Multiobjective Model Predictive Control Based on Convex Piecewise Affine Costs

A. Bemporad and D. Muñoz de la Peña

Abstract— **This paper proposes a novel model predictive control (MPC) scheme based on multiobjective optimization. At each sampling time, the MPC control action is chosen among the set of Pareto optimal solutions based on a time-varying and state-dependent decision criterion. After recasting the optimization problem associated with the multiobjective MPC controller as a multiparametric multiobjective linear problem, we show that it is possible to compute each Pareto optimal solution as an explicit piecewise affine function of the state vector and of the vector of weights to be assigned to the different objectives in order to get that particular Pareto optimal solution. Furthermore, we provide conditions for selecting Pareto optimal solutions so that the MPC control loop is asymptotically stable, and show the effectiveness of the approach in simulation examples.**

Index Terms— **Model predictive control, multiobjective optimization, multiparametric programming.**

I. INTRODUCTION

Multiobjective control problems are based on the optimization of multiple and often conflicting performance criteria in order to take into account different control specifications. Approaches to multiobjective control were proposed in the nineties in [1] and, more recently, in the context of model predictive control (MPC) in [2], where the authors, rather than looking for Pareto optimal solutions in the standard multiobjective setting [3], look for the optimal control sequence that minimizes the max of a finite number of objectives.

In this paper we consider a multiobjective MPC formulation where the optimal control sequence corresponds to one of the Pareto optimal solutions. As multiple Pareto solutions may exist, we provide conditions for selecting a Pareto solution that is optimal for a desired weighted sum of the different objectives and that preserves closed-loop asymptotic stability. To address computational issues, in this paper we also investigate multiparametric multiobjective linear programming (mp-moLP) to handle multiobjective MPC problems with convex piecewise affine cost functions. Multiparametric programming has been largely investigated in the last eight years for providing *explicit* MPC solutions [4].

For addressing the multiparametric multiobjective problem, in this paper we exploit the fact that Karush-Kuhn-Tucker (KKT) conditions for multiobjective optimization map into standard KKT conditions for a scalar optimization problem where the cost function is the weighted sum of the objectives, and where different weights provide different Pareto optimal solutions [3]. By exploiting this result, mp-moLP can be rephrased as a multiparametric LP problem with parameters (weights) in the cost function, and parameters (current states) in the right-hand-side of the constraints. Multiparametric LP problems with parameters in both the cost function *and* the rhs of the constraints have been addressed in [5]. Alternatively, by looking at the KKT conditions, such problems can be treated as multiparametric linear complementarity (mp-LC) problems (see the work of [6] and references therein).

The results of this paper are extended in [7].

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II. PROBLEM FORMULATION

Consider the problem of regulating a process modeled by the following linear discrete-time system

$$x(t+1) = Ax(t) + Bu(t) \tag{1}$$

under the linear input and state constraints

$$x(t) \in \mathcal{X},\ u(t) \in \mathcal{U}, \tag{2}$$

where $x \in \mathbb{R}^n$ is the state vector, $u \in \mathbb{R}^m$ is the input vector, and $t \in \mathbb{N}$ denotes the time step. We assume that $\mathcal{X} \subseteq \mathbb{R}^n$ and $\mathcal{U} \subseteq \mathbb{R}^m$ are convex full-dimensional polyhedral sets containing the origin in their interior.

The standard model predictive control (MPC) approach is to introduce a performance criterion to be minimized repeatedly at each time step. In this paper, instead of minimizing a single performance index, we consider the case of having $l+1$ different performance indices with $l \in \mathbb{N}$, and follow a multiobjective optimization approach. Multiobjective optimization is the process of simultaneously optimizing two or more (possibly) conflicting objectives subject to certain constraints. Consider the following multiobjective optimal control problem

$$\min_{U} \quad J(U,x) \tag{3a}$$

subject to

$$\begin{array}{l} x_{k+1} = Ax_k + Bu_k \\ x_0 = x \\ x_k \in \mathcal{X},\ k = 1,\dots,N \\ u_k \in \mathcal{U}\ k = 0,\dots,N-1 \\ x_N \in \Omega, \end{array} \tag{3b}$$

where $J(U,x) = [J_0(U,x), J_1(U,x), ..., J_l(U,x)]' : \mathbb{R}^s \times \mathbb{R}^n \to \mathbb{R}^{l+1}$ is a vector function, $l \geq 1$, $s = Nm$, $U = [u_0', \dots, u_{N-1}']'$ is the sequence of future control moves to be optimized, x_k is the k-steps ahead predicted state from the initial state $x = x(t)$, and Ω is a terminal polyhedral set

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containing the origin in its interior. Each performance index is defined as

$$J_i(U,x) = \sum_{k=0}^{N-1} L_i(x_k,u_k) + F_i(x_N), \tag{4}$$

where the stage costs $L_i : \mathbb{R}^{n+m} \to \mathbb{R}$ and the terminal costs $F_i : \mathbb{R}^n \to \mathbb{R}$, $i = 0,\dots,l$, satisfy the following assumption.

