OPTIMAL CONTROL OF UNCERTAIN PIECEWISE AFFINE/MIXED LOGICAL DYNAMICAL SYSTEMS

M.P. Silva*, **A. Bemporad**†, **M.A. Botto***, **J. Sá da Costa***

* Technical University of Lisbon, Instituto Superior Técnico,
Department of Mechanical Engineering, GCAR/IDMEC,
Av. Rovisco Pais, 1049-001 Lisboa, Portugal
Fax: +351 218 498 097 E-mail: miguel.silva@dem.ist.utl.pt
† Università di Siena, Dipartimento di Ingegneria dell'Informazione,
Via Roma 56, 53100 Siena, Italy
Fax: +39 02 700 543345 E-mail: bemporad@dii.unisi.it

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Abstract

This paper proposes an approach to extend the mixed logical dynamical modelling framework for synthesizing robust optimal control actions for constrained piecewise affine systems subject to bounded additive input disturbances. Rather than using closed-loop dynamic programming arguments, robustness is achieved here with an open-loop optimization strategy, such that the optimal control sequence optimizes nominal performance while robustly guaranteeing that safety/performance constraints are respected. The proposed approach is based on the robust mode control concept, which enforces the control input to generate trajectories such that the mode of the system, at each time instant, is independent of the disturbances.

1 Introduction

Hybrid systems are defined as systems composed of continuous and discrete variables [1]. Piecewise affine (PWA) systems [11], mixed logical dynamical (MLD) systems [2], linear complementarity (LC) systems [4] and max-min-plus-scaling (MMPS) systems [9] are different approaches, though equivalent models of hybrid dynamical systems. Their equivalence, shown in [5], allows one to interchange analysis and synthesis tools among them. Specifically, MLD systems, which are systems described by interdependent physical laws (with linear dynamics), logic rules (if-then-else rules) and operating constraints are very pertinent in the real-world engineering environment. In fact, this model is a compromise between applicability and complexity.

As parametric uncertainties and input disturbances are always present on real-life applications, analysis and synthesis procedures when additive input disturbances and/or parametric uncertainties are present on the continuous linear dynamics of the hybrid model, are topics that are starting to deserve the attention of researchers. The research works of [6–8] are some examples of analysis/synthesis procedures of uncertain piecewise linear hybrid dynamical systems. Backward reachability computations [8] (seminally proposed in [3]), as well as polytopic set algebra [6,12] are the tools proposed to deal with the nonlinearity and non-convexity properties of the hybrid/PWA system. Specifically, in [7] it is considered the robust time-optimal, robust optimal and robust receding horizon control problems, and tools from computational geometry, dynamic and parametric programming are used to obtain explicit state-feedback control laws. However, as expected due to the complexity of the tackled problems, the algorithms based directly on the results presented in [7] might be to inefficient to be realizable for large or complex systems, as the authors remark.

The approach pursued in this paper follows a different direction from the cited references. Instead of computing state feedback laws based on reachability computations, computational geometry tools and dynamic programming, the goal of this paper is to extend the MLD framework to uncertain PWA systems and to present a computational procedure, based on mixed-integer programming, to obtain open-loop control sequences that guarantee the fulfillment of dynamic and operating constraints and optimize a nominal performance criterion. Although the complexity of the problem proposed to tackle is not comparable to the one solved e.g. in [7], its possible extension to a receding horizon strategy based on closed-loop prediction is stimulating.

The paper has the following structure. In Section 2, the problem to solve is presented. The MLD prediction/synthesis model is developed in Section 3. The control synthesis algorithm is described in Section 4, and in Section 5 some conclusions are drawn.

2 Problem statement

Consider a discrete-time PWA system, subject to bounded additive input disturbances, used for synthesis purposes, such that safety and/or performance constraints are respected. The robust constrained optimal control problem for the uncertain system is defined as follows.

