ROBUST OPTIMAL CONTROL OF LINEAR HYBRID SYSTEMS: AN MLD APPROACH

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Abstract: A methodology for synthesizing robust optimal input trajectories for constrained linear hybrid systems subject to bounded additive disturbances is presented. The computed control sequence optimizes nominal performance while robustly guarantees that safety/performance constraints are respected. Specifically, for hybrid systems representable in the piecewise affine form, robustness is achieved with an open-loop optimization strategy based on the mixed logical dynamical modelling framework.

Keywords: Hybrid systems, optimal constrained control, robust control, mixed logical dynamical systems, piecewise affine systems, mixed-integer optimization

1. INTRODUCTION

With the spread of dynamical systems integrated with logical/discrete decision components and a market competition pressure to achieve fast and "optimal" designs, the study of dynamical processes having continuous and discrete variables, designated as hybrid systems, has gained increasing attention, and their study has recently seen a rapid development, thanks to the interaction between the computer science and the control engineering communities, motivated by the effect on applications e.g. embedded systems, chemical and biotechnologic processes, aerospace, manufacturing, robotics, automotive applications, etc. (Antsaklis, 2000). In the industrial context, the synthesis of control schemes for hybrid systems is usually approached with heuristic rules, mainly driven by engineering insight and experience, with a consequently long design and verification process. Therefore, the development of new tools to design control/supervisory schemes for hybrid systems and to analyze their stability, safety and performance is of great importance, and the growing interest of researchers and practioners on hybrid systems led to the development of new optimal control techniques suitable to this class of dynamic systems. By thinking of a hybrid system as one characterized by a set of operating modes, each one evolving according to time-driven dynamics, and switching between modes

[1] This research was partially supported by the FCT PRAXIS XXI/BD/21385/99, FCT project POCTI/EME/44336/2002, by the Italian Ministry of Education, University, and Research, and by the European Union under project "Computation and Control".

through discrete events which may be controlled or uncontrolled, the control of the switching times and the choice among several feasible modes, gave rise to a new rich class of optimal control problems. Several researchers presented new techniques to obtain solutions for some subclasses of this new class of optimal control problems, e.g. (Bemporad and Morari, 1999; Branicky *et al.*, 1998; Cassandras *et al.*, 2001; Hedlund and Rantzer, 1999; Kerrigan and Mayne, 2002; Mayne and Rakovic, 2003), just to mention a few. Some of the presented techniques extend classical optimal control principles, others apply dynamic programming techniques, while others also use tools from computational geometry and optimization such as, parametric programming, convex and mixed integer optimization. The choice of a suitable modelling framework, which is the fundamental infrastructure for the study of dynamic systems, is a trade-off between two conflicting criteria: the modelling power and the decisive power. Unfortunately, analysis and control problems for "simple" hybrid systems have an intrinsic high computational load (Blondel and Tsitsiklis, 2000). Specifically, for piecewise linear systems, a class of hybrid systems proposed by Sontag (1981), most analysis and control problems are undecidable or NP-hard. However, in spite of that inherent complexity, the progress of microprocessor technology, as well as of efficient software tools, make possible to tackle low and medium-size complex computational problems, when decidable. The crescent interest on hybrid systems can be shown by the variety of proposed modelling frameworks e.g.: Generalized Hybrid Dynamical Systems (Branicky *et al.*, 1998), Petri Nets (David and Alla, 1994), PieceWise Affine (PWA) systems (Sontag, 1981), Linear and Extended Linear Complementary (ELC) systems (Schutter and Moor, 1999), Max-Min-Plus-Scaling (MMPS) systems (Schutter and van den Boom, 2001), Mixed Logic Dynamical (MLD) systems (Bemporad and Morari, 1999). Heemels *et al.* (2001) proved the equivalence among PWA, ELC, MMPS, and MLD systems, allowing to interchange analysis and synthesis tools among them. Specifically, the MLD framework allows to model systems described by interdependent physical laws (with linear dynamics), logic rules (if-then-else rules) and operating constraints. In fact, the MLD framework is a compromise between modelling power and complexity. Another important characteristic of the MLD model is its optimization-oriented structure, allowing to "smoothly" extend existing optimal control methodologies developed for continuous-valued dynamics to the hybrid setting. More recently, the important topic of uncertainty in the system, which may occur due to parametric uncertainties and/or effect of disturbances, present on the continuous linear dynamics of the hybrid model, was also recently studied by some authors. Backward reachability computations (Lin *et al.*, 2002), as well as polytopic set algebra (Kerrigan *et al.*, 2002) are the tools proposed to deal with the nonlinearity and non-convexity properties of the hybrid/PWA system. In (Kerrigan and Mayne, 2002) it is considered the robust time-optimal, robust optimal and robust receding horizon control problems, and tools from computational geometry, dynamic and parametric programming are used to obtain explicit state-feedback control laws. However, as expected due to the complexity of the tackled problems, the algorithms might be too inefficient to be realizable for large or complex systems, as the authors remark. In order to take advantage of the optimization-oriented characteristic of the MLD framework, the method presented in this paper extends it to systems subject to bounded additive disturbances, presenting a computational procedure, based on mixed-integer programming, to obtain open-loop control sequences that guarantee the fulfillment of dynamic and operating constraints and optimize a nominal performance criterion. The paper has the following structure. In Section 2, the problem is formulated. The control synthesis algorithm is presented in Section 3. Section 4 is devoted to an illustrative example, and in Section 5 some conclusions are drawn.

