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Stabilizing Embedded MPC with Computational Complexity Guarantees

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Abstract— **This paper describes a model predictive control (MPC) approach for discrete-time linear systems with hard constraints on control and state variables. The finite-horizon optimal control problem is formulated as a quadratic program (QP), and solved using a recently proposed dual fast gradient-projection method. More precisely, in a finite number of iterations of the mentioned optimization algorithm, a solution with bounded levels of infeasibility and suboptimality is determined for an alternative problem. This solution is shown to be a feasible suboptimal solution for the original problem, leading to exponential stability of the closed-loop system. The proposed strategy is particularly useful in embedded control applications, for which real-time constraints and limited computing resources can impose tight bounds on the possible number of iterations that can be performed within the scheduled sampling time.**

I. INTRODUCTION

Model predictive control (MPC) is an approach to design feedback control laws that optimize a given closed-loop performance index, taking into account the presence of constraints on input and state variables (see, e.g., [1] and the references therein). In the past, MPC could be only applied to slow processes (e.g., chemical processes), due to the computation time required for solving an optimization problem at each sampling interval [2]. However, thanks to the increase of computing power of modern digital devices, and to the study of fast algorithms for on-line optimization, the use of MPC has gradually moved to fast processes, and the interest towards MPC is increasing in application fields like mechatronics, automotive, and aerospace.

In the most common case of linear MPC, the problem formulation is based on linear prediction models, linear constraints on inputs and states, and quadratic cost functions. The resulting optimization problem to be solved on line is translated into a quadratic program (QP), for which fast solvers have been introduced, mainly based on active-set methods [3]–[5], and interior-point methods [6], [7]. However, one of the main issues in the practical implementation of embedded controllers is the certification of the worst-case execution time, in order to satisfy requirements for hard real-time systems. For this reason, the recent research line on *real-time* MPC aims at designing optimization algorithms that give an acceptable (usually, suboptimal) solution of the QP problem in a number of iterations that can be *bounded a priori*. In addition, the algorithm should be as simple as possible (to be software-certifiable), and should require a small amount of memory (to allow its use in microcontrollers and programmable or application-specific integrated circuits, see, e.g., [8]).

A first possibility to meet the above requirements is the so-called *explicit MPC* approach introduced in [9], where the optimal control input is obtained during the design phase as an explicit function of the state vector, by means of parametric optimization. Such a function is piecewise affine and continuous, and can be easily implemented in embedded control systems, giving a precise estimate on the worst-case execution time. However, the use of explicit MPC is limited to relatively small problems (typical reasonable values are one or two control inputs, prediction horizon shorter than ten steps, up to twelve states+references).

Alternatively, optimization algorithms with guaranteed and explicitly stated non-asymptotic convergence rate have been proposed with the goal of providing a solution of the QP in a prescribed time. Recently, different variants of fast gradient methods, first proposed by Nesterov [10], [11] have been applied to MPC [12]–[15]. In [16] the authors proposed an accelerated dual gradient projection method based on [10], called GPAD (see also [17], [18]). Although GPAD is a *dual* method, bounds on the maximum number of iterations required to achieve specified levels of *primal* suboptimality and infeasibility are provided as complexity certificates. In [19], a non-accelerated version of GPAD is proposed for embedded MPC in hardware platforms with fixed-point arithmetic, where guidelines are provided for selecting the minimum number of fractional and integer bits that guarantee convergence to an approximate solution within a prespecified tolerance on primal suboptimality and infeasibility, after a specific number of iterations.

The main motivation of the present work is to use these bounds to derive a modification of the original MPC problem, in order to be able to apply the algorithm in [18] (or any other algorithm which guarantees bounded infeasibility and suboptimality, e.g. [19]) and obtain an asymptotically stable closed-loop system within a given domain of attraction.The problem of guaranteeing stability in the presence of suboptimal solutions has been pioneered in [20], and subsequently studied for instance in [21]–[23]. An important result assuming a-priori bounds on the level of suboptimality was presented in [24], where the authors proved the convergence of the closed-loop system (in the general framework of MPC for piecewise-continuous systems) to a polytopic set containing the origin. Following this research direction, we consider the case in which the solution might be (slightly) infeasible, with the maximum constraint violation bounded

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a priori by the feasibility tolerance set in the QP solution code, a rather frequent situation in many QP solvers.

After introducing some preliminary results in Section II, Section III states the standard linear MPC problem. Assuming to achieve the optimal solution at each sampling instant, the classical results on recursive feasibility and stability are recalled from [1]. Then, in Section IV, an alternative MPC problem is stated, in order to cope with the presence of solutions with bounded infeasibility and suboptimality. If these bounds are under a certain threshold, the origin of the closed-loop system is proved exponentially stable, and a region of attraction is determined. Also, it is proven that the alternative solution is a feasible (though suboptimal) solution of the standard MPC problem, and tends to the latter as the bounds tend to zero. In Section V, a slightly modified version of the QP solver in [18] is applied to the problem introduced in Section IV, showing that the resulting MPC law can provide stability and invariance for the closed-loop system within a guaranteed and *a-priori* determined computation time. Simulation results are presented in Section VI. Finally, conclusions are drawn in Section VII. Due to space limitation, the proofs of lemmas and theorems are omitted, and are available upon request to the authors.

