SAT-Based Branch & Bound and Optimal Control of Hybrid Dynamical Systems

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Abstract. A classical hybrid MIP-CSP approach for solving problems having a logical part and a mixed integer programming part is presented. A Branch and Bound procedure combines an MIP and a SAT solver to determine the optimal solution of a general class of optimization problems. The procedure explores the search tree, by solving at each node a linear relaxation and a satisfiability problem, until all integer variables of the linear relaxation are set to an integer value in the optimal solution. When all integer variables are fixed the procedure switches to the SAT solver which tries to extend the solution taking into account logical constraints. If this is impossible, a "no-good" cut is generated and added to the linear relaxation. We show that the class of problems we consider turns out to be very useful for solving complex optimal control problems for linear hybrid dynamical systems formulated in discrete-time. We describe how to model the "hybrid" dynamics so that the optimal control problem can be solved by the hybrid MIP+SAT solver, and show that the achieved performance is superior to the one achieved by commercial MIP solvers.

1 Introduction

In this paper we consider the general class of *mixed logical/convex* problems:

$$\min_{z,\nu,\mu} f(z) \qquad \text{(Convex function)} \tag{1a}$$

$$\text{s.t. } g_c(z) \le 0,\ h_c(z) = 0 \qquad \text{(Continuous constraints)} \tag{1b}$$

$$g_m(z,\mu) \le 0,\ h_m(z,\mu) = 0 \qquad \text{(Mixed constraints)} \tag{1c}$$

$$g_L(\nu,\mu) = \texttt{TRUE} \qquad \text{(Logic constraints)} \tag{1d}$$

$$z \in \mathbb{R}^{n_z},\ \nu \in \{0,1\}^{n_\nu},\ \mu \in \{0,1\}^{n_\mu},$$

where $g_c : \mathbb{R}^{n_z} \to \mathbb{R}^{q_{gc}}$, $g_m : \mathbb{R}^{n_z+n_\mu} \to \mathbb{R}^{q_{gm}}$ are convex functions, $h_c : \mathbb{R}^{n_z} \to \mathbb{R}^{q_{hc}}$, $h_m : \mathbb{R}^{n_z+n_\mu} \to \mathbb{R}^{q_{hm}}$ are affine functions, and $g_L : \{0,1\}^{n_\nu \times n_\mu} \to \{0,1\}^{n_{CP}}$ is a Boolean function.

An MIP solver provides the solution to (1) after solving a sequence of relaxed convex problems, typically standard linear or quadratic programs (LP, QP). A potential drawback of MIP is (a) the need for converting the logic constraints

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(1d) into mixed-integer inequalities, therefore losing most of the original discrete structure, and (b) the fact that its efficiency mainly relies upon the tightness of the continuous LP/QP relaxations.

Such a drawback is not suffered by techniques for solving constraint satisfaction problems (CSP), i.e., the problem of determining whether a set of constraints over discrete variables can be satisfied. Under the class of CSP solvers we mention constraint logic programming (CLP) [1] and SAT solvers [2], the latter specialized for the satisfiability of Boolean formulas.

While CSP methods are superior to MIP approaches for determining if a given problem has a feasible (integer) solution, the main drawback is their inefficiency for solving optimization, as they do not have the ability of MIP approaches to solve continuous relaxations (e.g., linear programming relaxations) of the problem in order to get upper and lower bounds to the optimum value.

For this reason, it seems extremely interesting to integrate the two approaches into one single solver. Some efforts have been done in this direction [3, 4, 5, 6, 7], showing that such mixed methods have a tremendous performance in solving mathematical programs with continuous (quantitative) and discrete (logical/symbolic) components, compared to MIP or CSP individually. Such successful results have stimulated also industrial interest: ILOG Inc. is currently distributing OPL (Optimization Programming Language), a modeling and programming language which allows the formulation and solution of optimization problems, using both MIP and CSP techniques, combining to some extent the advantages of both approaches; European projects with industrial participants, such as LISCOS [8], developed and are developing both theoretical insights and software tools for applying the combined approach of MIP and CSP to industrial case studies.

In this paper, we focus on combinations of convex programming (e.g., linear, quadratic, etc.) for optimization over real variables, and of SAT-solvers for determining the satisfiability of Boolean formulas. The main motivation for our study stems from the need for solving complex optimal control problems of theoretical and industrial interest based on "hybrid" dynamical models of processes that exhibit a mixed continuous and discrete nature. Hybrid models are characterized by the interaction of continuous models governed by differential or difference equations, and of logic rules, automata, and other discrete components (switches, selectors, etc.). Hybrid systems can switch between many operating modes where each mode is governed by its own characteristic continuous dynamical laws. Mode transitions may be triggered internally (variables crossing specific thresholds), or externally (discrete commands directly given to the system). The interest in hybrid systems is mainly motivated by the large variety of practical situations where physical processes interact with digital controllers, as for instance in embedded control systems.