2. PROBLEM DEFINITION

The goal of this paper is to propose a methodology to solve a discrete-time finite-horizon optimal control problem, using the MLD framework, of an uncertain linear hybrid system. As the PWA modelling framework is rather intuitive and descriptive, we assume that the system is described within this framework. Therefore, consider the following discrete-time PWA system perturbed by bounded additive disturbances, and subject to performance / safety / operational constraints $\mathcal{C}(.)$,

$$x(k+1) = A_i x(k) + B_i u(k) + e_i + W_i v(k) \text{ , if } \begin{bmatrix} x(k) \\ u(k) \end{bmatrix} \in \Omega_i \triangleq \{ \begin{bmatrix} x(k) \\ u(k) \end{bmatrix} : \ F_i x(k) + G_i u(k) \leq h_i \} \quad (1a)$$

$$\begin{bmatrix} x(k) \\ u(k) \end{bmatrix} \in \mathcal{C}(k) \triangleq \{ \begin{bmatrix} x(k) \\ u(k) \end{bmatrix} : \ K(k)x(k) + L(k)u(k) \leq m(k) \}, \forall k \in \{0, \ldots, N\} \quad (1b)$$

where $u(k) \in \mathbb{U} \subset \mathbb{R}^m$, $x(k) \in \mathbb{X} \subset \mathbb{R}^n$ and $v(k) \in \mathbb{V} \subset \mathbb{R}^p$ denote the input, state and disturbance vectors, respectively, at time k; the index i represents the current discrete mode ($i \in \{1, \ldots, s\}$); the partitions Ω_i are convex polytopes (i.e. closed and bounded polyhedra) in the input+state space; $\Omega = \bigcup_{i=1}^{s} \Omega_i$, $\mathring{\Omega}_i \bigcap \mathring{\Omega}_j = \emptyset$, $\forall \ i \neq j$, where $\mathring{\Omega}_i$ denotes the interior of the polytope Ω_i; $\mathbb{U}$,

$\mathbb{X}$ are convex polytopes, as well as $\mathbb{V}$, according to the typical unknown-but-bounded characterization of disturbances, with $0 \in \mathbb{V}$; $\mathcal{C}(k)$ denotes sets of (possibly time-varying) constraints on the input+state space; A_i, B_i, W_i, F_i, G_i, $K(k)$ and $L(k)$ are real matrices of appropriate dimensions, h_i and $m(k)$ are real vectors, and e_i is the affine real vector, for all $i = 1, \ldots, s$.

As shown by Bemporad and Morari (1999), the PWA system can be expressed as an MLD system, by transforming logical facts involving continuous variables into linear inequalities involving integer and continuous variables. The technique allows to arrive at an equivalent MLD model of the PWA dynamics, where the first inequality concerns the continuous/discrete interface, the second inequality concerns the discrete/continuous interface, and the last inequality represents operational constraints,

$$x(k+1) = B_{z_x} z_x(k) + B_{z_u} z_u(k) + B_\delta \delta(k) + \\ + B_{z_v} z_v(k) \qquad \text{(2a)}$$

$$E_\delta^{c/d} \delta(k) \leq E_x^{c/d} x(k) + E_u^{c/d} u(k) + e^{c/d} \qquad \text{(2b)}$$

$$E_{z_x}^{d/c} z_x(k) + E_{z_u}^{d/c} z_u(k) + E_\delta^{d/c} \delta(k) + E_{z_v}^{d/c} z_v(k) \\ \leq E_x^{d/c} x(k) + E_u^{d/c} u(k) + E_v^{d/c} v(k) + e^{d/c} \qquad \text{(2c)}$$