Assumption 1: For all $i = 0,\dots,l$ function $L_i : \mathbb{R}^{n+m} \to \mathbb{R}$ is jointly convex with respect to (x,u), function $F_i : \mathbb{R}^n \to \mathbb{R}$ is convex with respect to x, $L_i(0,0) = 0$, $F_i(0) = 0$, and there exist $\mathcal{K}$-functions[1] σ, σ_2 such that $L_i(x,u) \geq \sigma(\|x\|)$ for all $u \in \mathcal{U}$, $F_i(x) \geq \sigma(\|x\|)$ and $F_i(x) \leq \sigma_2(\|x\|)$ for some norm $\|\cdot\|$.

In general, the performance indices $J_i(U,x)$ are conflicting and it is not possible to obtain a solution that optimizes all the objectives at the same time. In order to obtain an optimal input trajectory U, an additional decision criterion must be taken into account that provides a trade-off between the different performance indices. In this work we propose to choose the optimal input trajectory among the set of Pareto optimal solutions of (3).

Definition 1 ([3]): A feasible point U^p is Pareto optimal if and only if there exists no other feasible point U such that $J_i(U,x) \leq J_i(U^p,x)$, $\forall i = 0,\dots,l$ and $J_j(U,x) < J_j(U^p,x)$ for at least one index $j \in \{0,\dots,l\}$.

Finding the set of Pareto optimal solutions of a multiobjective optimization problem (i.e., solving the multiobjective optimization problem) can be a hard task. For the class of problems considered here it is possible to use the so-called *weighting method* to solve the Problem (3) [3], [8] through the *scalarization* of the multiobjective problem (3)

$$U^*(x,\alpha) = \begin{array}{ll} \arg\min\limits_U & \alpha' J(U,x) \\ \text{s.t.} & \text{(3b)} \end{array} \tag{5}$$

where $\alpha = [\alpha_0,\dots,\alpha_l]' \in \mathbb{R}^{l+1}$ is a weight vector, $\alpha_i \geq 0$, $\forall i = 0,\dots,l$, $\sum_{i=0}^{l} \alpha_i = 1$.

As described in [8, Chapter 4.7.4], for each given $\alpha > 0$[2] the solution $U^*(x,\alpha)$ of (5) is also a Pareto optimal solution of (3), usually different for different values of $\alpha \in \mathbb{R}^{l+1}$. For convex vector optimization problems as in (3), it is also true that for every Pareto optimal point U^p there exists a vector $\alpha \geq 0$ such that $U^p = U^*(x,\alpha)$. Hence, the corresponding solutions of (5) for all possible weight vectors α cover the whole set of Pareto optimal solutions of (3). We can either restrict $\alpha > 0$ in (5) or, alternatively, tolerate possibly non-Pareto optimal solutions by leaving $\alpha \geq 0$. Both choices are not harmful, as will be discussed in Remark 1. In order to uniquely define a Pareto optimal solution to the multiobjective MPC problem (3), at each time step t a weight vector $\alpha(t)$ must be selected. The optimal future input trajectory associated with the MPC controller is then given by the optimizer of (3) for that particular weight vector.

[1] A $\mathcal{K}$ function $\sigma : \mathbb{R} \to \mathbb{R}$ is a strictly increasing function of its argument and satisfies $\sigma(0) = 0$.

[2] Vector inequalities denote the corresponding set of element-wise comparisons.

A. Proposed multiobjective MPC scheme

In MPC design the performance index is used to tune the properties of the closed-loop system (stability, robustness, speed of convergence to the target state, etc.). In general, different choices of weights in the performance index provide a different closed-loop response. In this paper we propose to use a time-varying target weight $\alpha_d(t) \in \mathbb{R}^{l+1}$ as an additional tuning parameter. On the other hand, arbitrary switching of $\alpha(t)$ may lead to instability, so the objective of the proposed MPC controller is to choose $\alpha(t)$ as close as possible to the desired $\alpha_d(t)$ at each sampling time t in a way that closed-loop stability is guaranteed. Hence, let

$$\alpha^*(x,\alpha_d,J_a) = \arg\min_\alpha \; f(\alpha - \alpha_d) \tag{6a}$$

$$\text{s.t. } V^*(x,\alpha) \leq J_a \tag{6b}$$

$$\sum_{i=0}^{l} \alpha_i = 1 \tag{6c}$$

$$\alpha_i \geq 0, \; i = 0,\dots,l, \tag{6d}$$

where $V^* : \mathbb{R}^{n+l+1} \to \mathbb{R}$ is the *value function* associated with Problem (5), $V^*(x,\alpha) = \alpha' J(U^*(x,\alpha),x)$, $f : \mathbb{R}^{l+1} \to \mathbb{R}$ is a convex function that penalizes the deviation of α from the target weight vector α_d and J_a is a value that depends on the optimal solution of the MPC at the previous time step, defined below in (7).