Problem 1. *Given an initial state x_0 and a final time N, find (if it exists) the control sequence $\mathbf{u} \triangleq \{u(0), \ldots, u(N-1)\}$ which (i) transfers the state from x_0 to a given final set $\mathcal{C}(N)$ that* *contains a target state x_f, and (*ii*) minimizes the performance index*

$$J_N(x_0,\mathbf{u}) \triangleq \sum_{k=0}^{N-1} \|\bar{x}(k)-x_f\|^2_{Q,2} + \\ +\|u(k)-u_f\|^2_{R,2} + \|\bar{x}(N)-x_f\|^2_{P,2} \tag{1}$$

subject to:

$$x(0)=\bar{x}(0)=x_0 \tag{2a}$$

$$\bar{x}(k+1)=A_i\bar{x}(k)+B_iu(k)+e_i\ ,\ \text{if}\ \left[\begin{smallmatrix}\bar{x}(k)\\u(k)\end{smallmatrix}\right]\in\Omega_i \tag{2b}$$

$$x(k+1)=A_ix(k)+B_iu(k)+e_i+W_iv(k)\ ,\ \text{if} \\ \left[\begin{smallmatrix}x(k)\\u(k)\end{smallmatrix}\right]\in\Omega_i\triangleq\{\left[\begin{smallmatrix}x(k)\\u(k)\end{smallmatrix}\right]:\ F_ix(k)+G_iu(k)\le h_i\} \tag{2c}$$

$$\left[\begin{smallmatrix}x(k)\\u(k)\end{smallmatrix}\right]\in\mathcal{C}(k)\triangleq\{\left[\begin{smallmatrix}x(k)\\u(k)\end{smallmatrix}\right]:\ K(k)x(k)+ \\ +L(k)u(k)\le m(k)\},\ \forall v(k)\in\mathcal{V},\ \text{for}\ k=0,\ldots,N \tag{2d}$$

where $u(k)\in\mathbb{U}\subset\mathbb{R}^m$, $x(k)\in\mathbb{X}\subset\mathbb{R}^n$ and $v(k)\in\mathbb{V}\subset\mathbb{R}^q$ denote the input, state and disturbance vectors, respectively, at time k. The index i represents the current system mode ($i\in\{1,\ldots,s\}$). The partitions Ω_i are convex polytopes (i.e. closed and bounded polyhedra) in the input+state space. Moreover, $\Omega=\bigcup_{i=1}^{s}\Omega_i$, $\overset{\circ}{\Omega}_i\bigcap\overset{\circ}{\Omega}_j=\emptyset$, $\forall\ i\neq j$, where $\overset{\circ}{\Omega}_i$ denotes the interior of the polytope Ω_i. $\mathbb{U}$, $\mathbb{X}$ are convex polytopes, as well as $\mathbb{V}$, according to the typical unknown-but-bounded characterization of disturbances, with $0\in\mathbb{V}$. $\mathcal{C}(k)$ denotes sets of (possibly time-varying) constraints on the input+state space. A_i, B_i, W_i, F_i, G_i, $K(k)$ and $L(k)$ are real matrices of appropriate dimensions, h_i and $m(k)$ are real vectors, and e_i is the affine real vector, for all $i=1,\ldots,s$. Furthermore, $\|x\|^2_{Q,2}\triangleq x'Qx$, $\bar{x}(k)$ represents the nominal trajectory, Q, R and P are symmetric positive definite matrices, and x_f, u_f are given desired target vectors.

The main goal of this paper is to solve Problem 1 using the MLD framework, which was developed to model hybrid systems and synthesize optimal and receding horizon control laws [2]. As $\mathbb{X}$, $\mathbb{U}$ are bounded, the PWA system can be expressed as an MLD system, which has the structure given in Figure 1 [13]. As the PWA system (2c) does not include discrete inputs or states, no discrete inputs or states are needed to model it within the MLD framework (only binary auxiliary variables).

Consider the discrete-time PWA dynamics described by (2c). An equivalent MLD model of the dynamics is presented next, where the first inequality concerns the continuous/discrete interface and the second inequality concerns the discrete/continuous interface

$$x(k+1)=B_{z_x}z_x(k)+B_{z_u}z_u(k)+B_\delta\delta(k)+B_{z_v}z_v(k) \tag{3a}$$

$$E_\delta^{c/d}\delta(k)\le E_x^{c/d}x(k)+E_u^{c/d}u(k)+e^{\,c/d} \tag{3b}$$

$$E_{z_x}^{d/c}z_x(k)+E_{z_u}^{d/c}z_u(k)+E_\delta^{d/c}\delta(k)+E_{z_v}^{d/c}z_v(k)\le \\ \le E_x^{d/c}x(k)+E_u^{d/c}u(k)+E_v^{d/c}v(k)+e^{\,d/c} \tag{3c}$$