II. Basic notations, definitions, and results

Let $\mathbb{R}_{>0}$, $\mathbb{R}_{\geq 0}$, $\mathbb{N}_{>0}$ and $\mathbb{N}_{\geq 0}$ denote the sets of positive reals, non-negative reals, positive integers and non-negative integers, respectively. Given two integers $a \leq b$, let $\mathbb{N}_{[a,b]} \triangleq \{a, a+1, ..., b\}$, and $\mathbb{N}_b \triangleq \{0, 1, ..., b\}$. Given a vector $v \in \mathbb{R}^n$, let $\|v\|$ denote its Euclidean norm. Given two vectors $u, v \in \mathbb{R}^n$, the notation $u \leq v$ refers to component-wise inequalities. Given a matrix $M \in \mathbb{R}^{n \times n}$, M' is its transpose, $\rho(M)$ its spectral radius, and let its positive definiteness be indicated as $M \succ 0$. If M is also symmetric, we use $\lambda_{\min}(M)$ and $\lambda_{\max}(M)$ to indicate its minimum and maximum eigenvalues, respectively. Also, we define $\mathbf{1}_n \triangleq [1 \ \ldots \ 1]' \in \mathbb{R}^n$. Given a set $\mathcal{X} \subseteq \mathbb{R}^n$, its interior is denoted by $\mathrm{int}(\mathcal{X})$. Given $\lambda \in \mathbb{R}_{\geq 0}$, we define $\lambda\mathcal{X} \triangleq \{x \in \mathbb{R}^n \ : \ x = \lambda a, \ a \in \mathcal{X}\}$.

Consider a discrete-time autonomous nonlinear system

$$x(t+1) = \varphi(x(t)) \tag{1}$$

where $x \in \mathbb{R}^{n_x}$ is the state vector, and $\varphi(\cdot)$ is a nonlinear function.

Definition 1: For a given $\lambda \in \mathbb{R}$ with $0 < \lambda \leq 1$, the set $\mathcal{X} \subseteq \mathbb{R}^{n_x}$ with $0 \in \mathrm{int}(\mathcal{X})$ is called a *λ-contractive set* for system (1) if, for all $x \in \mathcal{X}$, one has $\varphi(x(t)) \in \lambda\mathcal{X}$. A 1-contractive set is called *positively invariant*.

III. Problem statement

The controlled plant is described by the following discrete-time LTI state-space model

$$x(t+1) = Ax(t) + Bu(t) \tag{2}$$

where $t \in \mathbb{N}_{\geq 0}$, $x \in \mathbb{R}^{n_x}$, $u \in \mathbb{R}^{n_u}$, and it is assumed that the state vector $x(t)$ is available for feedback at each time t. The state and input values can be represented in a single vector

$$z \triangleq \begin{bmatrix} x \\ u \end{bmatrix} \in \mathbb{R}^{n_z}, \ n_z \triangleq n_x + n_u$$

and are required to satisfy the constraint

$$z \in \mathcal{Z} \triangleq \{z \in \mathbb{R}^{n_z} \ : \ F_z z \leq \mathbf{1}_{s_z}\} \tag{3}$$

with $F_z = \begin{bmatrix} F & G \end{bmatrix}$, and $F \in \mathbb{R}^{s_z \times n_x}$, $G \in \mathbb{R}^{s_z \times n_u}$, $s_z \in \mathbb{N}_{>0}$. Note that (3) implies that $\mathcal{Z}$ is nonempty, compact, and $0 \in \mathrm{int}(\mathcal{Z})$. Moreover, the representation of $\mathcal{Z}$ in (3) is without loss of generality [25], since it can represent any polytope that contains the origin in its interior.

Assumption 1: the pair (A, B) is stabilizable.

A. Standard MPC problem

The problem of regulating $x(t)$ to the origin can be solved by a standard MPC law for linear systems. In particular, we refer to the procedure hereafter described as a starting point for the theoretical development of the paper. Given two weight matrices $Q \in \mathbb{R}^{n_x \times n_x}$ and $R \in \mathbb{R}^{n_u \times n_u}$, defined such that $Q = Q' \succ 0$ and $R = R' \succ 0$, we define the *stage cost* $\ell(x, u) \triangleq \frac{1}{2}(x'Qx + u'Ru)$. Also, we define the linear *auxiliary control law*, $\kappa(x) \triangleq Kx$, where $K \in \mathbb{R}^{n_u \times n_x}$ is the gain associated to the infinite-horizon linear quadratic regulator (IH-LQR) defined by matrixes A, B, Q, and R. The application of the auxiliary control law determines the closed-loop system