The proposed multiobjective MPC algorithm is summarized by Algorithm II.1, which is executed at each time step t, where, with a slight abuse of notation, we have set $\alpha^*(t) = \alpha^*(x(t),\alpha_d(t),J_a(t))$ and $U^*(t) = U^*(x(t),\alpha^*(t))$. Note that the multiobjective MPC controller can be thought as a stabilizing MPC controller with a time-varying and possibly state-dependent performance index.

1. Acquire $x(t)$, compute $\alpha_d(t)$;
2. Let $U^*(t-1) = [u_0', u_1', \dots, u_{N-1}]'$, set $U_s(t) = [u_1', u_2', \dots, u_{N-1}', (Kx_N)']'$, where x_N is the optimal state for time step $t-1+N$ predicted at time $t-1$ starting from $x(t-1)$;
3. Evaluate
$$J_a(t) = \alpha^*(t-1)' J(U_s(t), x(t)); \tag{7}$$
4. Compute $\alpha^*(t)$ by solving (6) for $x = x(t)$, $\alpha_d = \alpha_d(t)$, $J_a = J_a(t)$;
5. Compute $U^*(t)$ by solving (5) for $x = x(t)$, $\alpha = \alpha^*(t)$;
6. Set $u(t)$ equal to the first optimal of $U^*(t)$;
7. End.

Algorithm II.1: Multiobjective MPC algorithm

B. Stability properties

Closed-loop stability properties are guaranteed by following a standard terminal region/terminal constraint approach [9].

Theorem 1: Let L_i and F_i, $i = 0,\dots,l$, satisfy Assumption 1. Assume that there exists a linear feedback $u = Kx$

such that the following conditions hold:

$$F_i((A+BK)x) - F_i(x) + L_i(x, Kx) \leq 0, \quad i = 0, \dots, l \tag{8a}$$

$$x \in \Omega \rightarrow (A+BK)x \in \Omega, \tag{8b}$$

$$K(x) \in U,\ \forall x \in \Omega. \tag{8c}$$

Then, if Problem (3) is feasible at $t = 0$, Problem (3) is feasible at all time steps $t \geq 0$ and system (1) in closed-loop with the MPC controller defined by Algorithm II.1 is asymptotically stable.

Proof: See [7] for a complete proof. ■

Since now on, we assume that in (4) the costs are convex and piecewise affine. For example, as in [10],

$$L_i(x,u) = \|Q_i x\|_\infty + \|R_i u\|_\infty,\ F_i(x) = \|P_i x\|_\infty \tag{9}$$

satisfy Assumption 1 if $Q_i \in \mathbb{R}^{q_i \times n}$, $R_i \in \mathbb{R}^{r_i \times m}$ are matrices with full column-rank, $\forall i = 0, \dots, l$ [11]. It is easy to enforce the assumptions of Theorem 1 when using stage and terminal costs as in (9). In fact, in the special case of matrix A stable, one can apply the techniques reported in [10] to find a common matrix P that satisfies condition (8) for $K = 0$. More generally, as proposed in [12], one can use nonlinear optimization to find a set of matrices P_i and a gain K satisfying $\|P_i(A+BK)P_i^{-L}\|_\infty + \|Q_i P_i^{-L}\|_\infty + \|R_i K P_i^{-L}\|_\infty \leq 1$, where $P_i^{-L} = [P_i' P_i]^{-1} P_i'$.

III. Multiparametric multiobjective linear programming

Let Assumption 1 hold and assume $L_i : \mathbb{R}^{n+m} \rightarrow \mathbb{R}$ and $F_i : \mathbb{R}^n \rightarrow \mathbb{R}$ are convex and piecewise affine functions. Following an approach similar to the one in [10], Problem (3) can be recast into a multiparametric multiobjective linear program

$$\begin{array}{ll} \min\limits_z & Cz \\ & Gz \leq b + Sx \end{array} \tag{10}$$

where $z \in \mathbb{R}^d$, $d = s + (2N-1)(l+1)$, is the vector of optimization variables (which includes U and additional slack variables, see [10]), $x \in \mathbb{R}^n$ is a vector of parameters, $C \in \mathbb{R}^{(l+1)\times d}$ defines the linear vector function of dimension $l+1$, where each row of matrix C defines a different scalar objective function

