ROBUST MODEL PREDICTIVE CONTROL: PIECEWISE LINEAR EXPLICIT SOLUTION

Alberto Bemporad[†‡], Francesco Borrelli[†], Manfred Morari[†]

[†] Automatic Control Laboratory, ETH - ETL I 26, 8092 Zurich, Switzerland
fax: +41-1-632 1211, e-mail: `bemporad,borrelli,morari@aut.ee.ethz.ch`
[‡] Dip. Ingegneria dell'Informazione, Università di Siena, Via Roma 56, 53100 Siena, Italy
fax: +39-0577-233 609, e-mail: `bemporad@dii.unisi.it`

`http://control.ethz.ch/~hybrid`

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Abstract

For discrete-time linear time-invariant systems with input disturbances and constraints on inputs and states, we develop an algorithm to determine explicitly, as a function of the initial state, the solution to robust optimal control problems based on min-max optimization. We show that the optimal control sequence is a piecewise linear function of the initial state. Thus, when the optimal control problem is solved at each time step according to a moving horizon scheme, the on-line computation of the resulting MPC controller is reduced to a simple linear function evaluation. In this paper the uncertainty is modeled as an additive norm-bounded input disturbance vector. The technique can be also extended to robust control of constrained systems affected by polyhedral parametric uncertainty.

1 Introduction

Model Predictive Control (MPC) has become the accepted standard for complex constrained multivariable control problems in the process industries. Here at each sampling time, starting at the current state, an open-loop optimal control problem is solved over a finite horizon. At the next time-step the computation is repeated starting from the new state and over a shifted horizon, leading to a moving horizon policy. The solution relies on a linear dynamic model, respects all input and output constraints, and optimizes a linear or quadratic performance index. Over the last decade a solid theoretical foundation for MPC has emerged so that for real-life large scale MIMO applications controllers with non-conservative stability guarantees can be designed routinely and with ease. The big drawback of MPC is the relatively formidable on-line computational effort which limits its applicability to relatively slow and/or small problems. Rather than solving the optimization problem on line, recently, Bemporad et al. proposed an approach where all computation is moved off line, for linear systems with a quadratic performance index [8], linear systems with a linear performance index [4], and hybrid systems with a linear performance index [5]. The idea stems from observing that the linear part of the objective and the right hand side of the constraints in the optimization problem depend linearly on the state vector $x(t)$, which is treated as a vector of parameters. Then the optimization problem can be recast as a multiparametric program [14] and can be solved off-line by using the appropriate solver [8, 10, 12]. The off-line solution is shown to be a piecewise linear function of the state and therefore the on-line computation reduces to a simple function evaluation.

A fundamental question about MPC is its *robustness* with respect to model uncertainty and noise. When we say that a control system is robust we mean that stability is achieved and the performance specifications are met for a specified range of model variations and a class of noise signals (uncertainty range). To be meaningful, any statement about "robustness" of a particular control algorithm must make reference to a specific uncertainty range as well as specific stability and performance criteria. Although a rich theory has been developed for the robust control of *linear systems*, very little is known about the robust control of *linear systems with constraints*. Recently, this type of problem has been addressed in the context of MPC, see the survey [7]. Two strategies are possible: (i) define a nominal model and a nominal disturbance (e.g., the null disturbance), and optimize nominal performance subject to robust constraints (i.e., the constraints must be satisfied for any possible realization of the disturbance); or (ii) solve a min-max problem to optimize robust performance (the minimum over the control input of the maximum over the possible disturbance). Min-max robust MPC was first proposed by Campo and Morari [11], and further developed in [2] and [25] for SISO FIR plants. Kothare *et al.* [20] optimize robust performance for polytopic/multi-model and structured feedback uncertainty, Scokaert and Mayne [24] for input disturbances only, and Lee and Yu [22] for linear time-varying and time-invariant state-space models depending on a vector of parameters $\theta \in \Theta$, where Θ is either an ellipsoid or a polyhedron. However solving a min-max problem is computationally very demanding.