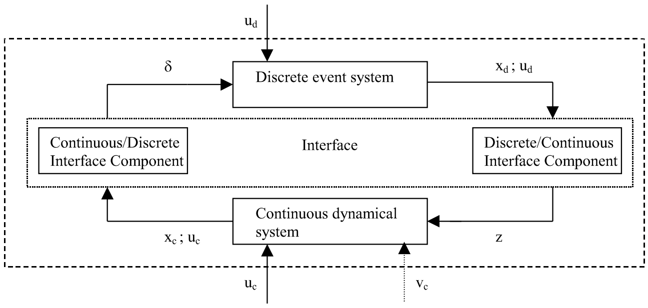

Figure 1: A generic MLD system: The c/d subscripts means continuous/discrete signals, v_c represents a continuous disturbance signal, δ and z are auxiliary discrete and continuous variables, respectively.

where $\delta(k)$ is an auxiliary vector with binary (zero/one) entries that defines the mode i (or equivalently partition i) of the system ($\dim[\delta]=(s\times1)$), moreover if mode i is active then $\delta_i(k)=1$ and $\delta_j(k)=0$, $\forall j\neq i$, $i,j\in\{1,\ldots,s\}$ (these constraints are incorporated in inequality (3b)); $z_{xi}(k)=\delta_i(k)x(k)$ are auxiliary continuous variable ($\dim[z_{xi}]=\dim[x]=(n\times1)$); using the Kronecker product to abbreviate notation, $z_x(k)=\delta(k)\otimes x(k)$, ($\dim[z_x]=(s\cdot n\times1)$); $z_u(k)=\delta(t)\otimes u(k)$ ($\dim[z_u]=(s\cdot m\times1)$) and $z_v(k)=\delta(k)\otimes v(k)$ ($\dim[z_v]=(s\cdot p\times1)$). Notice also, that $B_{z_x}\equiv[\,A_1\ A_2\ \ldots\ A_s\,]$, $B_{z_u}\equiv[\,B_1\ B_2\ \ldots\ B_s\,]$, $B_\delta\equiv[\,e_1\ e_2\ \ldots\ e_s\,]$, and $B_{z_v}\equiv[\,W_1\ W_2\ \ldots\ W_s\,]$. The system defined by equations (3) can be simulated if the system is well-posed [2] (i.e. all auxiliary variables are uniquely defined for all ($x(k)$,$u(k)$,$v(k)$)) and disturbances and control inputs are exactly known. However, the purpose of the paper is to develop a computational procedure, using the MLD framework, to determine an open-loop finite horizon optimal control sequence based on the prediction of future states, assuming that the disturbances are unknown and the only information about them are their bounds. Yet, the MLD framework was developed to deal with deterministic systems [2] and its extension to uncertain systems rises some problems. Clearly, when bounded disturbances are present, the predicted state $x(k)$ is set-valued. The computation of these sets is not an easy task since the system is nonlinear. The effect of the bounded disturbance on the state is dependent on the trajectory of the system, which, of course, also depends on the initial state and input. As the dynamic mode (i.e. the active partition) changes, the effect of the perturbation is different. Therefore, the influence of the disturbance can not be predicted independently of the trajectory of the system, as for linear systems. Furthermore, the set that defines the state, at each time step k, is in general a non-convex set, specifically, it is a union of convex sets when $\mathbb{V}$ is convex. Therefore, the typical approach for linear systems (e.g. as used in [10]), of using the extreme disturbance realizations can not be directly applied in this case.

2.1 The concept of robust mode control

To overcome the aforementioned problem, we propose to restrict the admissible control sequences to only those that guarantee that, for every value of the disturbance, the mode of the system is unique at each time step k. In fact, the uncertainty associated with the uncertain PWA system dynamics can be divided into two types of uncertainties: state-uncertainty and mode-uncertainty. By state uncertainty we mean that the exact value of the state at a given time step k is not known, but the mode is known. By mode uncertainty we mean that neither the mode nor the state are exactly known at a given time step k. State uncertainty is a disadvantageous, but intrinsic, property of the uncertain system which generates convex uncertainty sets. Mode uncertainty generates, as time evolves, unions of convex sets (which, in general, are non-convex). However, mode uncertainty is not an intrinsic property of the uncertain system, since a cautious control action may avoid it. Hence, the main property of the *robust mode control* is to generate control sequences such that the mode of the system is "certain" and, for each admissible control sequence, the uncertainty associated with the state is defined by a convex set. Although this extra structure on the problem has some computational advantages in terms of set descriptions, clearly, the main disadvantage of this restriction is the smaller domain of feasible control sequences (this disadvantage could be mitigated by extending the robust mode control concept to a closed-loop prediction policy).