$$C = [c_0\ \dots\ c_l]',\ c_i \in \mathbb{R}^d,\ i = 0, \dots, l.$$

According to the weighting method, the set of Pareto optimal points of Problem (10) can be fully characterized from the corresponding solutions of the following optimization problem

$$\begin{array}{ll} \min\limits_z & \alpha' Cz \\ & Gz \leq b + Sx \end{array} \tag{11}$$

for all possible weight vectors α such that $\alpha = [\alpha_0, \dots, \alpha_l]' \in \mathbb{R}^{l+1}$, $\alpha_i \geq 0$, $\forall i = 0, \dots, l$, $\sum_{i=0}^{l} \alpha_i = 1$. Problem (11) is equivalent to

$$\begin{array}{ll} \min\limits_z & (c_0' + \mu' C_\mu) z \\ & Gz \leq b + Sx, \end{array} \tag{12}$$

where in order to get rid of the equality constraint $\sum_{i=0}^{l} \alpha_i = 1$ we have expressed $\alpha_0 = 1 - \sum_{i=1}^{l} \alpha_i$, $\mu = [\alpha_1\ \dots\ \alpha_l] \in \mathbb{R}^l$, and $C_\mu = [(c_1 - c_0)\ \dots\ (c_l - c_0)]' \in \mathbb{R}^{l \times d}$.

Most of the multiparametric LP solvers only handle parameters either in the cost function or in the rhs of the constraints (which, by duality, is totally equivalent). In this paper we are going to characterize the explicit solution of this class of problems by exploiting the KKT conditions of Problem (12)

$$(c_0 + C_\mu' \mu) + G' \lambda = 0 \tag{13a}$$

$$\lambda'(Gz - b - Sx) = 0 \tag{13b}$$

$$Gz - b - Sx \leq 0 \tag{13c}$$

$$\lambda \geq 0 \tag{13d}$$

$$\mu \geq 0. \tag{13e}$$

By assuming that all the components of z are lower-bounded [3] by a quantity $z_{\min}$, Problem (12) can be recast as the multiparametric linear complementarity problem (mp-LCP)

$$\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} - \begin{bmatrix} 0 & -G \\ G' & 0 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} b - Gz_{\min} \\ c_0 \end{bmatrix} + \begin{bmatrix} 0 & S \\ C_\mu' & 0 \end{bmatrix} \begin{bmatrix} \mu \\ x \end{bmatrix},\ \left[\begin{smallmatrix} w_1 \\ w_2 \end{smallmatrix}\right]' \left[\begin{smallmatrix} z_1 \\ z_2 \end{smallmatrix}\right] = 0, \tag{14}$$

where $z_2 = z - z_{\min}$, $z_1 = \lambda$, w_2 are the Lagrange multipliers associated with the constraints $z \geq z_{\min}$, and w_1 is the vector of slack variables satisfying $Gz + w_1 = b + Sx$. Problem (14) can be solved by existing mp-LCP solvers [6]. To the authors' knowledge, however, no algorithms exist that are specialized for the structure (12), in which some of the parameters (μ) appear only in the cost function, the remaining parameters (x) only appear in the right hand side (rhs) of the constraints. For this reason, in the following paragraphs we characterize the particular structure of the solution to (12), that will be exploited in Section III-D to evaluate the proposed multiobjective MPC law.

The optimal multiparametric solution of the optimization problem (12) is fully defined by the combinations of constraints that are active at (one of) the optimizer(s) $z^*(x, \mu)$. The following concepts of degeneracy play a fundamental role in solving multiparametric problems of the form (12).

Definition 2: For a fixed $(\mu, x) \in \mathbb{R}^{l+n}$, the LP problem (12) is said to be *primal degenerate* if there exists an optimizer $z^*(x, \mu)$ such that the number of active constraints at the optimizer is larger than the number d of optimization variables. In this case more than one basis describes the optimal primal solution. *Dual degeneracy* occurs when the dual problem of (12) is primal degenerate. In this case more than one primal solution $z^*(x, \mu)$ is optimal.

Definition 3: Given an active set of constraints $\tilde{G}z = \tilde{b} + \tilde{S}x$, the linear independence constraint qualification (LICQ) property is said to hold if the set of active constraint gradients are linearly independent, i.e., the associated submatrix $\tilde{G}$ has full row rank.