Several authors focused on the problem of solving optimal control problems for hybrid systems. For continuous-time hybrid systems, most of the literature either studied necessary conditions for a trajectory to be optimal, or focused

on the computation of optimal/suboptimal solutions by means of dynamic programming or the maximum principle [9, 10, 11].

The hybrid optimal control problem becomes less complex when the dynamics is expressed in discrete-time, as the main source of complexity becomes the combinatorial (yet finite) number of possible switching sequences. In particular, in [12, 13, 14] the authors have solved optimal control problems for discrete-time hybrid systems by transforming the hybrid model into a set of linear equalities and inequalities involving both real and (0-1) variables, so that the optimal control problem can be solved by a mixed-integer programming (MIP) solver.

At the light of the benefits and drawbacks of the previous work in [12, 13, 14] for solving control and stability/safety analysis problems for hybrid systems using MIP techniques, we follow a different route that uses the aforementioned approach combining MIP and CSP techniques.

We build up a new modeling approach for hybrid dynamical systems directly tailored to the use of the hybrid MIP+SAT solver for solving optimal control problems, and show its computational advantages over pure MIP methods. A preliminary work in this direction appeared in [15], where generic constraint logic programming (CLP) was used for handling the discrete part of the optimal control problem.

The paper is organized as follows. In Section 2.1 optimal control problems of discrete-time hybrid models are introduced and in Section 2.2 are reformulated to the general class (1). Section 3 introduces the new solution algorithm for the general class (1). An example of optimal control problem of a hybrid model showing the benefits of the solution algorithm, compared to pure MIP approaches [12, 14] is shown in section 4.

2 Motivating Application

2.1 Optimal Control of Discrete-Time Hybrid Systems

Following the ideas in [14], a hybrid system can be modeled as the interconnection of an automaton (AUT) and a switched affine system (SAS) through an event generator (EG) and a mode selector (MS). The discrete-time hybrid dynamics is described as follows [14]:

$$\begin{aligned}
\text{(AUT)}\quad x_l(k+1) &= f_l(x_l(k), u_l(k), e(k)),\\
y_l(k) &= g_l(x_l(k), u_l(k), e(k)), && \text{(2a)}\\
\text{(SAS)}\quad x_c(k+1) &= A_{i(k)}x_c(k) + B_{i(k)}u_c(k) + f_{i(k)},\\
y_c(k) &= C_{i(k)}x_c(k) + D_{i(k)}u_c(k) + g_{i(k)}, && \text{(2b)}\\
\text{(EG)}\quad [e_j(k) = 1] &\longleftrightarrow [a_j^T x_c(k) + b_j^T u(k) \le c_j] && \text{(2c)}\\
\text{(MS)}\quad i(k) &= \begin{bmatrix}\delta_1\\ \vdots\\ \delta_s\end{bmatrix} = f_{\text{MS}}(x_l(k), u_l(k), i(k-1)) && \text{(2d)}
\end{aligned}$$

The automaton (or finite state machine) describes the logic dynamics of the hybrid system. We will only refer to "synchronous automata", where transitions are clocked and synchronous with the sampling time of the continuous dynamical equations. The dynamics of the automaton evolves according to the logic update functions (2a) where $k \in \mathbb{Z}^+$ is the time index, $x_l \in \mathcal{X}_l \subseteq \{0,1\}^{n_l}$ is the logic state, $u_l \in \mathcal{U}_l \subseteq \{0,1\}^{m_l}$ is the exogenous logic input, $y_l \in \mathcal{Y}_l \subseteq \{0,1\}^{p_l}$ is the logic output, $e \in \mathcal{E} \subseteq \{0,1\}^{n_e}$ is the endogenous input coming from the EG, and $f_l : \mathcal{X}_l \times \mathcal{U}_l \times \mathcal{E} \longrightarrow \mathcal{X}_l$, $g_l : \mathcal{X}_l \times \mathcal{U}_l \times \mathcal{E} \longrightarrow \mathcal{Y}_l$ are deterministic boolean functions.

The SAS describes the continuous dynamics and it is a collection of affine systems (2b) where $x_c \in \mathcal{X}_c \subseteq \mathbb{R}^{n_c}$ is the continuous state vector, $u_c \in \mathcal{U}_c \subseteq \mathbb{R}^{m_c}$ is the exogenous continuous input vector, $y_c \in \mathcal{Y}_c \subseteq \mathbb{R}^{p_c}$ is the continuous output vector, $i(k) \in \mathcal{I} \triangleq \left\{ \left[1\,0\cdots 0\right]^T, \cdots, \left[0\cdots 0\,1\right]^T \right\} \subseteq \{0,1\}^s$ is the "mode" in which the SAS is operating, $\sharp\mathcal{I} = s$ is the number of elements of $\mathcal{I}$, and $\{A_i, B_i, f_i, C_i, D_i, g_i\}_{i\in\mathcal{I}}$ is a collection of matrices of opportune dimensions. The mode $i(k)$ is generated by the mode selector, as described below. A SAS of the form (2b) preserves the value of the state when a switch occurs. Resets can be modeled in the present discrete-time setting as detailed in [14].