For systems affected by additive norm-bounded input disturbances, in this paper we show how the solution to robust optimal control problems based on a min-max formulation, with a performance index expressed as the sum over prediction time of the ∞-norm of the input command and of the deviation of the state from the desired value, can be determined explicitly, as a function of the initial state, by using a multiparametric mixed-integer linear program (mp-MILP) solver. We show that the optimal control profile is a piecewise linear function of the initial state and therefore the on-line computation reduces to a simple function evaluation. The technique can be also extended to other robust MPC schemes and to systems affected by polyhedral parametric uncertainty.

2 Problem Formulation

Consider the following discrete-time linear time-invariant system

$$\begin{cases} x(t+1) &= Ax(t) + Bu(t) + Ev(t) \\ y(t) &= Cx(t) \end{cases} \quad (1)$$

along with the constraints[1]

$$F_1 x(t) \le f_1,\ F_2 x(t) + G_2 u(t) \le f_2 \quad (2)$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, and $y(t) \in \mathbb{R}^p$ are the state, input, and output vector respectively, the pair (A, B) is stabilizable, and the rows of G_2 are all nonzero. The vector $v(t)$ is an unknown input disturbance entering the system, and we only assume that the bounds on $v(t)$ are known, namely that $v(t) \in \mathcal{V}$, where $\mathcal{V} \subset \mathbb{R}^p$ is a given polyhedral set containing the origin, $\mathcal{V} = \{v : Mv \le L\}$, $L \ge 0$.

Assume that a full measurement of the state $x(t)$ is available at the current time t. Most formulations of robust MPC based on min-max optimization [2, 11, 24, 25] require that the problem

$$\min_{u_t,\dots,u_{t+N-1}} \left\{ \max_{v_t,\dots,v_{t+N-1}} \left\{ \psi_{\mathcal{P}_\lambda}(x_{t+N|t}) + \sum_{k=1}^{N-1} \|Qx_{t+k|t}\|_\infty \right\} + \sum_{k=0}^{N-1} \|Ru_{t+k}\|_\infty \right\} \quad (3)$$

$$\begin{aligned} \text{subj. to } & F_1 x_{t+k|t} \le f_1,\ k = 1,\dots,N && (4a)\\ & F_2 x_{t+k|t} + G_2 u_{t+k} \le f_2,\ k = 0,\dots,N-1 && (4b)\\ & Mv_{t+k} \le L,\ k = 0,1,\dots,N-1 && (4c)\\ & x_{t+k+1|t} = Ax_{t+k|t} + Bu_{t+k} + Fv_{t+k},\ k \ge 0 && (4d)\\ & x_{t|t} = x(t) && (4e)\\ & x_{t+N|t} \in \mathcal{P}_\lambda && (4f) \end{aligned}$$

is solved at each time t, where $x_{t+k|t}$ denotes the predicted state vector at time $t + k$, obtained by applying the input sequence $u_t, \dots, u_{t+N-1}$ to model (1) starting from the state $x(t)$ and subject to the disturbance sequence $v_t, \dots, v_{t+N-1}$, $\|Vx\|_\infty \triangleq \max_{j=1,\dots,m} (V^j x)$, and V^i is the i-th row of $V \in \mathbb{R}^{m\times n}$.

In (3)–(4), we assume that $Q, R \in \mathbb{R}^{n\times n}$ are non-singular matrices. Note that the distinction between input and input/state constraints in (2) allows to impose pure state constraints in (4) only for $k \ge 1$, so that the controller may be feasible also for $x(t)$ outside the constraint set.

[1] Typically in MPC formulations constraints are expressed in the form $y_{\min} \le y(t) \le y_{\max}$ ($u_{\min} \le u(t) \le u_{\max}$) where $y_{\min}, y_{\max}$ ($u_{\min}$, $u_{\max}$) are $p(m)$-dimensional vectors. Such constraints can be equivalently expressed in the form (2).