The robust mode control concept, however, allow us to extend the MLD framework for designing robust control inputs for uncertain PWA systems. Next, the restrictions and properties of the robust mode control sequence are explicitly determined on a MLD-based modelling context. Consider the *nominal* MLD system, obtained from (3) for null disturbances (i.e. $v(k) = 0$ and $z_v(k) = 0,\ \forall k$). Consider also that system (3) is completely well-posed [2], i.e. there exist uniquely defined mappings $\mathcal{D} : R^n \times R^m \to \{0,1\}^s$, $\mathcal{Z}_x : R^n \times R^m \to R^{n.s}$ and $\mathcal{Z}_u : R^n \times R^m \to R^{m.s}$, such that the $\delta(k)$, $z_x(k)$ and $z_u(k)$ variables are uniquely defined for given $x(k)$ and $u(k)$ (when $\left[\begin{smallmatrix} x(k) \\ u(k) \end{smallmatrix}\right] \in \Omega$), such that $\delta(k) = \mathcal{D}\left(x(k), u(k)\right)$, $z_x(k) = \mathcal{Z}_x\left(x(k), u(k)\right)$ and $z_u(k) = \mathcal{Z}_u\left(x(k), u(k)\right)$. Finally, consider that when $v(i) = 0,\ \forall i \in \{0, \ldots, k-1\}$ and an input sequence u_0^{k-1} is applied to the MLD system, then the state trajectory is denoted by $\bar{x}^{\mathbf{u}}$. This input sequence and the null disturbance sequence define the nominal trajectory of the system, i.e. $\bar{x}(k) = F(k; x_t, u_0^{k-1}, 0_0^{k-1})$, where $F(.)$ denotes the system dynamics defined by equations (3). Therefore, by well-posedness of δ, z_x and z_u a unique mode trajectory is obtained: $\delta_0^{k-1} \triangleq \{\delta(0), \ldots, \delta(k-1)\}$.

To guarantee that the value of the δ variables are independent of the presence of the bounded disturbances v, the δ variables must verify the following condition

$$\begin{aligned} \delta(k) &= \mathcal{D}\left(\bar{x}(k), u(k)\right) \\ &= \mathcal{D}\left(x(k), u(k)\right),\ \forall v(k) \in \mathbb{V},\ \forall k. \end{aligned} \tag{4}$$

If the input sequence u_0^k is such that (4) is respected, then it defines a robust mode input sequence, and so the mode of the system, i.e. $\delta(j)$, is independent of the disturbance realization, for all instants $j \in \{0, \ldots, k\}$, and the state at time step $k+1$, which is set-valued, can be decomposed as follows

$$\begin{aligned} & x^{\mathbf{u}}(k+1) = F\left(k+1; x_0, \mathbf{u}_0^k, \mathbb{V}_0^k\right) \\ =\ & F\left(k+1; x_0, \mathbf{u}_0^k, \mathbf{0}_0^k\right) + \tilde{F}^\delta\left(k+1; 0, \mathbf{0}_0^k, \mathbb{V}_0^k\right) \\ =\ & \bar{x}^{\mathbf{u}}(k+1) + \tilde{x}^{\mathbf{u}}(k+1) \end{aligned} \tag{5}$$

where the first term represents the nominal trajectory and the second term denotes the convex uncertainty set associated with the state, which depends on δ_0^k and on $\mathbb{V}_0^k \triangleq \mathbb{V} \times \mathbb{V} \ldots \mathbb{V}$.

3 The MLD prediction/synthesis model

To obtain the MLD-based prediction/synthesis model in the presence of disturbances, we include the constraint given by equation (4), into the continuous/discrete interface equations (3b) of the MLD model. In effect, this means that the inequality must hold, component-wise, for all possible values of the uncertainty. Hence, let $\mathbf{v}^l \triangleq \left\{v^l(0), \ldots, v^l(N-1)\right\}$ denote a disturbance sequence that take values at the vertices of the polytope $\mathbb{V}_0^{N-1}$, let $l \in \mathcal{L}_v$ index these realizations, and let $\mathbf{x}^l$ denote the state associated with the respective disturbance realization (and the same notation for the other variables). For the sake of completeness, consider also that the safety/performance constraints defined by inequality (2d) are already included in the formulation. Hence, the MLD prediction/synthesis model, which encompasses the robust mode control concept, has the following form (the inequalities should be understood component-wise)

$$x^l(k+1) = B_{z_x} z_x^l(k) + B_{z_u} z_u(k) + B_\delta \delta(k) + B_{z_v} z_v^l(k) \tag{6a}$$