The event generator (EG) is a mathematical object that generates a Boolean vector according to the satisfaction of a set of *threshold events* (2c) where $_j$ denotes the j-th component of the vector, and $a_j \in \mathbb{R}^{n_c}$, $b_j \in \mathbb{R}^{m_c}$, $c_j \in \mathbb{R}$ define the hyperplane in the space of continuous states and inputs.

The mode selector (MS) selects the dynamic mode $i(k) \in \mathcal{I} \subseteq \{0,1\}^s$, also called the *active mode*, of the SAS and it is described by the logic function (2d) where $f_{MS} : \mathcal{X}_l \times \mathcal{U}_l \times \mathcal{I} \longrightarrow \mathcal{I}$ is a Boolean function of the logic state $x_l(k)$, of the logic input $u_l(k)$, and of the active mode $i(k-1)$ at the previous sampling instant. We say that a *mode switch* occurs at step k if $i(k) \neq i(k-1)$. Note that contrarily to continuous time hybrid models, where switches can occur at any time, in our discrete-time setting a mode switch can only occur at sampling instants.

A finite-time optimal control problem for the class of hybrid systems is formulated as follows:

$$\min_{\{x(k+1),u(k)\}_{k=0}^{T-1}} \sum_{k=0}^{T-1} \ell_k(x(k+1) - r_x(k+1), u(k) - r_u(k)) \tag{3a}$$

$$\text{s.t. dynamics (2a), (2b), (2c), (2d)} \tag{3b}$$

$$h_D(x(0), \{x(k+1), u(k), e(k), i(k)\}_0^{T-1}) \leq 0 \tag{3c}$$

$$h_A(x(0), \{x(k+1), u(k), e(k), i(k)\}_0^{T-1}) \leq 0 \tag{3d}$$

where T is the control horizon, $\ell_k : \mathbb{R}^{n\times m} \to \mathbb{R}$ is a nonnegative convex function, $n = n_c + n_l$, $m = m_c + m_l$, $r_x \in \mathbb{R}^n$, $r_u \in \mathbb{R}^m$ are given reference trajectories to be tracked by the state and input vectors, respectively.

The constraints of the optimal control problem can be classified as *dynamical constraints* (3b), representing the discrete-time hybrid system, *design constraints* (3c), artificial constraints imposed by the designer to fulfill the required spec-

ifications, and *ancillary constraints* (3d), an a priori additional and auxiliary information for determining the optimal solution which does not change the solution itself, rather help the solver to find it more easily.

2.2 Problem Reformulation

Problem (3) can be solved via MILP when the costs ℓ_k are convex piecewise linear functions, for instance $\ell_k(x,u) = \|Q_x x\|_\infty + \|Q_u u\|_\infty$, where Q_x, Q_u are full-rank matrices and $\|\cdot\|_\infty$ denotes the infinity-norm ($\|Qx\|_\infty = \max_{j=1,\ldots,n} |Q^j x|$, where Q^j is the j-th row of Q) [13], or via MIQP (mixed integer quadratic programming) when $\ell_k(x,u) = x'Q_x x + u'Q_u u$, where Q_x, Q_u are positive (semi)definite matrices [12]. In this paper we wish to solve problem (3) by using MIP and SAT techniques in a combined approach, taking advantage of SAT for dealing with the purely logic part of the problem. In order to do this, we need to reformulate the problem in a suitable way.
The automaton and mode selector parts of the hybrid system are described as a set of Boolean constraints so they do not require transformations. The event generator and SAS parts can be equivalently expressed, by adopting the so-called "big-M" technique [16], as a set of continuous and mixed constraints. Problem (3) can be cast as the mixed logical/convex program

$$\min_{\substack{\{x(k+1),\,u(k),\\ w(k),\,\delta(k)\}\\ k=0,\ldots,T-1}} \quad \sum_{k=0}^{T-1} \ell_k(x(k+1) - r_x(k+1), u(k) - r_u(k)) \tag{4a}$$

$$\text{s.t.} \quad Ax_c(k) \le b,\ x_c(k+1) = \sum_{i=1}^{s} w_i(k) \tag{4b}$$

$$M_1 x_c(k) + M_2 u_c(k) + M_3 w(k) \le M_4 e(k) + M_5 \delta(k) + M_6 \tag{4c}$$

$$g(x_l(k+1), x_l(k), u_l(k), e(k), \delta(k)) = \texttt{TRUE} \tag{4d}$$

$$w(k) = [w_1(k) \ldots w_s(k)]',\ w_i(k) \in \mathbb{R}^{n_c},\ \delta(k) \in \{0,1\}^s,$$

where $\{x_c(k+1), u_c(k), w(k)\}_{k=0}^{T-1}$ are the continuous optimization variables, $\{x_l(k+1),\ u_l(k),\ \delta(k),\ e(k)\}_{k=0}^{T-1}$ are the binary optimization variables, $x_c(0)$, $x_l(0)$ is a given initial state, constraints (4b), (4c) represent the EG and SAS parts (2c), (2b), and the purely continuous or mixed constraints from (3c), (3d), while (4d) represents the automaton (2a), the mode selector (2d), possible purely Boolean constraints from (3c), (3d). Matrices M_i, $i = 1\ \ldots\ 6$, are obtained by the big-M translation.

