in [-\Delta W, \Delta W]$.

On-line stage. Each constraint $c(t+k|t,\mu) \in \mathcal{C}$ can be easily rewritten as the linear constraint $\mu_{1k} \leq \mu \leq \mu_{2k}$, where

$$\mu_{1k} = \begin{cases} \frac{C^- - c_{1k}(t)}{c_{2k}} & \text{if} \quad c_{2k} > 0 \\ \frac{C^+ - c_{1k}(t)}{c_{2k}} & \text{if} \quad c_{2k} < 0 \\ -\infty & \text{if} \quad c_{2k} = 0, \text{ and } c_{1k}(t) \in \mathcal{C} \end{cases}$$

$$\mu_{2k} = \begin{cases} \frac{C^+ - c_{1k}(t)}{c_{2k}} & \text{if} \quad c_{2k} > 0 \\ \frac{C^- - c_{1k}(t)}{c_{2k}} & \text{if} \quad c_{2k} < 0 \\ +\infty & \text{if} \quad c_{2k} = 0, \text{ and } c_{1k}(t) \in \mathcal{C}. \end{cases}$$

Defining $\mu^- := \max_{k \leq k^o} \mu_{1k}$, $\mu^+ := \min_{k \leq k^o} \mu_{2k}$, and provided that $\mu^- \leq \mu^+$, (9) can be trivially solved by minimizing the parabola $J(\mu)$ over the closed interval $\mathcal{M}(t) = [\mu^-, \mu^+]$. If $\mu^- > \mu^+$, or $c_{2k} = 0$ and $H_c\Phi^k(x(t) - x_{w(t)}) \notin \mathcal{C}$, then no solution exists. In this case $v(t) = v(t|t-1, \mu_{t-1})$ as described earlier. From the computational point of view, the above algorithm appear to be comparable with [12], and much less demanding than those required by Receding Horizon controllers [2] and [6].

5 Simulation results

Example 1 The control strategy described in the previous section is used to control the AFTI-16 aircraft continuous-time model reported in [10]. An elevator and a flaperon are the two inputs of the plant. These inputs are subject to the physical constraints

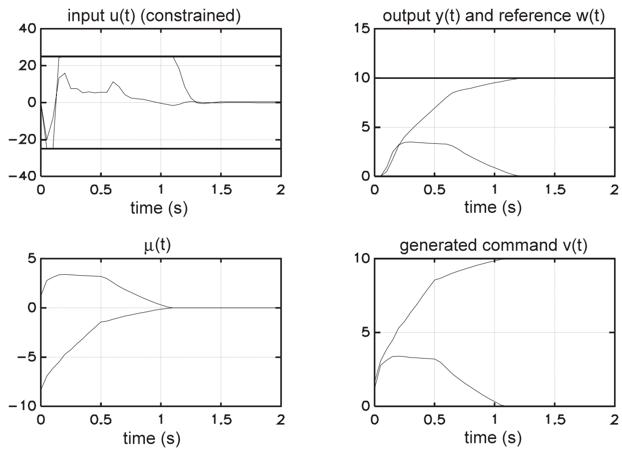

Fig. 4.1. Example 1. Response with the command governor.

$$|u_i(t)| \leq 25^{\text{o}}, \ \ i = 1, 2.$$

Then, the constrained vector $c(t)$ here equals $u(t)$. The angle of attack and the pitch angle form the output $y(t)$. The task is to have zero offset for step commands and avoid input saturations, which are liable to cause closed-loop instability. The continuous-time model is sampled every $T_s = .05s$ and a zero-order hold is used. For the resulting discrete-time model $x(t+1) = A_p x(t) + B_p u(t)$, $y(t) = C_p x(t)$, we find the LQ control law minimizing the cost $\sum_{k=0}^{\infty}[\|y(k)\|^2 + \|\Delta u(k)\|^2_{\psi_u}]$ with $\psi_u = \text{Diag}(10\bar{\rho}, \bar{\rho})$, $\bar{\rho} = .00005$. Input increments $\Delta u(k) = u(k) - u(k-1)$ are used in the cost so as to have zero offset in steady state. The underlying LQ control law is then $\Delta u(t) = G_1 u(t) + G_2 x(t)$, where

$$G_1 = \begin{bmatrix} 2.1661 & 0.1529 \\ 1.7823 & 1.5267 \end{bmatrix}$$

$$G_2 = \begin{bmatrix} 0.0000 & -2.5873 & -1.8325 & -14.4541 \\ 0.0006 & -47.0374 & -1.8538 & 23.6086 \end{bmatrix}$$

The reference w is premultiplied by the matrix

$$Z = \begin{bmatrix} -12.6314 & -4.0555 \\ -0.7915 & 35.5459 \end{bmatrix}$$

so as to have unity closed-loop dc-gain. By setting

$$\Phi := \begin{bmatrix} I_2 - G_1 & -G_2 \\ B_p & A_p \end{bmatrix}, \ G := \begin{bmatrix} Z \\ 0_{4\times 2} \end{bmatrix}$$

$$H := \begin{bmatrix} D_p & C_p \end{bmatrix}, \ H_c := \begin{bmatrix} I_2 & 0_{2\times 4} \end{bmatrix}, \ D := 0_{2\times 2}$$

and replacing x with the closed-loop state $[u' \ x']'$, Eq. (1) is obtained.