$$E_\delta^{c/d}\delta(k) \leq E_x^{c/d} x^l(k) + E_u^{c/d} u(k) + e^{c/d}\ ,\forall l \in \mathcal{L}_v\ ,\forall k \tag{6b}$$

$$\begin{aligned} & E_{z_x}^{d/c} z_x^l(k) + E_{z_u}^{d/c} z_u(k) + E_\delta^{d/c}\delta(k) + E_{z_v}^{d/c} z_v^l(k) \leq \\ & \quad \leq E_x^{d/c} x^l(k) + E_u^{d/c} u(k) + E_v^{d/c} v^l(k) + e^{d/c} \end{aligned} \tag{6c}$$

$$E_x^{ctr} x^l(k) + E_u^{ctr} u(t) \leq e^{ctr}\ ,\forall l \in \mathcal{L}_v\ ,\forall k. \tag{6d}$$

Now, Problem 1, with the restriction that the input is a robust mode control sequence, can be formulated with the MLD framework using the above equations. However, this formulation substantially increases the number of constraints and variables of the associated optimal control problem, so we adopt here a different approach. By equation (5), we conclude that the state at a given time k can be separated into a nominal value and a convex uncertainty set. As the optimization problem will be solved by a branch-and-bound strategy, the approach adopted here is to compute externally the uncertain terms. Therefore, by equation (5), the MLD prediction/synthesis model, which encompasses the robust mode control concept, can also be formulated as follows

$$\bar{x}(k+1) = B_{z_x}\bar{z}_x(k) + B_{z_u}z_u(k) + B_\delta\delta(k), \quad \bar{x}(0) = x_0 \tag{7a}$$

$$E_\delta^{c/d}\delta(k) \le E_x^{c/d}\bar{x}(k) + \min[E_x^{c/d}\tilde{x}(k)] + E_u^{c/d}u(k) + e^{c/d} \tag{7b}$$

$$E_{z_x}^{d/c}\bar{z}_x(k) + E_{z_u}^{d/c}z_u(k) + E_\delta^{d/c}\delta(k) \le E_x^{d/c}\bar{x}(k) + + E_u^{d/c}u(k) + e^{d/c} \tag{7c}$$

$$E_x^{ctr}\bar{x}(k) + \max[E_x^{ctr}\tilde{x}(k)] + E_u^{ctr}u(k) \le e^{ctr} \tag{7d}$$

Notice that the inequalities should be valid, component-wise, for all possible values of $\tilde{x}(k)$, and so a worst-case computation of $E_x^{c/d}\tilde{x}(k)$ and of $E_x^{ctr}\tilde{x}(k)$ must be done, which is denoted, respectively, by $\min[E_x^{c/d}\tilde{x}(k)]$ and $\max[E_x^{ctr}\tilde{x}(k)]$. As a matter of fact, the uncertainty set, $\tilde{x}(k)$, associated with the MLD-based prediction/synthesis model, is a function of $\mathbb{V}_0^{k-1}$ and also of δ_0^{k-1} (see (5)), where the former can be viewed as external parameters, the latter is composed of optimization variables. A method to compute the terms associated with the uncertainty is presented next.

3.1 Computation of the worst-case uncertainty terms

The restriction of the input sequence to a robust mode input sequence allows that the computation of $\tilde{x}(k)$ can be based on $\tilde{x}(k) = B_{z_x}\tilde{z}_x(k-1) + B_{z_v}z_v(k-1)$, $\tilde{x}(0) = 0$ (expression (6a) minus expression(7a), at time k). This equality can be expressed as explicitly dependent on $\delta(k-1)$, $v(k-1)$ and $\tilde{x}(k-1)$ as follows

$$\begin{aligned}\tilde{x}(k) &= B_{z_x}\left(\delta(k-1)\otimes\tilde{x}(k-1)\right) + \\ &\quad +B_{z_v}\left(\delta(k-1)\otimes v(k-1)\right), \quad \tilde{x}(0) = 0\end{aligned} \tag{8}$$