$$x(t+1) = A_K x(t) \tag{4}$$

where $A_K \triangleq A + BK$. By virtue of Assumption 1, $\rho(A_K) < 1$. The *terminal cost* is defined as $V_f(x) \triangleq \frac{1}{2}x'Px$, where $P = P' \succ 0$ is the solution of the algebraic Riccati equation associated the above-mentioned IH-LQR. Finally, we define the *terminal set* as

$$\mathcal{X}_f \triangleq \{x \in \mathbb{R}^{n_x} \ : \ F_f x \leq \mathbf{1}_{s_f}\}, \tag{5}$$

with $F_f \in \mathbb{R}^{s_f \times n_x}$, $s_f \in \mathbb{N}_{>0}$, which is assumed to be a (not necessarily maximal) positively invariant set in

$$\mathcal{X}_K \triangleq \{x \in \mathbb{R}^{n_x} | (F + GK)x \leq \mathbf{1}_{s_z}\}, \tag{6}$$

for the closed loop system (4), i.e.,

$$x \in \mathcal{X}_f \Rightarrow \begin{bmatrix} x \\ Kx \end{bmatrix} \in \mathcal{Z}, \ A_K x \in \mathcal{X}_f. \tag{7}$$

The MPC control law is determined by solving the following optimization problem:

$$V^\star(x) = \min\{V_N(\mathbf{z}) | \mathbf{z} \in \mathcal{S}_N(x)\}, \tag{8}$$

where $\mathbf{z} \triangleq \begin{bmatrix} z_0' & \cdots & z_{N-1}' & x_N' \end{bmatrix}'$, $z_k \triangleq \begin{bmatrix} x_k' & u_k' \end{bmatrix}'$, the finite-horizon cost function is

$$V_N(\mathbf{z}) = \sum_{k=0}^{N-1} \ell(x_k, u_k) + \ell_N(x_N), \tag{9}$$

the parametric constraint set is

$$\mathcal{S}_N(x) = \{\mathbf{z} \in \mathcal{A}(x) \,|\, z_k \in \mathcal{Z}, k \in \mathbb{N}_{N-1}, x_N \in \mathcal{X}_f\},$$

with

$$\mathcal{A}(x) = \{\mathbf{z}|\ x_0 = x, x_{k+1} = Ax_k + Bu_k, k \in \mathbb{N}_{N-1}\},$$

while $N \in \mathbb{N}_{>0}$ is the length of the *prediction horizon*. The set of states x for which a feasible solution of (8) with horizon length N exists is referred to as $\mathcal{D}_N$. It is well known that for $x \in \mathcal{D}_N$, the (unique) optimal state-input sequence, $\mathbf{z}^\star(x)$, can be obtained by solving a quadratic program (QP). Then, according to the *receding horizon* principle, only the first control move $u_0^\star(x)$ is applied to the system at time t, while the optimization process with the same prediction horizon N is repeated at time $t+1$. The following result (which is an extension of the result of Theorem 2.2.4 in [1] to the case of mixed constraints) holds:

Theorem 1: Let Assumption 1 hold for system (2) and apply, at each $t \in \mathbb{N}_{\geq 0}$, the control law

$$u(t) = u_0^\star(x(t)). \tag{10}$$

Then, the set $\mathcal{D}_N$ is positively invariant for the closed-loop system (2), (10). Moreover, the origin is an exponentially stable equilibrium point of the closed-loop system with domain of attraction equal to $\mathcal{D}_N$.

The above described MPC strategy requires finding, at each time instant t, the optimal solution of (8). However, in practice it can happen that there is no guarantee that the control sequence obtained from the QP solver is optimal, not even feasible, especially if the QP solution algorithm cannot exceed a pre-determined number of iterations, due to real-time requirements. In the next section, we formulate an alternative problem with respect to (8). This latter is modified, in order to determine a stabilizing control law, assuming that known bounds on suboptimality and constraint violation are given.

IV. Alternative MPC problem

Given a scalar $\epsilon \in \mathbb{R}_{\geq 0}$ (which is a design parameter whose meaning will be clarified in Section V) and $k \in \mathbb{N}_{>0}$, we define

$$\begin{aligned}\mathcal{Z}_k^\epsilon &\triangleq (1-k\epsilon)\mathcal{Z}\\ &= \{z \in \mathbb{R}^{n_z}|\ F_z z \leq (1-k\epsilon)\mathbf{1}_{s_z}\} \subseteq \mathcal{Z}.\end{aligned} \tag{11}$$