The set $\mathcal{P}_\lambda$ is a positively invariant set for system (1) that can be computed as in [9, 19] and the function $\psi_{\mathcal{P}_\lambda}(x)$ is its Minkowski functional [9] . Let $U^*(t) = \{u_t^*, \dots, u_{t+N-1}^*\}$ be the optimal solution of (3)–(4). Then, at time t,

$$u(t) = u_t^* \quad (5)$$

is applied as input to system (1). The optimization (3)–(4) is repeated at time $t+1$, based on the new state $x(t+1)$, yielding a *moving* or *receding horizon* control strategy.

The two main issues regarding this policy are (*i*) ensure robust constraint fulfillment, i.e., guarantee that the constraints (2) are satisfied for all $v(t) \in \mathcal{V}$, and (*ii*) stability of the resulting closed-loop system, in the sense that the state reaches a robustly invariant set around the origin.

2.1 Closed-Loop Prediction

The min-max formulation (3), (4) is based on an *open-loop* prediction, in contrast to the formulation of [3, 20, 22] where *closed-loop* prediction is adopted by letting $u_{t+k} = Fx_{t+k|t} + \bar{u}_{t+k}$, where $\bar{u}_{t+k}$ are new degrees of freedom. The benefits of closed-loop prediction can be understood by viewing the optimal control problem as a dynamic game between the disturbance and the input: in open-loop prediction the whole disturbance sequence plays first, then the input sequence is left with the duty of counteracting the worst disturbance realization. By letting the whole disturbance sequence play first, the effect of the uncertainty may grow over the prediction horizon and may easily lead to infeasibility of the outer min problem. On the contrary, in closed-loop prediction schemes the disturbance and the input play one move at a time, which makes the effect of the disturbance more easily mitigable [3, 7]

In order to achieve closed-loop robust MPC, we modify (3) as

$$\min_{u_t} \left\{ \|Ru_t\|_\infty + \max_{v_t} \left\{ \|Qx_{t+1|t}\|_\infty + \min_{u_{t+1}} \left\{ \dots + \min_{u_{t+N-1}} \left\{ \|Ru_{t+N-1}\|_\infty + \max_{v_{t+N-1}} \left\{ \psi_{\mathcal{P}_\lambda}(x_{t+N|t}) \right\} \dots \right\} \right\} \right\} \quad (6)$$

We will refer to (3), (4) as OL-RMPC (open-loop robust MPC), and to (6), (4) as CL-RMPC (closed-loop robust MPC), respectively. Note that according to our terminology, the strategy proposed in [24] is an OL-RMPC scheme.

3 Explicit Solution to the Robust MPC Problem

It is clear that in most applications the min-max formulations above are computationally prohibitive to be solved on-line. In [22] the authors propose to solve CL-RMPC via dynamic programming by discretizing the state-space, and it is therefore limited to very simple prediction models. By removing the number of degrees of freedom in the choice of the optimal input moves, other CL-RMPC strategies have been proposed, e.g., in [3, 20, 21].

In this paper we aim at finding a solution to the min-max MPC problem by tackling the problem from the different perspective proposed earlier in [4, 5, 8] for the deterministic linear and hybrid case, where the solution to the MPC optimization problem is found off-line as an explicit function of the current state, by using multiparametric programming tools [8, 10, 12].

3.1 Multiparametric Programming

The operations research community has addressed parameter variations in mathematical programs at two levels: *sensitivity analysis*, which characterizes the change of the solution with respect to small perturbations of the parameters, and *parametric programming*, where the characterization of the solution for a full range of parameter values is sought. More precisely, programs which depend only on one scalar parameter are referred to as *parametric programs*, while problems depending on a vector of parameters are referred to as *multi-parametric programs*.