Problem (4) belongs to the general class (1) in which all constraints depend on the state initial condition $[x_c(0)'\ x_l(0)']'$ of the hybrid system. In the hybrid optimal control problem at hand, z collects all the continuous variables ($x_c(k+1)$, $u_c(k)$, $k = 0,\ldots,T-1$), the auxiliary variables needed for expressing the SAS dynamics, possibly slack variables for upper bounding the cost function in (4a) [13], μ collects the integer variables that appear in mixed constraints ($e(k)$, $\delta_i(k)$, $k = 0,\ldots,T-1$, $i = 1,\ldots,s$), and ν collects the integer variables such as $x_l(k)$, $u_l(k)$ that only appear in logic constraints. Note that in general if

the objective function in the the form $f(z, \mu)$ we could consider the new objective function ϵ, $\epsilon \in \mathbb{R}$, and an additional constraint $f(z, \mu) \leq \epsilon$ which is a mixed convex constraint that could be included in (1c).

3 SAT-Based Branch&Bound

3.1 Constraint Satisfaction and Optimization

While optimization is primarily associated with mathematics and engineering, CSP was developed (more recently) in the computer science and artificial intelligence communities. The two fields evolved more or less independently until a few years ago. Yet they have much in common and are applied to solve similar problems. Most importantly for the purposes of this paper, they have complementary strengths, and the last few years have seen growing efforts to combine them [4, 3, 17, 5, 18].

The recent interaction between CSP and optimization promises to affect both fields. In the following subsections we illustrate an approach for merging them into a single problem-solving technology, in particular by combining convex optimization and satisfiability of Boolean formulas (SAT).

Convex Optimization. Convex optimization is very popular in engineering, economics, and other application domains for solving nontrivial decision problems. Convex optimization includes linear, quadratic, and semidefinite programming, for which several extremely efficient commercial and public domain solvers are nowadays available. An excellent reference to convex optimization is the book by Boyd and Vandenberghe [19].

SAT Problems. An instance of a satisfiability (SAT) problem is a Boolean formula that has three components:

- A set of n variables: $x_1, x_2, \ldots, x_n$.
- A set of literals. A literal is a variable $(Q = x)$ or a negation of a variable $(Q = \neg x)$.
- A set of m distinct clauses: $C_1, C_2, \ldots, C_m$. Each clause consists of only literals combined by just logical *or* $(\vee)$ connectives.

The goal of the satisfiability problem is to determine whether there exists an assignment of truth values to variables that makes the following Conjunctive Normal Form (*CNF*) formula satisfiable:

$$C_1 \wedge C_2 \wedge \ldots \wedge C_m,$$

where $\wedge$ is a logical "and" connective. For a survey on SAT problems and related solvers the reader is referred to [2].

3.2 A SAT-Based Hybrid Algorithm

The basic ingredients for an integrated approach are (1) a solver for convex problems obtained from relaxations over continuous variables of mixed integer convex programming problems of the form (4a)-(4b)-(4c), and (2) a SAT solver for testing the satisfiability of Boolean formulas of the form (4d). The relaxed model is used to obtain a solution that satisfies the constraint sets (1b) and (1c) and optimizes the objective function (1a). The optimal solution of the relaxation may fix some of the (0-1) variables to either 0 or 1. If all the (0-1) variables in the relaxed problem have been assigned (0-1) values, the solution of the relaxation is also a feasible solution of the mixed integer problem. More often, however, some of the (0-1) variables have fractional parts, so that further "branching" and solution of further relaxations is necessary. To accelerate the search of feasible solutions one may use the fixed (0-1) variables to "infer" new information on the other (0-1) variables by solving a SAT problem obtained by constraint (1d). In particular, when an integer solution of μ is found from convex programming, a SAT problem then verifies whether this solution can be completed with an assignment of ν that satisfies (1d).

The basic branch&bound (B&B) strategy for solving mixed integer problems can be extended to the present "hybrid" setting where both convex optimization and SAT solvers are used. In a B&B algorithm, the current best integer solution is updated whenever an integer solution with an even better value of the objective function is found. In the hybrid algorithm at hand an additional SAT problem is solved to ensure that the integer solution obtained for the relaxed problem is feasible for the constraints (1d) and to find an assignment for the other logic variables ν that appear in (1d). It is only in this case that the current best integer solution is updated.

The B&B method requires the solution of a series of convex subproblems obtained by branching on integer variables. Here, the non-integer variable to branch on is chosen by selecting the variable with the largest fractional part (i.e., the one closest to 0.5), and two new convex subproblems are formed with that variable fixed at 0 and at 1, respectively. When an integer feasible solution of the relaxed problem is obtained, a satisfiability problem is solved to complete the solution. The value of the objective function for an integer feasible solution of the whole problem is an upper bound (UB) of the objective function, which may be used to rule out branches where the optimum value attained by the relaxation is larger than the current upper bound.