Assumption 2: The nonnegative scalar ϵ satisfies

$$\epsilon < \min\left\{\tfrac{1}{N}, 1-\rho(A_K)\right\}. \tag{12}$$

The first term in the min in (12) implies that $0 \in \text{int}(\mathcal{Z}_N^\epsilon)$, while the second term implies that $\rho((1-\epsilon)^{-1}A_K) < 1$, which by [26, Lemma 3], [27, Theorem 2.1], ensures that the maximal $(1-\epsilon)$–contractive set for (4) contains the origin in its interior. As $\rho(A_K) < 1$, an ϵ satisfying (12) always exists. A different terminal set with respect to $\mathcal{X}_f$ is defined as

$$\mathcal{X}_f^\epsilon \triangleq \left\{x \in \mathbb{R}^{n_x}|\ F_f^\epsilon x \leq \mathbf{1}_{s_f^\epsilon}\right\}, \tag{13}$$

with $F_f^\epsilon \in \mathbb{R}^{s_f^\epsilon \times n_x}$ and $s_f^\epsilon \in \mathbb{N}_{>0}$, and is assumed to be a (not necessarily maximal) $(1-\epsilon)$-*contractive set* in

$$\mathcal{X}_K^\epsilon \triangleq \{x \in \mathbb{R}^{n_x}|\ (F+GK)x \leq (1-N\epsilon)\mathbf{1}_{s_z}\}, \tag{14}$$

for the closed-loop system (4), i.e.,

$$x \in \mathcal{X}_f^\epsilon \Rightarrow \begin{bmatrix} x \\ Kx \end{bmatrix} \in \mathcal{Z}_N^\epsilon,\ A_K x \in (1-\epsilon)\mathcal{X}_f^\epsilon. \tag{15}$$

Notice that by Assumption 2, such a set exists. The set $(1-\epsilon)\mathcal{X}_f^\epsilon$ can be described as

$$(1-\epsilon)\mathcal{X}_f^\epsilon = \left\{x \in \mathbb{R}^{n_x}|\ F_f^\epsilon x \leq (1-\epsilon)\mathbf{1}_{s_f^\epsilon}\right\} \tag{16}$$

which is analogous to the definition of the set $\mathcal{Z}_k^\epsilon$ in (11). We are now ready to introduce the modified finite-horizon optimal control problem

$$V_\epsilon^\star(x) = \min\{V_N(\mathbf{z})|\ \mathbf{z} \in \mathcal{S}_N^\epsilon(x)\}, \tag{17}$$

where

$$\mathcal{S}_N^\epsilon(x) = \left\{\mathbf{z} \in \mathcal{A}(x)\ \middle|\ z_k \in \mathcal{Z}_{k+1}^\epsilon, k \in \mathbb{N}_{N-1}, x_N \in (1-\epsilon)\mathcal{X}_f^\epsilon\right\}.$$

The set $\mathcal{D}_N^\epsilon$ is defined as the set of states x for which there exists a feasible solution for (17). For every $x \in \mathcal{D}_N^\epsilon$, the unique optimal solution of (17) is denoted by $\mathbf{z}_\epsilon^\star(x)$.

For every $x \in \mathcal{D}_N^\epsilon$, we suppose that a vector $\bar{\mathbf{z}}(x) \in \mathbb{R}^{Nn_z+n_x}$ can be computed, satisfying the following assumption:

Assumption 3: For every $x \in \mathcal{D}_N^\epsilon$, vector $\bar{\mathbf{z}}(x) = \begin{bmatrix}\bar{z}_0' & \cdots & \bar{z}_{N-1}' & \bar{x}_N'\end{bmatrix}'$ is such that

$$V_N(\bar{\mathbf{z}}(x)) - V_\epsilon^\star(x) \leq \delta, \tag{18a}$$

$$\bar{\mathbf{z}}(x) \in \mathcal{A}(x), \tag{18b}$$

$$\bar{z}_k \in \mathcal{Z}_k^\epsilon, \quad k \in \mathbb{N}_{N-1}, \tag{18c}$$

$$\bar{x}_N \in \mathcal{X}_f^\epsilon, \tag{18d}$$

$$\bar{\mathbf{z}}(x) = \mathbf{z}_\epsilon^\star(x), \quad \text{if } x \in \mathcal{X}_f^\epsilon, \tag{18e}$$

where $\bar{z}_k \triangleq [\bar{x}_k'\ \bar{u}_k']'$, $k \in \mathbb{N}_{N-1}$ and $\delta \in \mathbb{R}_{\geq 0}$ is a constant to be determined as a tuning parameter (see Section V), analogously to ϵ. For each $x \in \mathcal{D}_N^\epsilon$, let $\mathcal{Z}_{\epsilon,\delta}(x)$ denote the set of all vectors $\bar{\mathbf{z}}(x) \in \mathbb{R}^{Nn_z+n_x}$ satisfying (18), and $\mathcal{U}_{\epsilon,\delta}(x)$ the set of all $\bar{u}_0(x)$ corresponding to vectors $\bar{\mathbf{z}}(x)$.

Remark 1: Conditions (18c)-(18d) imply that the sequence $\bar{\mathbf{z}}(x)$ leads to a maximum violation of each of the $Ns_z + s_f^\epsilon$ linear inequalities, $z_k \in \mathcal{Z}_{k+1}^\epsilon$, $k \in \mathbb{N}_{N-1}$, $x_N \in (1-\epsilon)\mathcal{X}_f^\epsilon$ which is equal to ϵ. Also notice that $\mathcal{Z}_{\epsilon,\delta}(x)$ (and consequently $\mathcal{U}_{\epsilon,\delta}(x)$) is nonempty for any $x \in \mathcal{D}_N^\epsilon$, since it contains $\mathbf{z}_\epsilon^\star(x)$.