In this paper we deal with multi-parametric programs of the form

$$J(x) = \min_{z \triangleq [z_c, z_d]} \quad c'z \quad \text{s.t.} \quad Gz \leq W + Sx \tag{7}$$

where $z_c \in \mathbb{R}^{n_c}$ are continuous optimization variables, $z_d \in \{0,1\}^{n_d}$ are integer optimization variables and $x \in \mathbb{R}^s$ is a vector of parameters. We refer to (7) as a (right-hand-side) *multi-parametric mixed integer linear program* (mp-MILP) [1, 12]. Solving (7) amounts to determining the form of the value function J and the optimizer z^* as a function of x, for all x in a given polyhedral set $X \subseteq \mathbb{R}^n$ of parameters. The following theorem recalls some known properties of the optimal value function $J(x)$ and of the optimizer $z^*(x)$.

Theorem 1 *Consider the multi-parametric mixed integer linear program (7). Let $X_f \subseteq X$ be the set of parameters for which (7) is feasible. The optimizer $z_c^*(x) : X_f \mapsto \mathbb{R}^{n_c}$, $z_d^*(x) : X_f \mapsto \{0,1\}^{n_c}$ and the optimal solution $J(x) : X_f \mapsto \mathbb{R}$ are piecewise linear functions of x.*

3.2 Off-Line Algorithms for the Explicit Solution of Robust MPC

In this section we show how multiparametric algorithms can be employed to solve min-max problems in explicit form.

Lemma 1 *Let $J(z,x) : \mathbb{R}^m \times \mathbb{R}^n \mapsto \mathbb{R}$ be a piecewise linear (possibly nonconvex) function of (z,x),*

$$J(z,x) = L_i z + H_i x + K_i \text{ for } \left[\begin{smallmatrix} z \\ x \end{smallmatrix}\right] \in \mathcal{R}_i \tag{8}$$

where $\{\mathcal{R}_i\}_{i=1}^s$ are polyhedral sets with disjoint interiors. Consider the multi-parametric optimization problem

$$\min_z \quad J(z,x) \quad \text{s.t.} \quad Gz \leq W + Sx. \tag{9}$$

The explicit solution $z^*(x) : \mathbb{R}^m \mapsto \mathbb{R}^m$ to problem (9) can be determined by an mp-MILP solver.

Proof. By following the approach of [6] to transform piecewise linear functions into a set of mixed-integer linear inequalities, introduce the auxiliary binary variables $\delta_i \in \{0,1\}$, defined as

$$[\delta_i = 1] \leftrightarrow \left[\left[\begin{smallmatrix} z \\ x \end{smallmatrix}\right] \in \mathcal{R}_i \right], \tag{10}$$

and satisfying the exclusive-or condition $\sum_{i=1}^s \delta_i = 1$, and set

$$J(z,x) = \sum_{i=1}^s w_i \tag{11}$$

$$w_i \triangleq [L_i z + H_i x + K_i]\delta_i \tag{12}$$

By transforming (10)–(12) into mixed-integer linear inequalities [6], it is easy to rewrite (9) as a multi-parametric MILP. □

In the special case where $J(z,x)$ is a convex function (i.e., $\mathcal{R} \triangleq \cup_{i=1}^s \mathcal{R}_i$ is a convex set and J is convex over $\mathcal{R}$), the following lemma can be easily proved.

Lemma 2 *Let $J(z,x) : \mathbb{R}^m \times \mathbb{R}^n \mapsto \mathbb{R}$ be a convex piecewise linear function of (z,x). Then the multi-parametric optimization problem (9) is an mp-LP.*

Proof. As $J(z,x)$ is a convex piecewise linear function, it follows that $J(z,x) = \max_{i=1,\dots,s}\{L_i z + H_i x + K_i\}$ [23]. Then, it is easy to show that (9) and the mp-LP

$$\begin{array}{ll} \min_{z,\varepsilon} & \varepsilon \\ \text{s.t.} & Gz \leq W + Sx \\ & \varepsilon \geq L_i z + H_i x + K_i,\ i = 1,\dots,s. \end{array} \tag{13}$$

are equivalent. □

Theorem 2 *CL-RMPC can be solved in explicit piecewise linear form*

$$\begin{array}{ll} u(t) = & F_i x(t) + g_i, \text{ if} \\ & x(t) \in \mathcal{X}_i \triangleq \{x :\ T_i x \leq S_i\},\ i = 1,\dots,s \end{array} \tag{14}$$

where $\mathcal{X} \triangleq \cup_{i=1}^s \mathcal{X}_i$ is the set of states for which (3)–(4) is feasible, by solving 2N mp-MILPs.