Let P denote the set of convex and SAT subproblems to be solved. The proposed SAT-based B&B method can be summarized as follows:

1. **Initialization**. $UB = \infty$, $P = \{(p^0, SAT^0)\}$. The convex subproblem p^0 is generated by using (1a),(1b), (1c) along with the relaxation $\mu \in [0,1]^{n_\mu}$, and the SAT subproblem SAT^0 is generated by using (1d).
2. **Node selection**. If $P = \emptyset$ then go to 7.; otherwise select and remove a (p, SAT) problem from the set P; The criterion for selecting a problem is called *node selection rule*.

3. **Logic inference**. Solve problem SAT. If it is infeasible go to step 2.
4. **Convex reasoning**. Solve the convex problem p, and:
   4.1. If the problem is infeasible or the optimal value of the objective function is greater than UB then go to step 2.
   4.2. If the solution is not integer feasible then go to step 6.
5. **Bounding**. Let $\mu^* \in \{0,1\}^{n_\mu}$ be the integer part of the optimal solution found at step 4.; to extend this partial solution, solve the SAT problem finding ν such that $g(\nu, \mu^*)$ =TRUE. If the SAT problem is feasible then update UB; otherwise add to the LP problems of the set P the "no-good" cut [3]
$$\sum_{i \in T^*} \mu_i - \sum_{j \in F^*} \mu_j \leq B^* - 1,$$
   where $T^* = \{i | \mu_i^* = 1\}$, $F^* = \{j | \mu_j^* = 0\}$, and $B^* = |T^*|$. Go to step 2.
6. **Branching**. Among all variables that have fractional values, select the one closest to 0.5. Let μ_i be the selected non-integer variable, and generate two subproblems $(p \cup \{\mu_i = 0\},\ SAT\&\{\neg\mu\})$, $(p \cup \{\mu_i = 1\},\ SAT\&\{\mu\})$ and add them to set P; go to step 2.
7. **Termination**. If $UB = \infty$, then the problem is infeasible. Otherwise, the optimal solution is the current value UB.

Remark 1. At each node of the search tree the algorithm executes a three-step procedure: logic inference, solution of the convex relaxation, and branching. The first step and the attempted completion of the solution do not occur in MIP approaches but they are introduced here by the distinction of mixed (0-1) variables μ and pure (0-1) variables ν. The logic inference and the attempted completion steps do not change the correctness and the termination of the algorithm but they improve the performance of the algorithm because of the efficiency of the SAT solver in finding a feasible integer solution.

Remark 2. The class of problems (1) is similar to the MLLP framework introduced by Hooker in [20],

$$\min\ c'x \tag{5a}$$
$$\text{s.t. } p_j(y,h) \rightarrow [A^j(x) \geq a^j],\ j \in J \tag{5b}$$
$$q_i(y,h),\ i \in I, \tag{5c}$$

where $x \in \mathbb{R}^{n_x}$, $y \in \{0,1\}^{n_y}$, $h \in \mathcal{D} \subset \mathbb{Z}^{n_h}$, (5b) is the continuous part, and (5c) is the logic part. If we consider the only y variables as discrete variables and a liner cost function, constraints (1b), (1c) represent the linearization of (5b), and constraints (1d) are equivalent to (5c).
There are however a few differences between frameworks (1) and (5). First, the relaxation problem of (1) is the same for each node in the search tree, while in (5) the relaxation depends on which left-hand side of (5b) is true. Second, in the class of problems (1) constraints of type

$$[\mu = 1] \longleftrightarrow [z_1 + z_2 \geq \alpha],$$

can not be introduced and they have to be converted into inequalities, becoming part of constraints (1c). Inference is done only in the logic part, by the SAT solver, and no information is derived by the continuous part. In the MLLP framework, instead, inferences are made in both ways.

Remark 3. The modeling framework (1) can also be solved by using a combined approach of MIP and CLP [15]. The role of constraint propagation is obviously to reduce as much as possible the domain sets of the μ variables that appear in the constraints managed by the CLP solver. In this way, the constraint propagation can reduce the search space removing some branches in the search tree that can not have feasible solutions. Moreover the constraint propagation together with choice points can help to find a completion of the solution trying to fix the ν variables.
The SAT solver behaves in a similar way to CP solver. The SAT inference is a feasibility check. If a partial assignment of the μ variables is infeasible for the set of constraints (1d) SAT is able to find the infeasibility easier and more quickly than a CLP solver. SAT solvers are also more efficient for finding a feasible assignment for the ν variables with respect to CLP solvers.
However the efficiency of SAT solvers relies upon the representation of the logic part of the problem. While CLP can be used both with logic formulas and linear constraints, as well as global constraints, SAT turns out to be useful only with Boolean formulas.

4 Numerical Results

In this section we show on an example of hybrid optimal control problem that the hybrid solution technique described in the previous sections has a better performance compared to commercial MIP solvers.