The following lemma will be needed to prove the main result of this section.

Lemma 2: If Assumptions 1, 2, 3 hold, then

$$V_N(\bar{\mathbf{z}}) = V_\epsilon^\star(x) = V^\star(x) = V_f(x), \quad x \in \mathcal{X}_f^\epsilon, \tag{19}$$

for any $\bar{\mathbf{z}} \in \mathcal{Z}_{\epsilon,\delta}(x)$. □

In order to prove the theoretical properties of the proposed control law, the following technical assumption is introduced.

Assumption 4: There exists $\xi \in \mathbb{R}_{>0}$, with $\xi < \lambda_{\min}(Q)$, such that

$$\mathcal{B}_\xi \triangleq \left\{x \in \mathbb{R}^{n_x}|\ \|x\|^2 \leq \frac{2\delta}{\lambda_{\min}(Q) - \xi}\right\} \subseteq \mathcal{X}_f^\epsilon.$$

Analogously to the considerations made for Assumption 2, for any given $\mathcal{X}_f^\epsilon$ of the form (13) it is possible to satisfy Assumption 4 if the parameter δ is chosen to be small enough.

Theorem 3: Let Assumptions 1-4 be satisfied and consider the closed-loop system

$$x(t+1) = \varphi(x(t)) = Ax(t) + B\mu(x(t)), \qquad (20)$$

where $\mu(x(t)) \in \mathcal{U}_{\epsilon,\delta}(x(t))$. Then, the following hold:

(i) recursive feasibility for (17) is ensured, i.e., the set $\mathcal{D}_N^\epsilon$ is a positively invariant set for the closed-loop system;
(ii) $(x(t), \mu(x(t))) \in \mathcal{Z}$, $t \in \mathbb{N}_{\geq 0}$;
(iii) the origin is an exponentially stable equilibrium point for system (20) with domain of attraction $\mathcal{D}_N^\epsilon$. □

Remark 2: Note that (18e) plays a fundamental role in proving the exponential stability of the origin, and the assumption that (18e) holds will be justified in Section V. However, in case (18e) does not hold, it would be still possible to prove the convergence of the closed-loop system to a polytope including the origin, analogously to Theorem 16 in [24]. □

In the particular case where $\delta = 0$, i.e., $V_N(\bar{\mathbf{z}}) \leq V_\epsilon^\star(x)$, for any $\bar{\mathbf{z}} \in \mathcal{Z}_{\epsilon,\delta}(x)$ (which is possible since $\bar{\mathbf{z}}$ may not belong to $\mathcal{S}_\epsilon(x)$), Assumption 4 is not needed anymore, and the result in Theorem 3 can be simplified as follows.

Corollary 1: Consider system (2) fulfilling Assumption 1, and apply the control law $u(t) = \mu(x(t)) \in \mathcal{U}_{\epsilon,0}(x(t))$, being Assumptions 2-3 satisfied. Then, all the assertions of Theorem 3 are valid. □

Remark 3: Note that, if $\epsilon = 0$, one has $\mathcal{Z}_k^\epsilon \equiv \mathcal{Z}$ for all $k \in \mathbb{N}_{\geq 0}$, and $\mathcal{X}_f^\epsilon$ could be chosen equal to $\mathcal{X}_f$, also implying $\mathcal{D}_N^\epsilon \equiv \mathcal{D}_N$. This would lead to automatic satisfaction of Assumption 2. Also, assuming $\delta = 0$, Assumption 3 would be satisfied, with conditions (18c)-(18d) coinciding with $z_k \in \mathcal{Z}$, $k \in \mathbb{N}_{N-1}$, $x_N \in \mathcal{X}_f$. Assumption 4 would be satisfied as well for any $\xi < \lambda_{\min}(Q)$, since $0 \in \mathcal{X}_f^\epsilon$. In conclusion, $\epsilon = \delta = 0$ would automatically lead to the satisfaction of Assumptions 2-4. Therefore, Theorem 1 is proven as a particular case of Theorem 3.

Remark 4: Note that any solution of (17) which satisfies Assumption 3 is also a *feasible*, but possibly suboptimal, solution of (8). Therefore, the alternative problem (17) can be seen as a way to obtain a suboptimal solution to the original problem (8), in the presence of known bounds on suboptimality, δ, and on constraint violation, ϵ.