Proof. According to a dynamic programming approach, consider the last maximization problem $V(N-1, u_{t+N-1}, x_{t+N-1|t}) = \max_{v_{t+N-1}} \{\psi_{\mathcal{P}_\lambda}(x_{t+N|t})\}$ which is nonconvex and piecewise linear with respect to the optimization vector v_{t+N-1} and the parameters u_{t+N-1}, $x_{t+N-1|t}$. By Lemma 1, it can be solved multipametrically by an mp-MILP solver. By Theorem 1, the value function V is a piecewise linear function of u_{t+N-1}, x_{t+N-1}. Then, by Lemma 1 the minimization problem

$$\begin{array}{rcl} J(N-1, x_{t+N-1|t}) & = & \min_{u_{t+N-1}} \{\|Ru_{t+N-1}\|_\infty + \\ & & V(N-1, u_{t+N|t}, x_{t+N|t})\} \end{array}$$

is again solvable multipametrically by an mp-MILP solver, and the value function J is a piecewise linear function of x_{t+N-1}. By iterating the above maximization and minimization procedures, by Theorem 1 the last optimizer $u_t^* = u(t)$ is a piecewise linear function of $x_{t|t} = x(t)$. □

Theorem 3 *OL-RMPC can be solved in explicit piecewise linear form (14), where $\mathcal{X} \triangleq \cup_{i=1}^{s} \mathcal{X}_i$ is the set of states for which (6)–(4) is feasible, by solving 2 mp-MILPs.*

Proof. The inner maximization problem in (3)–(4) is nonconvex and piecewise linear with respect to the optimization vector $v_t, \ldots, v_{t+N-1}$ and the parameters $u_t, \ldots, u_{t+N-1}, x_{t+N-1|t}$. By Lemma 1, it can be solved multipametrically by an mp-MILP solver. By Theorem 1, the value function V is a piecewise linear function of $u_t, \ldots, u_{t+N-1},\ x_{t+N-1}$. Then, by Lemma 1 the outer minimization problem is again solvable multipametrically by an mp-MILP solver, and the optimizer $u_t^*,\ldots,u_{t+N-1}^*$ is a piecewise linear function of $x_{t|t} = x(t)$, in particular $u_t^* = u(t)$. □

Remark 1 Numerical experience indicates that explicit solutions to CL-RMPC schemes are easier to solve than to OL-RMPC schemes. Such a result has an analogy in the deterministic finite-horizon unconstrained optimal LQ control. This can be either solved via dynamic programming and Riccati iterations, or by formulating and solving an unconstrained quadratic program, whose Hessian matrix has a size which depends on the prediction horizon N. The former approach is numerically more efficient. □

3.3 Multiparametric Solvers

The first method for solving multi-parametric linear programs was formulated by Gal and Nedoma [17], and later a few authors have dealt with multi-parametric linear [15, 16, 23], nonlinear [13], quadratic [8], and mixed-integer [1, 12] programming solvers. Parametric programming systematically subdivides the space of parameters into characteristic regions, which depict the feasibility and corresponding performance as a function of the parameters.