4.1 "Hybrid" Model

Consider a room with two bodies with temperatures T_1, T_2 and let T_{amb} be the room temperature (this example is an extension of the example reported in [21]). The room is equipped with a heater, close to body 1, delivering thermal power u_{hot} and an air conditioning system, close to body 2, draining thermal power u_{cold}. These are turned on/off according to some rules dictated by the closeness of the two bodies to each device. We want guarantee that the bodies are not cold or hot.

The discrete-time continuous dynamics of each body is described by the difference equation

$$\frac{T_i(k+1) - T_i(k)}{T_s} = -\alpha_i(T_i(k) - T_{amb}) + k_i(u_{\text{hot}}(k) - u_{\text{cold}}(k)) + cu_e(k), \quad (6)$$

where $i = 1, 2$, α_i, k_i, c are suitable constants, T_s is the sampling time, and $u_e(k)$ is an exogenous input that can be used to deliver or drain thermal power manually (e.g. by opening a window or by changing the water flow from a centralized heating system).

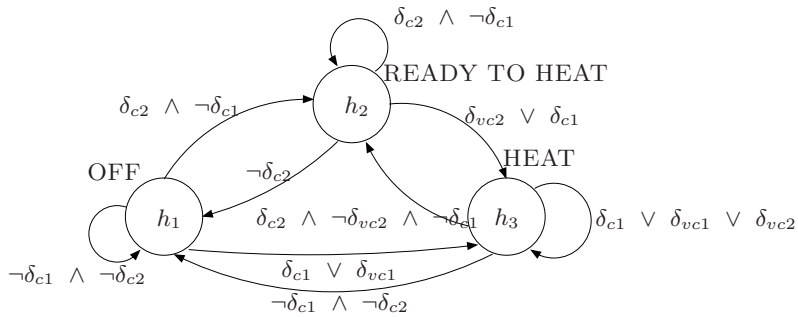


Fig. 1. Automaton regulating the heater

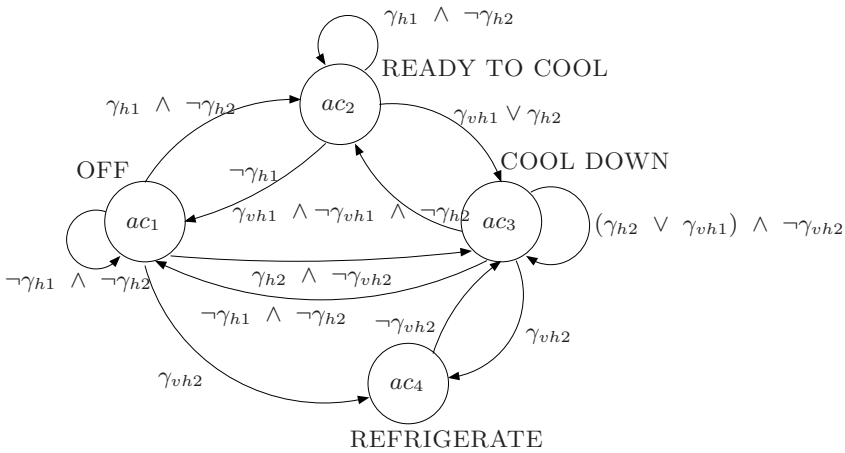


Fig. 2. Air conditioning system automaton

The automaton part of the system is described by the two automata represented in Figures 1 and 2, where δ_{ci},δ_{vci},γ_{hi} and γ_{vhi}, for $i = 1, 2$, are logic variables defined as follows

$$[\delta_{vci}(k) = 1] \longleftrightarrow [T_i(k) \leq T_{vci}], \tag{7a}$$
$$[\delta_{ci}(k) = 1] \longleftrightarrow [T_i(k) \leq T_{ci}], \tag{7b}$$
$$[\gamma_{hi}(k) = 1] \longleftrightarrow [T_i(k) \geq T_{hi}], \tag{7c}$$
$$[\gamma_{vhi}(k) = 1] \longleftrightarrow [T_i(k) \geq T_{vhi}], \tag{7d}$$

and where $T_{vci} \leq T_{ci} \leq T_{hi} \leq T_{vhi}$ are constant thresholds. The automaton for the heater (Figure 1) sets the heater in the "ready to heat" state if body 2 is cold, and will go in "heat" state if body 2 is very cold. If body 1 is cold or very cold the heater is turned on immediately. The automaton of the air conditioning (A/C) system (Figure 2) sets the air conditioning system in the "ready to cool" state if body 1 is hot, unless body 2 is cold, in other words, the A/C system is turned on only when body 1 is very hot. However, the draining thermal power

is half of the full power. The A/C system is set to the maximum power if the body 2 is very hot but it is immediately switched to half power as soon as body 2 is only hot (due to energy consumptions of the A/C system).