V. Description of the optimization algorithm

In this section we briefly summarize GPAD [16], a Dual Accelerated Gradient Projection algorithm based on [10], see also [17], [18]. Problem (17) can be expressed as

$$V_\epsilon^\star(x) = \min_{\mathbf{z} \in \mathcal{A}(x)} \{V_N(\mathbf{z}) | g^\epsilon(\mathbf{z}) \leq 0\}, \qquad (21)$$

where $g^\epsilon(\mathbf{z}) = (g_1^\epsilon(z_1), \cdots, g_N^\epsilon(x_N))$, with $g_k^\epsilon : \mathbb{R}^{n_x+nu} \to \mathbb{R}^{s_z}$, $k \in \mathbb{N}_{N-1}$, defined as $g_k^\epsilon(z) \triangleq F_z z - (1-(k+1)\epsilon)\mathbf{1}_{s_z}$, while $g_N^\epsilon(x) = F_f^\epsilon x - (1-\epsilon)\mathbf{1}_{s_f^\epsilon}$. The dual function

$$\Psi_\epsilon(\mathbf{y}, x) \triangleq \min_{\mathbf{z} \in \mathcal{A}(x)} V_N(\mathbf{z}) + g^\epsilon(\mathbf{z})' y$$

Algorithm 1 Accelerated Dual Gradient Projection (GPAD)

$\mathbf{y}_{(0)} = \mathbf{y}_{(-1)} = 0$. $\bar{z}_{(-1)} = 0$. $\theta_0 = \theta_{-1} = 1$. $\nu \leftarrow 0$
Step 1. $\mathbf{w}_{(\nu)} = \mathbf{y}_{(\nu)} + \theta_\nu(\theta_{\nu-1}^{-1} - 1)(\mathbf{y}_{(\nu)} - \mathbf{y}_{(\nu-1)})$.
Step 2. $\mathbf{z}_{(\nu)} = \arg\min_{\mathbf{z} \in \mathcal{A}(x)} V_N(\mathbf{z}) + \mathbf{w}'_{(\nu)} g^\epsilon(\mathbf{z})$
Step 3. $\bar{\mathbf{z}}_{(\nu)} = (1-\theta_\nu)\bar{\mathbf{z}}_{(\nu-1)} + \theta_\nu \mathbf{z}_{(\nu)}$
Step 4. If $\|[g^\epsilon(\bar{\mathbf{z}}_{(\nu)})]_+\|_\infty \leq \epsilon$, $\bar{\mathbf{z}}(x) \leftarrow \bar{\mathbf{z}}_{(\nu)}$ stop.
Step 5. $\mathbf{y}_{(\nu+1)} = \left[\mathbf{w}_{(\nu)} + \frac{1}{L_{\Psi_\epsilon}} g^\epsilon(\mathbf{z}_{(\nu)})\right]_+$
Step 6. $\theta_{\nu+1} = \frac{\sqrt{\theta_\nu^4 + 4\theta_\nu^2} - \theta_\nu^2}{2}$. $\nu \leftarrow \nu + 1$. Go to Step 1.

is Lipschitz-continuous, with Lipschitz constant equal to L_{Ψ_ϵ}. Algorithm 1 is based on the accelerated gradient method of [10] applied to the dual problem $\max_{\mathbf{y} \geq 0} \Psi_\epsilon(\mathbf{y}, x)$.

The only complicated part of Algorithm 1 is Step 2. If Problem (21) is posed in *condensed form*, i.e., the equality constraints corresponding to the state equations have been eliminated (off-line), then Step 2 consists of a matrix-vector product which requires $O(N^2)$ operations. One can do even better, by viewing Step 2 as an unconstrained linear-quadratic optimal control problem, and applying the modified Riccati approach proposed in [16], which requires only $O(N)$ flops to compute $\mathbf{z}_{(\nu)}$.

The following theorem provides an upper bound on the maximum number of iterations to compute a solution that satisfies Assumption 3, with $\delta = 0$. Specifically, since the initial dual iterate is equal to the zero vector, GPAD is doing always better than optimal, therefore one has to care only about ϵ-feasibility, and this is the only termination criterion employed at Step 4. The following theorem gives complexity and stability guarantees for Algorithm 1.

Theorem 4: For any $x \in \mathcal{D}_N^\epsilon$ Algorithm 1 will terminate after at most

$$\nu_\epsilon^\star = \left\lceil \left(\frac{8 L_{\Psi_\epsilon} \Delta_y^\epsilon}{\epsilon}\right)^{\frac{1}{2}} \right\rceil - 2 \qquad (22)$$

iterations, with $\bar{\mathbf{z}}(x) \in \mathcal{Z}_{\epsilon,0}(x)$, where

$$\Delta_y^\epsilon \triangleq \max_{x \in \mathcal{D}_N^\epsilon} \min_{\mathbf{y}_\epsilon^\star(x) \in \mathcal{Y}_\epsilon^\star(x)} \|\mathbf{y}_\epsilon^\star(x)\|_1 \qquad (23)$$

The corresponding MPC law $\mu(x) = \bar{u}_0(x)$ produced by Algorithm 1 renders the origin exponentially stable for the closed-loop system (20) with region of attraction $\mathcal{D}_N^\epsilon$. □

Bounds on dual optimal solutions such as the one of (23) are called *Uniform Dual Bounds* (UDBs) in [16], [18].