Two main approaches have been proposed for solving mp-MILP problems. In [1], the authors develop an algorithm based on branch and bound (B&B) methods. At each node of the B&B tree an mp-LP is solved where a certain number of integer variables is relaxed to continuous values in $[0, 1]$. The solution at the root node, where all the integer variables are relaxed, represents a valid lower bound, while the solution at a leaf node where all the integer variables have been fixed to 0 or 1 represents a valid upper bound. As in standard B&B methods, the complete enumeration of combinations of 0-1 integer variables is avoided by comparing the multiparametric solutions, and by fathoming the nodes where there is no improvement of the value function. In [12] an alternative algorithm was proposed, which only solves mp-LPs where the integer variables are fixed to the optimal value determined by an MILP, instead of solving mp-LP problems with relaxed integer variables. More in detail, problem (7) is alternatively decomposed into an mp-LP and an MILP subproblem. First an MILP problem is solved by considering also parameters as variables. Then an mp-LP is solved where the binary variables are fixed to the optimal values determined by the previous MILP. The solution of the mp-LP provides a parametric upper bound. A new integer vector is determined by solving an MILP that includes an additional constraint imposing a decrease of the value function with respect to the previous mp-LP (see [12] for more details). The algorithmic implementation of the mp-MILP [12] algorithm which is adopted in this paper relies on [10] for solving mp-LP problems, and on [18] for solving MILPs.

4 Illustrative Example

We compare the explicit solution of (1) nominal MPC [4], (2) robust MPC according to the OL-RMPC formulation in [24] and (3) robust feedback MPC according to the CL-RMPC formulation presented in Section 3.2, on the simple discrete-time state-space model

$$x(t+1) = x(t) + u(t) + v(t)$$

considered in [24]. The goal is to robustly regulate the system to the origin while minimizing the performance measure

$$L(x_t, u_t, u_{t+1}) = \sum_{k=0}^{1} |x_{t+k|t}| + |10u_k| \tag{15}$$

subject to the state constraints

$$-1.2 \leq x_{t|t+k} \leq 2,\ k = 0, 1, 2, \tag{16}$$

to the end region constraint

$$-1 \leq x_{t+2|t} \leq 1, \tag{17}$$

and under the hypothesis that the disturbances are norm-bounded

$$-1 \leq v_{t+k} \leq 1,\ k = 0, 1. \tag{18}$$

(1) *Nominal MPC*. We ignore the disturbance and solve explicitly the problem

$$\min_{u_t, u_{t+1}} L(x_t, u_t, u_{t+1}). \tag{19}$$

Using the approach of [4], the explicit MPC law is

$$u(t) = \begin{cases} 0 & \text{if} \quad -1 \leq x(t) \leq 1 \quad \text{(Region \#1)} \\ -x(t)+1 & \text{if} \quad 1 \leq x(t) \leq 2 \quad \text{(Region \#2)} \\ -x(t)-1 & \text{if} \quad -1.2 \leq x(t) \leq -1 \quad \text{(Region \#3)} \end{cases} \tag{20}$$

In Figure 1 the closed-loop system is simulated from the initial state $x_0 = -1.2$ and $v_k = -1/k,\ k \geq 1$ (as in [24]). Note that constraints violations are experienced during the transient.

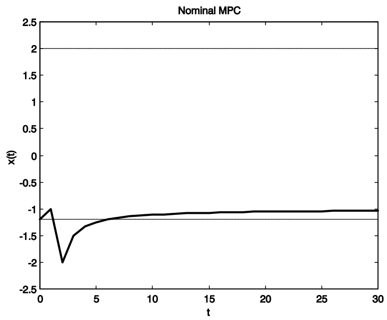

Figure 1: Closed loop trajectories of the nominal MPC (20)

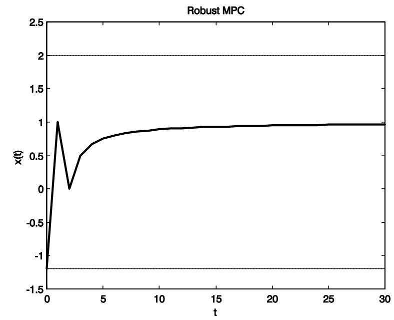

Figure 2: Closed loop trajectories of OL-RMPC (22)