The heater delivers thermal power and the A/C system drains thermal power according to the following rules:

$$u_{\text{hot}} = \begin{cases} u_H \text{ if } h_3 = 1 \\ 0 \quad \text{otherwise} \end{cases} \qquad u_{\text{cold}} = \begin{cases} u_C \text{ if } ac_4 = 1 \\ \frac{u_C}{2} \text{ if } ac_3 = 1 \\ 0 \quad \text{otherwise} \end{cases}. \tag{8}$$

By following the notation of (2a), we have $x_l = [h_1\ h_2\ h_3\ ac_1\ ac_2\ ac_3\ ac_4]' \in \{0,1\}^7$, $u_l = \emptyset$ and $e(k) = [\delta_{vc1}\ \delta_{vc2}\ \delta_{c1}\ \delta_{c2}\ \gamma_{h1}\ \gamma_{h2}\ \gamma_{vh1}\ \gamma_{vh2}]' \in \{0,1\}^8$.

The system has six modes: $(u_{\text{hot}}, u_{\text{cold}}) \in \{(0,0), (u_H,0), (0,u_C), (0,u_C/2), (u_H,u_C), (u_H,u_C/2)\}$. The mode selector function is defined as follows

$$i(k) = \begin{bmatrix} \neg h_3(k) \wedge \neg ac_4(k) \wedge \neg ac_3(k) \\ h_3(k) \wedge \neg ac_4(k) \wedge \neg ac_3(k) \\ \neg h_3(k) \wedge ac_4(k) \wedge \neg ac_3(k) \\ \neg h_3(k) \wedge \neg ac_4(k) \wedge ac_3(k) \\ h_3(k) \wedge ac_4(k) \wedge \neg ac_3(k) \\ h_3(k) \wedge \neg ac_4(k) \wedge ac_3(k) \end{bmatrix} \in \{0,1\}^6,$$

which only depends on logic states.

The SAS dynamics (6), i.e., the continuous part of the hybrid system, is translated into a set of inequalities using the Big-M technique, which provides the set of constraints

$$Ax_c(k) + Bu_c(k) + Cw(k) \leq D\delta(k) + E, \tag{9}$$

where $x_c = [T_1\ T_2]'$, $u_c = u_e$, $w(k) \in \mathbb{R}^3$ contains the auxiliary continuous variables needed to represent the conditions $u_{\text{hot}} = u_H$, $u_{\text{cold}} = u_C$, $u_{\text{cold}} = u_C/2$, and $\delta(k) = [h_3(k)\ ac_3(k)\ ac_4(k)] \in \{0,1\}^3$. Constraints (9) are obtained by employing the HYSDEL compiler [14], a dedicated "hybrid" system description language and compiler which translates a description of the problem into the mathematical mixed+logical dynamical (MLD) representation introduced in [12], a mathematical framework useful for defining optimal control problems as pure MIP problems.

Finally, the event generator is represented by (7a) and (7b). These are translated by HYSDEL into a set of linear inequalities:

$$G'_x x_c(k) + G'_u u_c(k) + D'e(k) \leq E', \tag{10}$$

where $e(k) = [\delta_{vc1}\ \delta_{vc2}\ \delta_{c1}\ \delta_{c2}\ \gamma_{h1}\ \gamma_{h2}\ \gamma_{vh1}\ \gamma_{vh2}]' \in \{0,1\}^8$.

4.2 Optimal Control Problem

The goal is to design an optimal control profile for the continuous input u_e that minimizes $\sum_{k=0}^{T} |T_i(k) - T_{amb}|$ subject to the hybrid dynamics and the following additional constraints:

– Continuous constraints on temperatures to avoid that they assume unacceptable values

$$-10 \leq T_1(k) \leq 50 \qquad -10 \leq T_2(k) \leq 50. \tag{11a}$$

These constraints may be interpreted as dynamical constraints due to physical limitations of the bodies.

– A continuous constraint on exogenous input to avoid excessive variations:

$$-10 \leq u_e(k) \leq 10. \tag{12}$$

This constraint may be interpreted as a design constraint of the form (3c).

4.3 Results

The above dynamics and constraints are also modeled in HYSDEL [14] to obtain an MLD model of the hybrid system in order to compare the performance achieved by the hybrid solver with the one obtained by employing a pure MILP approach.

The optimal control problem is defined over horizon of T steps as:

$$\min_{\{x,u,z,\delta,\epsilon_T\}} \sum_{k=0}^{T-1} \epsilon_T(k) \tag{13a}$$

$$\text{s.t. } \epsilon_T(k) \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} \geq \pm(T_i(k) - T_{amb}), \tag{13b}$$

$$\text{automata Figures 1, 2,} \tag{13c}$$

$$(9),\ (10) \tag{13d}$$

$$(11),\ (12) \tag{13e}$$

where $\{x, u, z, \delta, \epsilon_T\}=\{x(k), u(k), z(k), \delta(k), \epsilon_T(k)\}_{k=0}^{T-1}$, $\epsilon_T =[\epsilon_{T1}(0), \epsilon_{T2}(0), \ldots, \epsilon_{T1}(T-1), \epsilon_{T2}(T-1)]' \in \mathbb{R}^{2T}$.

Each part of the optimal control problem is managed by either the SAT solver or the LP solver: the cost function (13a), the inequalities (13b), (13d), and the additional constraints (13e) are managed by the LP solver, the logic part (13c) is managed by the SAT solver. Our simulations have been done describing and solving the problem within the Matlab environment and calling, through MEX interfaces, respectively, zCHAFF [22] for SAT and CPLEX [23] for LP.