In the absence of primal and dual degeneracy and assuming that the LICQ condition holds, given a combination of d active constraints $\tilde{G}z = \tilde{b} + \tilde{S}x$ in which all the rows of $\tilde{G}$

[3] This is always the case when the set $\mathcal{U}$ of admissible inputs is bounded.

are linearly independent, the set of the parameters (μ, x) for which this combination yields the optimal solution and the primal $z^*(x,\mu)$ and dual $\lambda^*(x,\mu)$ optimizers can be easily characterized as in [14]

$$\begin{aligned} z^*(x,\mu) &= \tilde{G}^{-1}(\tilde{b}+\tilde{S}x) && \text{(15a)}\\ \tilde{\lambda}^*(x,\mu) &= -(\tilde{G}')^{-1}(c_0+C'_\mu\mu) && \text{(15b)}\\ \hat{\lambda}^*(x,\mu) &= 0, && \text{(15c)} \end{aligned}$$

where $\hat{\lambda}$ are the Lagrange multipliers corresponding to inactive constraints $\hat{G}z \le \hat{b}+\hat{S}x$. We denote CR the region of the state and weight vector in which a particular solution is optimal. Note that although in each critical region CR z^* only depends on x while $\tilde{\lambda}$ only depends on μ, the critical region CR itself of validity of (15) depends on conditions involving both μ and x. More precisely, the critical region CR containing all and only parameters (μ, x) for which (15) is the optimal solution for the particular combination of active constraints we have chosen is $CR = CR_x \times CR_\mu$, where, by substituting (15) in (13c), (13d)

$$\begin{array}{l} CR_x = \{x: \ \hat{G}\tilde{G}^{-1}(\tilde{b}+\tilde{S}x)-\hat{b}-\hat{S}x \le 0\}\\ CR_\mu = \{\mu: \ (\tilde{G}')^{-1}(c_0+C'_\mu\mu) \le 0\}. \end{array} \tag{16}$$

Several methods have been proposed to characterize completely the optimal solution $z^*(\mu, x)$ on the remaining set $CR^{rest} = (\mathcal{M}\times\mathcal{X})\setminus CR$, where $\mathcal{M}$ and $\mathcal{X}$ are the set of parameters of interest over which we want to characterize the multiparametric solution. Unfortunately here we cannot adopt the approach described in [15], also used in the Hybrid Toolbox [16] for solving mp-LPs. In fact, such an approach relies on the satisfaction of the so-called "facet-to-facet" property [17], which in general does not hold for Problem (12) (see Figure 1 below). As an alternative, one can use the approach of [14] to partition CR^{rest}. The final result is an explicit description of the Pareto optimal points of the multiobjective linear problem (10) in the form of a PWA function of the parameters x and of the weights μ. In this paper we will adopt the approach of [6] to solve Problem (12) through its reformulation (14).

So far we have neglected degeneracy issues. Unfortunately, it is well known that explicit MPC based on piecewise affine cost functions often lead to degenerate multiparametric linear programs [10].

A. Dual degeneracy

Dual degeneracy may occur and can be tackled for instance as in [14]. Note that, however, full-dimensional critical regions of dual degeneracy in multiparametric multiobjective LPs are less likely to occur than in multiparametric LPs. In fact, assume that a region CR of dual degeneracy exists in the (μ, x) space and let (μ_0, x_0) in the interior of CR such that $\mathcal{B}((\mu_0,x_0),\tau) \subset CR$ for some $\tau > 0$, where $\mathcal{B}((\mu_0,x_0),\tau)$ is the Euclidean ball of radius τ centered in (μ_0, x_0). Let $\gamma = \mu-\mu_0 \in \mathbb{R}^l$ and consider the optimality conditions

$$\begin{aligned} \tilde{G}z &= \tilde{b}+\tilde{S}x && \text{(17a)}\\ -\tilde{G}'\lambda &= c_0+C'_\mu\mu. && \text{(17b)} \end{aligned}$$

By transposing and multiplying by z (17b) we get

$$-\lambda'\tilde{G}z = (c_0+C'_\mu\mu)z \tag{18}$$

and, by substituting (17a) in (18), $\begin{bmatrix} \tilde{G} \\ c'_0+\gamma'C_\mu \end{bmatrix} z = \begin{bmatrix} I \\ -\tilde{\lambda}' \end{bmatrix}(\tilde{b}+\tilde{S}x) - \begin{bmatrix} 0 \\ \mu'_0C_\mu \end{bmatrix}$. Assuming that the LP is not primal degenerate, by (17a) it must hold that $\tilde{G}$ has less than d linearly independent rows. Similarly, absence of primal degeneracy and presence of dual degeneracy imply that multiple solutions z are possible if and only if rank $\begin{bmatrix} \tilde{G} \\ c'_0+\gamma'C_\mu \end{bmatrix} < d$, $\forall\gamma \in \mathbb{R}^l$, $\|\gamma\| \le \tau$.

B. Primal degeneracy

Primal degeneracy occurs when more than d constraints are active at optimality and the dual solution is not unique. In this case matrix $\tilde{G}$ is not square and formulas (15) cannot be applied. We propose below two alternative approaches to handle primal degeneracy.