Remark 5: According to Theorem, 4 GPAD reaches ϵ-feasibility in $O\left(\sqrt{\frac{L_{\Psi_\epsilon}\Delta_y^\epsilon}{\epsilon}}\right)$ iterations, while the complexity estimate to achieve the same level of suboptimality for the dual cost, which is the standard result found in the literature (see, e.g., [10]) is of order $O\left(\sqrt{\frac{L_{\Psi_\epsilon}}{\epsilon}}\Delta_y^\epsilon\right)$, which may be much larger.

Remark 6: Notice that the bound on dual optimal solutions, Δ_y^ϵ, must be valid on the entire $\mathcal{D}_N^\epsilon$, in order to be able to guarantee stability and invariance of domain of

$\mathcal{D}_N^\epsilon$ for the closed-loop system. In [18], it is shown that a tight upper bound to Δ_y^ϵ can be computed by solving a Linear Program with Linear Complementarity Constraints (LPLCC) for which specialized efficient algorithms exist for its solution. Notice that the techniques proposed in [15], [16], lead to bounds which are valid only on a subset interior of $\mathcal{D}_N^\epsilon$, since they are based on Slater's condition, and thus cannot be used to derive an iteration bound on the entire set $\mathcal{D}_N^\epsilon$.

VI. Simulation examples

Consider the problem of regulating the state of the discrete-time unstable system

$$\begin{aligned} x(t+1) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t) \end{aligned}$$

to the origin, where

$$A = \begin{bmatrix} 1.09 & 0.22 \\ 0.49 & 0.02 \end{bmatrix}, B = \begin{bmatrix} 1.22 & 0.88 \\ -0.78 & -0.34 \end{bmatrix},$$

$$C = \begin{bmatrix} 1.34 & -0.16 \\ -3.19 & -0.56 \end{bmatrix}, D = \begin{bmatrix} 1.60 & 1.01 \\ -0.68 & 0.77 \end{bmatrix}.$$

The system is subject to input and output constraints, $\|u(t)\|_\infty \leq 1$, $\|y(t)\|_\infty \leq 1$. The state and input weight matrices are

$$Q = \begin{bmatrix} 5.44 & 5.80 \\ 5.80 & 7.01 \end{bmatrix}, \quad R = \begin{bmatrix} 1.14 & 0.68 \\ 0.68 & 0.62 \end{bmatrix}.$$

The IH-LQR gain is

$$K = \begin{bmatrix} -1.50 & -0.17 \\ 3.39 & 0.48 \end{bmatrix},$$

and $\rho(A_K) = 0.081$, therefore, according to Assumption 2, ϵ can take values smaller than $\min\{\frac{1}{N}, 0.919\}$.

Table I gives bounds on the maximum number of iterations on $\mathcal{D}_N^\epsilon$, according to Theorem 4, for $\epsilon \in \{10^{-3}, 5 \times 10^{-3}, 10^{-2}, 5 \times 10^{-2}\}$. For each value of ϵ, $\nu_\epsilon^\star$ is the theoretical bound given by Eq. (22), with a tight upper bound on Δ_y^ϵ (cf. (23)) computed by solving the LPLCC described in [18], while $\hat{\nu}_\epsilon^\star$ is the maximum number of iterations encountered by simulating the closed-loop system from 100 different initial states belonging to $\mathcal{D}_N^\epsilon$. One can observe that for the specific example, the theoretical bound is quite tight. For $\epsilon \in \{10^{-3}, 5 \times 10^{-3}, 10^{-2}\}$ it is less than twice the observed iteration bound, while for $\epsilon = 5 \times 10^{-2}$ it reaches a factor of around 2.7.

Another significant conclusion that can be drawn from Table I, is that the iteration bounds $\nu_\epsilon^\star$, $\hat{\nu}_\epsilon^\star$ decrease as ϵ increases. For embedded applications, this means that according to hardware specifications and sampling time, one can select the appropriate value of ϵ that will guarantee stability and invariance of the corresponding closed-loop system. However, the price to pay is a smaller region of attraction, $\mathcal{D}_N^\epsilon$, as can be seen in Figure 1.

Figures 2 and 3 depict the input and output trajectories corresponding to the feedback law obtained by Algorithm 1 with $\epsilon = 5 \times 10^{-2}$, and by solving the original MPC problem (8) (using `GUROBI` with maximum accuracy), respectively, starting from $x_0 = \begin{bmatrix} -0.101 & -3.548 \end{bmatrix}'$. As expected, the closed-loop system respects input and output constraints at all times. Furthermore, the infinite-horizon closed-loop cost is 52.98, very close to the one corresponding to the system obtained by solving problem (8), which is 52.80.