In all our simulations we have adopted depth first search as the *node selection rule*, to reduce the amount of memory used during the search.

For the initial condition $T_1(0) = 5°$ C, $T_2(0) = 2°$ C and for $T_{amb} = 25°$ C we have done simulations for different horizons (the obtained optimal solution is clearly the same both using the SAT-based B&B and the MILP), reported in Table 1.

We can see that the performance of the SAT-based B&B is always better than the one obtained by using the commercial MILP solver of CPLEX. In Table 1, we

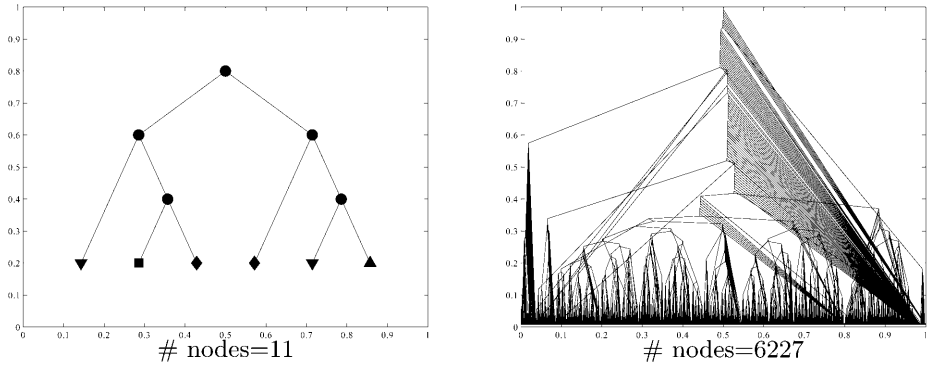
Table 1. Optimal control solution: comparison among SAT-based B&B, naive MILP, and CPLEX MILP

T	Bool.	SATbB&B			CPLEX		Naive MILP	
	Vars	(s)	LPs	SATs	(s)	LPs	(s)	LPs
5	82	0.09	5	6	0.03	18	0.48	23
10	157	0.18	5	6	0.13	79	3.7150	119
15	232	0.33	5	6	0.42	199	83.69	943
20	307	0.5110	6	8	0.5410	243	109.0870	2181
25	382	0.7620	8	10	0.8210	286	503.0030	3833
30	457	1.0520	9	12	1.0110	333	1072.3	6227
35	532	1.4420	10	13	1.7170	341	> 1200	–
40	607	1.8630	13	16	2.5030	374	> 1200	–
45	682	2.7740	15	20	3.8320	475	> 1200	–

Table 2. Computation time for solving a pure integer feasibility problem: comparison between SAT (zCHAFF) and MILP (CPLEX)

T	Bool. Vars	Constr	SAT (s)	MILP (s)
5	82	460	0	0.02
10	157	920	0.01	0.02
15	232	1380	0.02	0.03
20	307	1840	0.03	0.03
25	382	2300	0.04	0.05
30	457	2760	0.05	0.06
35	532	3220	0.06	0.07
40	607	3680	0.08	0.10
45	682	4140	0.09	0.13

also compare the performance of a "naive MILP" solver, that is obtained from the SAT-based B&B code by simply disabling SAT inference. The main reason is that the SAT B&B algorithm solves a much smaller number of LPs than an MILP solver. The "cuts" performed by the SAT solver, i.e. the infeasible SAT problems, obtained at step 3 of the algorithm turn out very useful to exclude subtrees containing no integer feasible solution, see Figure 3. Moreover, the time spent for solving the integer feasibility problem at the root node of the search treee described as SAT problem is much smaller than solving a pure integer feasibility problem, see Table 2. We can also see from Table 1 that the number of feasible SAT solved equals the number of LP solved plus one. This one more SAT is used to complete a feasible solution and it is very useful to further reduce the computation time.



(a) SAT-based algorithm (circle: feasible, square: integer feasible, diamond: infeasible, triangle (up): Bound phase, triangle (down): SAT cut)

(b) Naive MILP algorithm

Fig. 3. Comparison of the trees generated by the SAT-based and naive MILP algorithms (T=30)

The results were simulated on a PC Pentium IV 1.8 GHz running CPLEX 9.0 and zCHAFF 2003.12.04.

5 Conclusions

In this paper we have proposed a new unifying framework for MIP and CSP techniques based on the integration of convex programming and SAT solvers for solving optimal control problems for discrete-time hybrid systems. The approach consists of a logic-based branch and bound algorithm, whose performance in terms of computation time can be superior in comparison to pure mixed-integer programming techniques, as we have illustrated on an example.
Ongoing research is devoted to the improvement of the logic-based method by including relaxations of the automaton and MS parts of the hybrid system in the convex programming part, to the investigation of alternative relaxations of the SAS dynamics that are tighter than the big-M method, and to the use of SAT solvers for also performing domain reduction (cutting planes).

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