A desired set-point trajectory $w(t) \equiv [0 \ 10]'$ has been chosen. Fig. 1 depicts the trajectories that result when the CG ($\psi_v \approx 0_{2\times 2}$, $\psi_y = I_2$) is activated so as to constrain the two plant inputs within the prescribed saturation limits. With a Gaussian white sensor noise

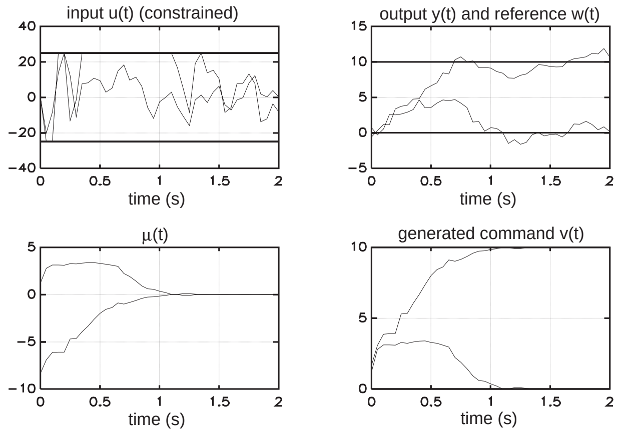

Fig. 5. Example 1. Response with the command governor and output measurement noise.

(with covariance I_2), and by evaluating the current state $x(t)$ with a Kalman filter, the results shown in Fig. 2 were found. The system behaviour has been simulated in $3.9s$ with Simulink 1.2 on a 486DX2/66 computer, using Matlab 3.5 standard quadratic programming routines. We chose $\Gamma_{11} = 0.995$, $\Gamma_{22} = 0.996$, and $\delta = 2.5$. Algorithm 1 found $k^o = 10$.

The earlier results on the same plant reported by [10] exhibit a poorer performance, above all in terms of settling times. A reason why the CG of this paper can give performances better than the CG's of [10] and [12] is mainly due to the fact that in the former a vector optimization of the same dimension as the reference vector is carried out, whereas in the latter methods only scalar optimization is used.

6 Conclusions

The command governor problem, viz. the one of on-line designing, given the reference to be tracked, a command sequence in such a way that a compensated control system can operate in a stable way with satisfactory tracking performance and no constraint violation, has been addressed by exploiting some ideas originating from predictive control. In this connection, the concept of a "virtual" command sequence is instrumental to synthesize a command governor having the stated properties along with a moderate computational burden. This is achieved by: first, linearly parameterizing the virtual command sequence by a vector of the same dimension as the reference, and defining the functional form of the sequence in accordance with a Lyapunov function argument so as to ease stability analysis; second, choosing at each sampling time the free parameter vector as the one minimizing a constrained quadratic command selection index. For linear dynamic systems with constraints, it has been shown how to use off-line an iterative algorithm so as to restrict to a fixed finite integer the infinite number of time-instants over which the prescribed constraints must be checked so as to decide admissibility of a virtual command input.

Though some related encouraging indications have been provided by simulations, an important future research topic is stability and performance robustness of the command governor against exogenous disturbances and modeling errors.

References

[1] D. Q. Mayne and E. Polak, "Optimization based design and control", *Preprints 12th IFAC World Congress*, Vol. 3, pp. 129-138, Sydney, Jul. 1993.

[2] D. Q. Mayne and H. Michalska, "Receding horizon control of nonlinear systems", *IEEE Trans. Automat. Control*, Vol. 35, pp. 814-824, 1990.

[3] J. B. Rawlings and K. R. Muske, "The stability of constrained receding-horizon control", *IEEE Trans. Automat. Control*, Vol. 38, pp. 1512-1516, 1993.

[4] E. Mosca, "*Optimal, Predictive, and Adaptive Control*", Prentice Hall, Englewood Cliffs, N. Y., 1995.

[5] D. W. Clarke, "Advances in Model-Based Predictive Control", *Advances in Model-Based Predictive Control*, Oxford University Press Inc., N. Y., 1994.

[6] P. O. M. Scokaert and D. W. Clarke, "Stability and feasibility in constrained predictive control", *Advances in Model-Based Predictive Control*, Oxford Press Inc., N. Y., 1994.

[7] A. Bemporad and E. Mosca, "Constraint fulfilment in feedback control via predictive reference management", *Proc. IEEE Conf. on Control Applications*, Glasgow, U. K., Aug. 1994.

[8] A. Bemporad and E. Mosca, "Constraint fulfilment in control systems via predictive reference management", *Proc. IEEE Control and Decision Conf.*, Lake Buena Vista, Florida, U.S.A, 1994.

[9] A. Bemporad and E. Mosca, "Reference management predictive control", *EURACO Workshop*, Florence, Italy, Sept. 1995.

[10] P. Kapasouris, M. Athans and G. Stein, "Design of feedback control systems for unstable plants with saturating actuators," *Proc. IFAC Symposium on Nonlinear Control System Design*, Pergamon Press, 1990.

[11] E. G. Gilbert and K. Tin Tan, "Linear systems with state and control constraints: the theory and applications of maximal output admissible sets", *IEEE Trans. Automat. Control*, Vol. 36, pp. 1008-1020, 1991.

[12] E. G. Gilbert, I. Kolmanovsky and K. Tin Tan, "Discrete-time reference governors and the nonlinear control of systems with state and control constraints", *Proc IEEE Control and Decision Conf.*, Lake Buena Vista, Florida, U.S.A, 1994.