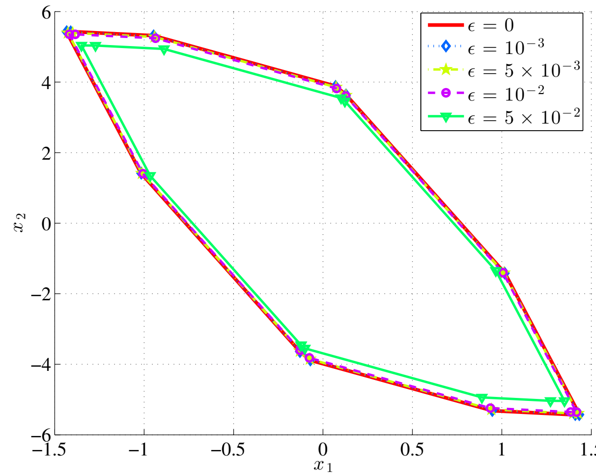

Fig. 1: Region of Attraction, D_{15}^ϵ, for $\epsilon \in \{10^{-3}, 5 \times 10^{-3}, 10^{-2}, 5 \times 10^{-2}\}$.

TABLE I: Complexity certification analysis ($\nu^\star$: Theoretical Iteration Bound, $\hat{\nu}^\star$: Practical Iteration Bound)

N	ϵ							
	10^{-3}		5×10^{-3}		10^{-2}		5×10^{-2}	
	$\nu_\epsilon^\star$	$\hat{\nu}_\epsilon^\star$	$\nu_\epsilon^\star$	$\hat{\nu}_\epsilon^\star$	$\nu_\epsilon^\star$	$\hat{\nu}_\epsilon^\star$	$\nu^\star$	$\hat{\nu}_\epsilon^\star$
5	6868	3626	3086	1634	2195	1165	1188	447
7	6914	3651	3107	1645	2210	1173	1196	450
9	6934	3661	3116	1649	2217	1177	1199	452
11	6944	3667	3121	1652	2220	1178	1202	453
13	6950	3670	3123	1653	2222	1179	1408	531
15	6954	3672	3125	1654	1980	1180	1569	736

VII. Conclusions

This paper has proposed an MPC approach for linear systems subject to mixed state-input constraints that is based on a very-simple-to-implement QP algorithm and with proved stability and suboptimality guarantees. Given the optimal control problem to be solved, the QP algorithm, that is based on dual gradient projection iterations, is applied to a modified problem with tightened constraints, in order to obtain a suboptimal solution of the original problem that enjoys guarantees of recursive feasibility and exponential stability. The solution is obtained at each time step within a finite number of iterations of the QP algorithm, and tends to the optimal solution of the original problem as the allowed number of iterations tends to infinity. Finally, simulation examples show the potential of the proposed approach in practical applications of embedded MPC under hard real-time constraints and low-cost control hardware.

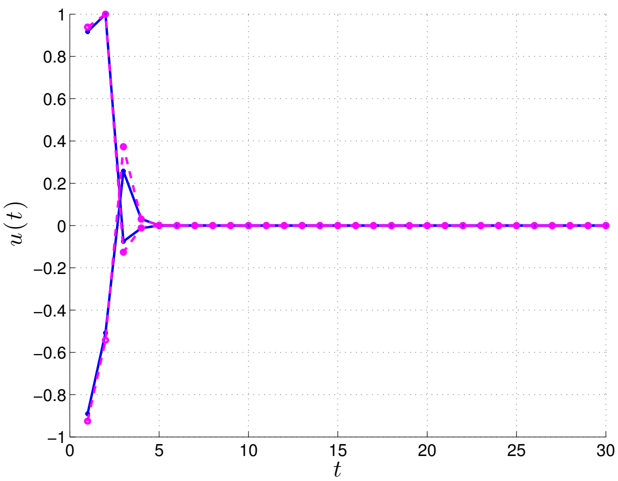

Fig. 2: Input trajectory for MPC law obtained using Algorithm 1 applied to the modified MPC problem (17) (magenta, dot-circle) and using the MPC law obtained by solving (8) with high accuracy (blue, solid-asterisk).

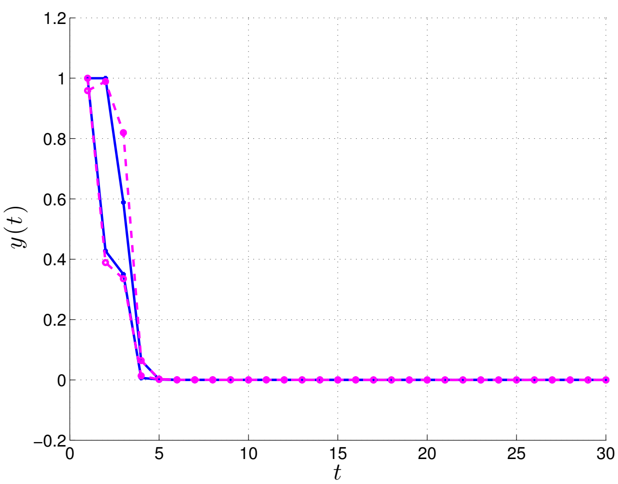

Fig. 3: Output trajectory for MPC law obtained using Algorithm 1 applied to the modified MPC problem (17) (magenta, dot-circle) and using the MPC law obtained by solving (8) with high accuracy (blue, solid-asterisk).

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