For compactness of notation consider the following representation

$$B_{z_x}^{\delta(k)}\tilde{x}(k) \triangleq B_{z_x}\left(\delta(k)\otimes\tilde{x}(k)\right) \tag{9}$$

$$B_{z_v}^{\delta(k)}v(k) \triangleq B_{z_v}\left(\delta(k)\otimes v(k)\right), \tag{10}$$

and consider also the following notation for the product of matrices

$$\begin{aligned}B_{z_x}^{\delta(j\to i)} &\triangleq B_{z_x}^{\delta(j)}B_{z_x}^{\delta(j-1)}\dots B_{z_x}^{\delta(i+1)}B_{z_x}^{\delta(i)}, \text{ if } i\le j \\ &\triangleq I_n, \quad \text{otherwise,}\end{aligned} \tag{11}$$

where I_n represents a square identity matrix with dimension n. Using the above notation, equation (8) can be expressed as a function of sequences δ_0^{k-1} and v_0^{k-1}

$$\begin{aligned}\tilde{x}(k) &= \sum_{i=0}^{k-1} B_{z_x}^{\delta(k-1\to i+1)}B_{z_v}^{\delta(i)}v(i), \text{ if } k\ge 1 \\ &= 0, \quad \text{if } k=0.\end{aligned} \tag{12}$$

Observing (7), the $\min$ and $\max$ componentwise values of $[E_x^i\tilde{x}(k)]$ (where $i \in \{c/d, ctr\}$), have to be computed. Clearly, the computation of the $\min$ and $\max$ componentwise values of the set $[\cdot]$ implies that the resultant set is a worst-case approximation of the original set. Considering that the sequence δ_0^{k-1} is known, the computation of $\min[E_x^{c/d}\tilde{x}(k)]$ ($\max[E_x^{ctr}\tilde{x}(k)]$) can be done by solving $n_{c/d}$ (n_{ctr}) linear programs, where $n_{c/d}$ (n_{ctr}) is the number of rows of $E_x^{c/d}$ (E_x^{ctr}). Pre-multiplying equality (12) by $E_x^{c/d}$ (E_x^{ctr}) the j^{th} minimization (maximization) linear program is defined as follows

$$\min[E_x^{c/d}\tilde{x}(k)]_j \ :$$

$$\min_{v_0^{k-1}\in\mathbb{V}_0^{k-1}} \left(E_x^{c/d}\right)_j \sum_{i=0}^{k-1} B_{z_x}^{\delta(k-1\to i+1)}B_{z_v}^{\delta(i)}v(i) \tag{13}$$

$$\max[E_x^{ctr}\tilde{x}(k)]_j \ :$$

$$\max_{v_0^{k-1}\in\mathbb{V}_0^{k-1}} \left(E_x^{ctr}\right)_j \sum_{i=0}^{k-1} B_{z_x}^{\delta(k-1\to i+1)}B_{z_v}^{\delta(i)}v(i) \tag{14}$$

where $(E_x^{c/d})_j$ and $(E_x^{ctr})_j$ denotes the j^{th} row vector of $E_x^{c/d}$ and of E_x^{ctr}, respectively. Therefore, at each time step k, with $k\ge 1$, we have to solve $n_{c/d}+n_{ctr}$ linear programs.

4 The control synthesis algorithm

At this stage, we have presented the main tools to solve the robust mode optimal control problem (RMOCP). However, as the RMOCP defines a mixed-integer quadratic optimization problem, we will adopt a branch-and-bound (B-B) strategy to solve it. A B-B technique solves a subproblem of the original problem at each node of the generated tree, so next we present the general formulation of one of these subproblems.

Problem 2. *Definition of subproblem $P2(\Delta_0^j)$.*
*Given an initial state x_0, a final time N, and a partial mode sequence Δ_0^j between time instants 0 and j, find (if it exists) the control sequence $\mathbf{u}\equiv\{u(0),u(1),\dots,u(N-1)\}$, and the auxiliary sequences δ_{j+1}^{N-1} (with the binary components relaxed to the 0-1 real interval), $\mathbf{z_u}$ and $\bar{\mathbf{z}}_\mathbf{x}$, which (*i*) transfer the state from x_0 to a given final set $\mathcal{C}(N)$ that contains a target state x_f and (*ii*) minimize the performance index*

$$\begin{aligned}&J_N(x_0,\mathbf{u},\delta_{j+1}^{N-1},\mathbf{z_u},\bar{\mathbf{z}}_\mathbf{x}) \triangleq \\ &\textstyle\sum_{k=0}^{N-1} \|\bar{x}(k)-x_f\|_{Q,2}^2 + \|u(k)-u_f\|_{R,2}^2 + \\ &+\|\bar{x}(N)-x_f\|_{P,2}^2\end{aligned} \tag{15}$$

subject to:

$$\bar{x}(k+1) = B_{z_x}\bar{z}_x(k) + B_{z_u}z_u(k) + B_\delta\delta(k), \quad \bar{x}(0) = x_0 \tag{16a}$$