1) Projection method: Given a set I of active constraints, the critical region $CR_I = CR_{xI}\times CR_{\mu I}$ can be obtained by choosing CR_{xI} as suggested in [14], and $CR_{\mu I}$ by projection of the polyhedron defined by

$$\begin{array}{l} C'_\mu\mu + c_0 + \tilde{G}'\tilde{\lambda} = 0\\ \tilde{\lambda} \ge 0 \end{array} \tag{19}$$

onto the μ-space.

2) Active constraint set selection: Projection can be a time consuming task, so we propose next an alternative method. In the case of primal degeneracy there may be multiple choices for the combination I of active constraints for which the corresponding submatrix $\tilde{G}$ has d linearly independent rows. As suggested in [18], one can select arbitrarily a combination I of active constraints and proceed. The drawback of this approach is that overlapping regions of primal degeneracy may be generated. Although overlaps do not change the solution (they only augment the memory space used for storing the solution), overlaps may be eliminated by processing the solution a posteriori.

C. Properties of the explicit solution

Lemma 1: Consider the multiparametric linear problem (12) with parameters $\mu \in \mathbb{R}^l$ in the cost function and $x \in \mathbb{R}^n$ in the r.h.s. of the constraints. Then the set F^* of parameters (μ, x) for which (12) has a solution is a convex polyhedron, the value function $V^*: F^* \to \mathbb{R}$ is continuous w.r.t. (μ, x), convex and piecewise affine w.r.t. μ for any given x and w.r.t. x for any given μ. Moreover, there exists a piecewise affine optimizer function $z^*: F^* \to \mathbb{R}^d$ of (μ, x) defined as

$$z^*(\mu,x) = \begin{array}{l} \phi_i x+\gamma_i \text{ if } \begin{bmatrix} H_i^\mu & 0\\ 0 & H_i^x \end{bmatrix}\begin{bmatrix}\mu\\ x\end{bmatrix} \le K_i,\\ i=1,\ldots,n_r. \end{array} \tag{20}$$

Proof: As $F^* = \{\mu \in \mathbb{R}^l : \ [1 \ \dots \ 1]\mu \le 1, \ \mu \ge 0\} \times \Pi_x\{(z,x) \in \mathbb{R}^{d+n}: \ Gz \le b+Sx\}$, where Π_x is the projection operator over the x-space, clearly F^* is a convex polyhedron. Piecewise affinity of z^* follows by construction, as by (15a) z^* depends in an affine way on μ and x, respectively (in case of dual degeneracy, affine solutions from (15) can be extracted as described in [14]) and therefore is piecewise affine with respect to (μ, x) once the partition of F^* in polyhedral cells is defined by the multiparametric solver. In each region CR_i, the value function $V^*(\mu, x) = (c_0 + \mu' C_\mu)(\phi_i x + \gamma_i)$ where $\phi_i x + \gamma_i = z^*(\mu, x)$ is the affine expression of the optimizer in CR_i. Convexity and continuity of V^* with respect to x for any given μ follow by the properties of the multiparametric LP (12) [19, p. 180], and, by duality, the same properties hold with respect to μ. As $V^* = (c_0 + \mu' C_\mu) z^*(\mu, x)$ is the composition of continuous functions, it is also continuous with respect to (μ, x). The particular structure of the critical regions $CR_i = CR_{xi} \times CR_{\mu i} = \{[\begin{smallmatrix} \mu \\ x \end{smallmatrix}] : \left[\begin{smallmatrix} H_i^\mu & 0 \\ 0 & H_i^x \end{smallmatrix}\right] [\begin{smallmatrix} \mu \\ x \end{smallmatrix}] \leq K_i\}$ in (20), in the absence of degeneracy, follows from (16). In case of dual (and/or primal) degeneracy, from (17) (and/or (19)) it also follows the same structure of CR_i by imposing primal and dual feasibility on z and λ. ■

Continuity of the optimizer w.r.t. (μ, x) could be proved through mathematical tools from point-to-set maps theory [20].

Remark 1: As discussed earlier, non-strictly positive values of μ may lead to non-Pareto optimal solutions. However, we can either restrict $\mu > 0$ in (22) or, alternatively, tolerate possibly non-Pareto optimal solutions by leaving $\mu \geq 0$. In the first case, the stability result of Theorem 1 still holds.

D. On-line selection of the weight vector

We consider now the on-line selection problem (6) of the weight vector $\alpha^*(x, \alpha_d, J_a)$ for the particular case of f convex and piecewise affine

$$f(\alpha - \alpha_d) = \max\{f_j^\alpha(\alpha - \alpha_d) + f_j^0\},\ j = 1, \ldots, n_f \quad (21)$$

A possible choice for f in (21) is $f(\alpha - \alpha_d) = \|\alpha - \alpha_d\|_\infty$.