(2) *OL-RMPC [24].* The min-max problem

$$\min_{u_t, u_{t+1}} \max_{v_t, v_{t+1}} L(x_t, u_t, u_{t+1}) \tag{21}$$

is solved explicitly as described in the section 3.2. The inner maximization problem returns the disturbance as an explicit function of the input profile and initial state:

$$\begin{bmatrix} v_{t+1|t} \\ v_{t+2|t} \end{bmatrix} = \begin{cases} \begin{bmatrix} 1 \\ 1 \end{bmatrix} & \text{if} \begin{bmatrix} -1 & -1 & 0 \\ -2 & -2 & -1 \\ -1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_{t|t} \\ u_t \\ u_{t+1} \end{bmatrix} \leq \begin{bmatrix} -2 \\ -1 \\ -1 \\ 0 \\ 0 \\ 0 \\ 1 \\ -1 \end{bmatrix} \\ -\begin{bmatrix} 1 \\ 1 \end{bmatrix} & \text{if} \begin{bmatrix} 2 & 2 & 1 \\ 1 & 1 & -0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_{t|t} \\ u_t \\ u_{t+1} \end{bmatrix} \leq \begin{bmatrix} 2 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 1 \\ -1 \end{bmatrix} \\ -\begin{bmatrix} 1 \\ 1 \end{bmatrix} & \text{if} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \\ -1 & -1 & 0 \end{bmatrix} \begin{bmatrix} x_{t|t} \\ u_t \\ u_{t+1} \end{bmatrix} \leq \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \\ 0 \\ -1 \end{bmatrix} \\ \begin{bmatrix} -1 \\ 1 \end{bmatrix} & \text{if} \begin{bmatrix} -1 & -1 & -1 \\ 1 & 1 & -0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} x_{t|t} \\ u_t \\ u_{t+1} \end{bmatrix} \leq \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \\ -1 \\ 0 \end{bmatrix} \end{cases}$$

that, by solving the outer minimization problem, leads to the explicit OL-RMPC law

$$u(t) = -x(t) + 1 \quad \text{for } -1.2 \leq x(t) \leq 2 \tag{22}$$

In Figure 2 the closed-loop system is simulated from the initial state $x_0 = -1.2$ and and $v_k = -1/k,\ k \geq 1$. As expected, in spite of the disturbances the control law drives the state to the end-region without violating the constraints.

However because the the min-max optimization is of open-loop nature, the problem becomes infeasible if the upper-bound on the state is reduced from 2 to 1.8.

(3) *CL-RMPC*. The min-max problem is formulated as in (6)–(4):

$$\min_{u_t} \max_{v_t} \min_{u_{t+1}} \max_{v_{t+1}} L(x_t, u_t, u_t + 1) \tag{23}$$

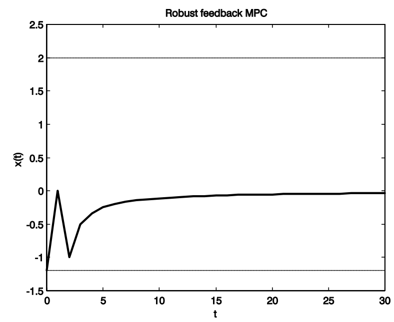

Figure 3: Closed loop trajectories of CL-RMPC (24)

and explicitly solved as described in 3.2 to obtain the explicit feedback robust MPC law

$$u(t) = -x(t) \quad \text{for } -1.2 \leq x(t) \leq 2 \tag{24}$$

In Figure 3 the closed-loop system is simulated from the initial state $x_0 = -1.2$ and and $v_k = -1/k,\ k \geq 1$. In spite of the disturbances, the control law drives the state to the end region without any constraint violation. Moreover, contrarily to OL-RMPC, problem (23) remains feasible even if the upper-bound on the state is reduced from 2 to 1.8.

The controllers (20), (22), and (24) were computed off-line in Matlab 5.3 on a Pentium III 650 in about 55 s.

5 Conclusions

For discrete-time linear time-invariant systems with input disturbances and constraints on inputs and states, we have presented a procedure to determine explicitly, as a function of the current state, the solution to robust receding horizon optimal control problems based on a min-max formulation with a performance index expressed in ∞-norm. We showed that the optimal control profile is a piecewise linear function of the initial state. The technique can be easily extended to system affected by parametric polyhedral uncertainty and to performance indices based on 1-norms.

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