$$E_\delta^{c/d}\delta(k) \le E_x^{c/d}\bar{x}(k) + \min[E_x^{c/d}\tilde{x}(k)] + E_u^{c/d}u(k) + e^{c/d} \tag{16b}$$

$$E_{z_x}^{d/c}\bar{z}_x(k) + E_{z_u}^{d/c}z_u(k) + E_\delta^{d/c}\delta(k) \le E_x^{d/c}\bar{x}(k) + + E_u^{d/c}u(k) + e^{d/c} \tag{16c}$$

$$E_x^{ctr}\bar{x}(k) + \max[E_x^{ctr}\tilde{x}(k)] + E_u^{ctr}u(k) \le e^{ctr}, \tag{16d}$$

where the terms $\min[E_x^{c/d}\tilde{x}(k)]$ *and* $\max[E_x^{ctr}\tilde{x}(k)]$ *are previously computed by Equations (13) and (14), respectively (for* $k > j+1$ *these terms are set to zero).*

In [2] the optimal control sequence is computed via a B-B algorithm, a quite efficient strategy for solving optimization problems with a combinatorial characteristic. It proceeds by traversing a tree in which each node is a subproblem of the initial problem in order to find a feasible leaf node with minimal value of the cost function. Before starting the description of the algorithm, consider the following notation: the optimal cost of each subproblem $P2(\Delta_0^j)$ is denoted by $V_N(P2(\Delta_0^j))$ and its solution by $\arg^*(P2(\Delta_0^j))$; the problem at the root node is denoted by $P2(root)$ (or by $P2(\Delta_0^{-1})$) and is obtained by relaxing all binary variables which compose δ_0^{N-1} and considering that disturbances are null for all time steps $k \in \{0,\ldots,N-1\}$, i.e. the $min[.]$ and $max[.]$ terms are set equal to zero. $V_N(P2)$ and $\arg^*(P2)$ have the correspondent interpretation when the robust mode optimal control problem is considered. Next, the pseudo code that defines the main steps of the algorithm is presented. Figure 2 schematically represents the tree associated with the proposed B-B based synthesis algorithm.

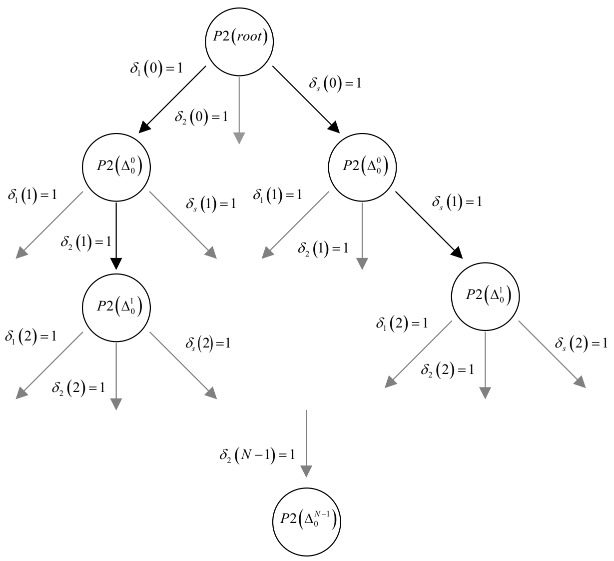

Figure 2: The synthesis algorithm based on a branch-and-bound strategy.

Algorithm 1. ***B-B based synthesis algorithm****.*
Initialize all data structures: $\mathcal{S} = \emptyset$ *(empty stack),* $j = -1$, $V_N(P2) = \infty$, $arg^*(P2) =$ *"infeasible", (note:* $P2(\Delta_0^{-1}) \equiv P2(root)$*);*

Push $P2(\Delta_0^j)$ *onto the top of stack* $\mathcal{S}$*;*
While $\mathcal{S} \neq \emptyset$ ***do***
  Pop $P2(\Delta_0^j)$ *off the top of the stack* $\mathcal{S}$ *and solve* $P2(\Delta_0^j)$*;*
  If $j = N-1$ *and* $V_N(P2(\Delta_0^{N-1})) < V_N(P2)$ ***then***
  assign $V_N(P2) = V_N(P2(\Delta_0^{N-1}))$,
  $arg^*(P2) = arg^*(P2(\Delta_0^{N-1}))$*;*
  Else
    If $V_N(P2(\Delta_0^j)) < V_N(P2)$ ***then*** *Subdivide* $P(\Delta_0^j)$ *into* s *subproblems, by generating the branches corresponding to all possible* s *vectors (modes) of* $\delta(j+1)$ *(fix the* i *component to one and the others to zero); sort the problems by decreasing value of* $V_N(P2(\delta_0^{j+1}))$ *and index the sorted problems at each node by the* l *variable, i.e. by* $P2^l(\Delta_0^{j+1})$*;*
      For $l = 1$ *to* s
        If $V_N(P2^l(\Delta_0^{j+1})) < V_N(P2)$ ***then***
        Push $P2^l(\Delta_0^{j+1})$ *onto the top of* $\mathcal{S}$*;*
        Endif;
      Endfor;
    Endif;
  Endifelse;
Endwhile;
Return *"optimal cost and optimal argument:"* $V_N(P2)$, $arg^*(P2)$.