Theorem 2: Let $I(x) \subseteq \{1, \ldots, n_r\}$ be the set of indices i of the regions CR_x defined in (16) to which x belongs. Given $\alpha_d = [\alpha_{d0}, \mu_{d1}, \ldots, \mu_{dl}] \in \mathbb{R}^{l+1}$, with $\alpha_{d0} = 1 - \sum \mu_{di}$, the solution to Problem (6) $\alpha^*(x, \alpha_d, J_a) = [1 - \sum_{i=1}^l \mu_i^*, \mu_1^*, \ldots, \mu_l^*]'$, where μ^* can be determined by solving the linear programming problem

$$\begin{array}{ll} \min_{\mu,\beta} & \beta \\ \text{s.t.} & \beta \geq f_j^\alpha \left[\begin{smallmatrix} 0 \\ \mu - \mu_d \end{smallmatrix}\right] + f_j^0,\ j = 1, \ldots, n_f \\ & (\phi_i x + \gamma_i)'(c_0' + C_\mu' \mu) \leq J_a,\ \forall i \in I(x) \\ & \sum_{i=1}^l \mu_i \leq 1 \\ & \mu_i \geq 0,\ i = 1, \ldots, l, \end{array} \quad (22)$$

with $l + 1$ variables and $n_f + \text{card}(I(x)) + 2$ constraints.

Proof: For a fixed x, the value function $V^*(\mu, x)$ is a piecewise affine and convex function of μ that, by the structure of the critical regions CR_i proved in Lemma 1, is defined over the regions $CR_{\mu i}$ indexed by $i \in I(x)$. Hence, thanks to the result of [13] for convex piecewise affine functions, for every fixed x the value function $V^*(\mu, x)$ by Lemma 1 can be evaluated as the maximum of the affine functions $\{(\phi_i x + \gamma_i)'(c_0' + C_\mu' \mu)\}_{i \in I(x)}$. ■

Unfortunately Problem (22) in general is not jointly convex with respect to (μ, β) and (x, μ_d, J_a), due to the fact that $V^*(\mu, x)$ may not be a jointly convex function of (μ, x). Problem (22) needs to be solved on-line for the given values of $x(t)$, $\mu_d(t)$, $J_a(t)$ and the corresponding set of constraints indexed by $I(x(t))$.

IV. Example

Consider a linear system (1) defined by matrices $A = [\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}]$, $B = [\begin{smallmatrix} 0.5 \\ 1 \end{smallmatrix}]$ subject to the constraints $|x(t)| \leq 10$, $|u(t)| \leq 10$. We consider two different objective functions based on the infinity norm defined by $Q_0 = [\begin{smallmatrix} 0.1 & 0 \\ 0 & 1 \end{smallmatrix}]$, $R_0 = 0.1$, $P_0 = [\begin{smallmatrix} 3.5669 & 1.3986 \\ 0.0001 & 3.1040 \end{smallmatrix}]$, and $Q_1 = [\begin{smallmatrix} 1 & 0 \\ 0 & 0.1 \end{smallmatrix}]$, $R_1 = 0.1$, $P_1 = [\begin{smallmatrix} 9.6085 & 1.1401 \\ -0.2965 & 9.4107 \end{smallmatrix}]$. These matrices satisfy the constraint $1 - \|P_i(A+BK)P_i^{-L}\|_\infty - \|Q_i P_i^{-L}\|_\infty - \|R_i K P_i^{-L}\|_\infty \geq 0$ with $P_i^{-L} = [P_i' P_i]^{-1} P_i'$, $K = [-0.5\ -1.4]$, for $i = 0, 1$. This implies that each cost function satisfies (8a) for the common local controller $u = Kx$, see [12]. In order to guarantee stability, we consider the terminal region

$$\Omega = \left\{ x \in \mathbb{R}^2 | \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & -1 \\ -1 & 0 \\ -0.5 & -1.4 \\ 0.5 & 1.4 \end{bmatrix} x \leq \begin{bmatrix} 10 \\ 10 \\ 10 \\ 10 \\ 10 \\ 10 \end{bmatrix} \right\}$$

defined by the positive invariant set of the system in closed-loop with the local controller. The set Ω satisfies (8b) and (8c) for the common local controller $u = Kx$. This implies that the cost function and the terminal region satisfy the assumptions of Theorem 1, so that the multi-objective MPC loop is asymptotically stable.

We compare three different controllers: (i) $h_0(x(t)) = E_0 U^*(x(t), [1\ 0]')$, (ii) $h_1(x(t)) = E_0 U^*(x(t), [0\ 1]')$, and (iii) $h_{mo}(x(t)) = E_0 U^*(t)$, where $U^*(x, \alpha)$ is the solution of (3b), $U^*(t)$ is the optimal input trajectory of the proposed multi-objective scheme and $E_0 = [I\ 0\ \ldots\ 0]$ is such that $E_0 U = u_0$. We use the target weight vector $\alpha_d(t) = [(1 - \mu_d(t))\ \mu_d(t)]'$ with

$$\mu_d(t) = \left\{ \begin{array}{l} 1 \text{ if } \|x(t)\|_2 > 10 \\ \|x(t)\|_2/10 \text{ otherwise.} \end{array} \right.$$

The control laws h_0 and h_1 correspond to the standard MPC controllers based on the cost functions $J_0(\cdot)$ and $J_1(\cdot)$ subject to the same set of constraints. Note that although both controllers clearly provide different closed-loop performance, they guarantee that the closed-loop system is asymptotically stable and they have the same feasibility region. For this particular example, h_0 provides a slower convergence to the origin than h_1 but is more robust with respect to measurement noise. Hence, when the state is far from the origin, it is desirable to drive the system fast to the equilibrium point, however, once the origin has been reached it is desirable to improve the robustness with respect to noise. This can be achieved by switching between both standard MPC controllers, however in this case, stability is not guaranteed. The proposed multi-objective MPC controller allows us to profit from the properties of both controllers, while guaranteeing closed-loop stability. The target weight vector $\alpha_d(t)$, which is a tuning parameter for the multi-objective controller h_{mo}, has been chosen in order to give priority to h_1 when the state is far from the origin, and to h_0 once is near the origin.

A set of simulations was carried out starting from different states inside the feasibility region of the controllers (note that this region is equal for all of them). In the simulations we consider random measurement noise, $u(t) = h_i(x(t)+w(t))$, $\|w(t)\|_\infty \leq 0.5$, where $w(t) = 0$ for all $t \leq 30$. In order to measure the robustness to measurement noise the

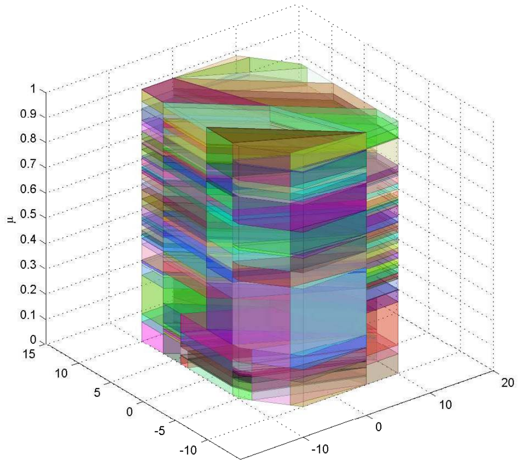

Fig. 1. Explicit solution for $N = 4$ with respect to the parameter vector $[x' \ \mu]'$

following values are measured: $SNR_u = \frac{\sum_{t=31}^{f} \|u(t)\|_2}{\sum_{t=31}^{f} \|w(t)\|_2}$, $SNR_x = \frac{\sum_{t=31}^{f} \|x(t)\|_2}{\sum_{k=31}^{f} \|w(t)\|_2}$, where f is the length of the simulation. Performance is evaluated according to the time t_r needed for the norm of the state to go below the 10% of its initial value. The average results over 100 simulations with $N = 4$ and $f = 100$ are:

Controller	SNR_u	SNR_x	t_r
h_0	0.4989	0.5934	7.62
h_1	1.1168	0.7972	3.1
h_{mo}	0.5797	0.5959	3.13

The results show that the multi-objective controller provides signal to noise ratios similar to h_0, with a time t_r similar to h_1. To carry out the simulations, the explicit solution of (5) was obtained using the LCP multiparametric solvers described in [6]. The resulting multi-parametric optimization problem is defined by three parameters, namely the two states and the weight vector μ. Figure 1 shows the regions of the PWA explicit solutions. This PWA function is has 290 regions. It can be seen that the partitions in the μ-space are orthogonal to the plane defined by the state vector parameters, consistently with (16). It is interesting to note that the facet-to-facet property [17] does not hold for this class of problems.

V. CONCLUSIONS

This paper has proposed an MPC formulation based on multiple performance criteria that enjoys closed-loop stability properties. Compared to standard MPC formulations based on a single performance index, the multiobjective criterion allows one to take into account several and often irreconcilable control specifications, such as high-bandwidth (closed-loop promptness) far away from convergence, and low-bandwidth (good noise rejection properties) near convergence. The corresponding optimization problem was solved as a multiparametric linear complementarity problem that provides the optimal Pareto solution as a piecewise affine function of the state vector and of the set of parameters that weight the different criteria in the equivalent scalarized problem. Thanks to such an explicit characterization of the solution, given a higher-level reference signal specified at each time step for the preferred weights, an optimal selection of the weights can be computed on-line by solving a simple convex programming problem, namely a linear programming problem in case all objectives are convex piecewise affine functions.

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