The proposed B-B algorithm is based on depth-first branching strategy, i.e. the nodes in the search tree are explored going one step deeper into the B-B tree at each iteration by choosing, on the nodes just created, the one with lower cost, and so the minimum lower bound of P2, i.e. the robust mode optimal control problem. Summarizing, the algorithm executes the following steps. At the root node, all binary variables δ_0^{N-1} are relaxed and disturbances are not considered. $P2(root)$ is solved, meaning to solve the nominal (without disturbances) and relaxed problem and so a quadratic programming (QP) algorithm is used. The cost $V_N(P2(root))$ is a lower bound on the optimal cost of the original problem P2, i.e. $V_N(P2)$, because $P2(root)$ is less constrained (the binary variables are relaxed and the effect of disturbances is not taken into account). The next level of the tree is composed with the nodes generated by imposing to each branch one of the possible s modes of $\Delta_0^0 \equiv \{\delta(0)\}$. At each node, the problem $P2(\Delta_0^0)$ is solved, i.e. the disturbances are considered null for $k \geq 1$ and the sequence δ_1^{N-1} is relaxed. These problems can be also solved by a QP algorithm. The optimal cost of $P2(\Delta_0^0)$, i.e. $V_N(P2(\Delta_0^0))$, is computed and compared with the cost of the best existing feasible solution. If the computed cost is higher or equal to the existing feasible one then the associated subtree is discarded, since the obtained cost is a lower bound of $V_N(P2)$ *on that path* (i.e., a feasible solution with the same initial discrete trajectory will always have cost greater or equal to that one). The subtree is also discarded if no solution is found, i.e. $V_N(P2(\Delta_0^j)) = \infty$. If, on the contrary, the computed cost is lower than the cost of the best existing feasible solution then another set of s branches are generated from that node. The algorithm proceeds choosing the node with lower cost, until all nodes have been investigated (or discarded).

Note that the proposed algorithm finds the optimal solution of the robust mode optimal control problem, which we denote by $P2$. In fact, the set of possible solutions of the root node problem, $P2(root)$, includes the optimal solution of the $P2$ problem. Therefore, if $P2(root)$ is infeasible implies that $P2$ is also infeasible. Moreover, the set of feasible solutions for any parent node problem includes the set of feasible solutions for any one of its s child node problems and the optimal value of each $P2(\Delta_0^j)$ problem, i.e. $V_N(P2(\Delta_0^j))$, is a valid (optimistic) bounding function, which allows to perform valid (conservative) cuts on the tree. Therefore, one of the leaf nodes generated by the algorithm corresponds to the optimal mode trajectory $(\Delta_0^{N-1})^*$ and the optimal solution of $P2(\Delta_0^{N-1})$ corresponding to that mode sequence equals the optimal solution of $P2$, i.e. $V_N(P2(\Delta_0^{N-1})) = V_N(P2)$. As a matter of fact, when disturbances are not present, the algorithm reduces to the one presented in [2]. Notice also that search heuristics, which were not discussed, may improve significantly the performance of the algorithm.

5 Conclusions

This paper presents a procedure to extend the MLD framework for synthesizing robust optimal control inputs of constrained PWA systems subject to bounded additive input disturbances. The control sequence minimizes, on a finite time interval horizon, a nominal quadratic performance index guaranteeing that the mode of the dynamics, at each time instant, is independent of the disturbances and that all safety/performance constraints are verified. The approach is based on the *robust mode control* concept, which is a restriction on the admissible control sequences. The open-loop prediction strategy, as well as the restriction on the control moves has a negative impact on the feasibility domain. Research is currently aimed at extending this technique to closed-loop prediction strategies, which are expected to greatly enlarge the domain of feasibility with respect to the presented open-loop approach.

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