$$E_x^{ctr} x(k) + E_u^{ctr} u(t) \leq e^{ctr}, \qquad \text{(2d)}$$

where inequalities should be understood componentwise and $\forall k \in \{0, \ldots, N\}$; $\delta(k)$ is an auxiliary vector with binary (zero/one) entries that defines the mode i (or equivalently partition i) of the system ($\dim[\delta] = (s \times 1)$), moreover if mode i is active then $\delta_i(k) = 1$ and $\delta_j(k) = 0$, $\forall j \neq i$, $i, j \in \{1, \ldots, s\}$ (these constraints are incorporated in inequality (2b)); $z_{xi}(k) = \delta_i(k) x(k)$ are auxiliary continuous variable ($\dim[z_{xi}] = \dim[x] = (n \times 1)$); using the Kronecker product to abbreviate notation, $z_x(k) = \delta(k) \otimes x(k)$, ($\dim[z_x] = (s \cdot n \times 1)$); $z_u(k) = \delta(k) \otimes u(k)$ ($\dim[z_u] = (s \cdot m \times 1)$) and $z_v(k) = \delta(k) \otimes v(k)$ ($\dim[z_v] = (s \cdot p \times 1)$). Notice also, that $B_{z_x} \equiv [\, A_1\ A_2\ \ldots\ A_s \,]$, $B_{z_u} \equiv [\, B_1\ B_2\ \ldots\ B_s \,]$, $B_\delta \equiv [\, e_1\ e_2\ \ldots\ e_s \,]$, and $B_{z_v} \equiv [\, W_1\ W_2\ \ldots\ W_s \,]$. Although well-poseness of (2) and knowledge of initial state, disturbances and control inputs allows the simulation of system behavior, the computation of optimal control sequences based on the prediction of future states, assuming that disturbances are unknown and the only information about them are their bounds, is a much harder task. In fact, the MLD framework was developed to deal with deterministic systems and its extension to uncertain systems rises some problems, since the predicted state $x(k)$ is set-valued, and owing to the nonlinearity of the system those sets are non-convex and the effect of the disturbance on the state is dependent on the trajectory of the system. Therefore, the typical approach used for linear systems, e.g. by Scokaert and Mayne (1998), of using the extreme disturbance realizations can not be directly applied in this case. To overcome the aforementioned problem, Silva *et al.* (2003) proposed to restrict the admissible control sequences such that, for every value of the disturbance, the mode of the system is unique at each time step k. The main property of this *robust mode control* sequence is to guarantee that the mode of the system is "certain" and, for each admissible control sequence, the uncertainty associated with the state is defined by a convex set. This extra structure on the dynamics of the controlled system is advantageous from a computational point of view, though it (may) lead to smaller domain of feasible control sequences, which could be compensated by extending the robust mode control concept to a closed-loop prediction policy. Within the MLD modelling framework, the robust mode control restriction is expressed by the following condition (Silva *et al.*, 2003)

$$\delta(k) = \mathcal{F}(x(k), u(k)), \ \forall v(.) \in \mathbb{V}, \qquad (3)$$

Notice that if the input sequence u_0^k is such that (3) is respected, then the mode of the system, i.e. $\delta(j)$, is independent of the disturbance realization, for all instants $j \in \{0, \ldots, k\}$, and the state at time step $k+1$, which is set-valued, can be decomposed as follows $x^{\mathbf{u}}(k+1) = \bar{x}^{\mathbf{u}}(k+1) + \tilde{x}^{\mathbf{u}}(k+1)$, where the first term represents the nominal trajectory and the second term denotes the convex uncertainty set associated with the state, which depends on δ_0^k and on $\mathbb{V}_0^k$.

By including the constraint (3) into (2b), we obtain the MLD-based prediction/synthesis model. To achieve a finite number of constraints, and by noticing that the state-set generated by the disturbance is convex, one considers the extreme disturbance realizations $\mathbf{v}^l \triangleq \{v^l(0), \ldots, v^l(N-1)\}$, indexed by $l \in \mathcal{L}_v$, i.e. all disturbance sequences that take values at the vertices of the polytope $\mathbb{V}_0^{N-1}$. We consider also that $\mathbf{x}^l$ denote the state associated with the respective disturbance realization, and we assume the same notation for the other variables. Hence, we define the robust mode optimal control problem as follows,

Problem 1. Given an initial state x_0 and a final time N, find (if it exists) the control sequence $\mathbf{u} \equiv \{u(0), u(1), \ldots, u(N-1)\}$, and the auxiliary sequences $\boldsymbol{\delta}$, $\mathbf{z_u}$ and $\mathbf{z_x}$, which (i) transfer the state from x_0 to a given final set $\mathcal{C}(N)$ that contains a target state x_f and (ii) minimize the performance index

$$J_N(x_0, \mathbf{u}, \boldsymbol{\delta}, \mathbf{z_u}, \mathbf{z_x}) \triangleq \sum_{k=0}^{N-1} \|\bar{x}(k) - x_f\|_{Q,2}^2 + \\ + \|u(k) - u_f\|_{R,2}^2 + \|\bar{x}(N) - x_f\|_{P,2}^2 \qquad (4)$$

subject, $\forall l \in \mathcal{L}_v$, $\forall k$, to:

$$x^l(k+1) = B_{z_x} z_x^l(k) + B_{z_u} z_u(k) + B_\delta \delta(k) + \\ + B_{z_v} z_v^l(k)\, x^l(0) = x_0 \quad (5a)$$

$$E_\delta^{c/d}\delta(k) \leq E_x^{c/d} x^l(k) + E_u^{c/d} u(k) + e^{c/d} \quad (5b)$$

$$E_{z_x}^{d/c} z_x^l(k) + E_{z_u}^{d/c} z_u(k) + E_\delta^{d/c}\delta(k) + E_{z_v}^{d/c} z_v^l(k) \\ \leq E_x^{d/c} x^l(k) + E_u^{d/c} u(k) + E_v^{d/c} v^l(k) + e^{d/c} \quad (5c)$$

$$E_x^{ctr} x^l(k) + E_u^{ctr} u(t) \leq e^{ctr} \quad (5d)$$

where $\|x\|_{Q,2}^2 \triangleq x'Qx$; $\bar{x}(k)$ represents the nominal trajectory; $\mathbf{z_x} \triangleq \{\mathbf{z_x^1}, \ldots, \mathbf{z_x^q}\}$, where q is the number of indices in $\mathcal{L}_v$; Q, R and P are symmetric positive definite matrices; and x_f and u_f are given desired target vectors.

3. SYNTHESIS ALGORITHM

Due to the presence of integer variables $\boldsymbol{\delta}$, Problem P1 defines a Mixed Integer Quadratic Programming (MIQP) problem, which can be efficiently solved by branch and bound based methods. Branch-and-bound is a strategy for solving optimization problems with a combinatorial characteristic. It proceeds by traversing a tree in which each node is a simpler (relaxed) "subproblem" of the original problem, such that the optimal solution of these subproblems defines lower bounds on the cost of the original problem. The algorithm presented nextly is based on this strategy, and can be seen as extension of the procedure proposed by Bemporad and Morari (1999), though the latter did not consider uncertainty on system dynamics. Before starting the description of the algorithm, consider the following notation. The optimal cost of problem P1 is generically denoted by $V_N(P)$ and its optimal solution by $\arg^*(P)$. Problem $P(\Delta_0^j))$ corresponds to problem P though fixing the first $j+1$ components of the sequence $\boldsymbol{\delta}$ to the values defined by Δ_0^j, and relaxing the others $N-j-1$ binary variables to the $[0,1]$ real interval. The problem at the root node is denoted by $P(root)$ (or by $P(\Delta_0^{-1})$), and is obtained relaxing all binary variables which compose $\boldsymbol{\delta}$. The optimal cost of each subproblem $P(\Delta_0^j)$ is denoted by $V_N(P(\Delta_0^j))$ and its solution by $\arg^*(P(\Delta_0^j))$. Next, the pseudo code that define the main steps of the algorithm to solve problem P1 is presented.

Algorithm 1.
Initialize all data structures: $\mathcal{S} = \emptyset$ (empty stack), $j=-1$, $V_N(P) = \infty$, $arg^*(P) =$ "infeasible", (note: $P(\Delta_0^{-1}) \equiv P(root)$);
Push $P(\Delta_0^j)$ onto the top of stack $\mathcal{S}$;
While $\mathcal{S} \neq \emptyset$ **do**
 Pop $P(\Delta_0^j)$ off the top of the stack $\mathcal{S}$ and solve $P(\Delta_0^j)$;
 If $arg^*(P(\Delta_0^j))$ is feasible (i.e. $\boldsymbol{\delta}_{j+1}^{N-1}$ is binary and the solution respects all constraints) and $V_N(P(\Delta_0^j)) < V_N(P)$ **then** assign $V_N(P) = V_N(P(\Delta_0^j))$, $arg^*\,(P) = arg^*(P(\Delta_0^j))$;
 Else
  If $V_N(P(\Delta_0^j)) < V_N(P)$ **then** Subdivide $P(\Delta_0^j)$ into s subproblems, by generating the branches corresponding to all possible s vectors (modes) of $\delta(j+1|t)$ (fix the i component to one and the others to zero); sort the problems by decreasing value of $V_N(P(\Delta_0^{j+1}))$ and index the sorted problems at each node by the l variable, i.e. by $P^l(\Delta_0^{j+1})$;
   For $l = 1$ to s
    If $V_N(P^l(\Delta_0^{j+1})) < V_N(P)$ **then** Push $P^l(\Delta_0^{j+1})$ onto the top of $\mathcal{S}$;
Endif; **Endfor**; **Endif**; **Endifelse**; **Endwhile**;
Return "Optimal cost and optimal argument of problem P:" $V_N(P)$, $arg^*(P)$.

The algorithm executes the following steps. At the root node, all binary variables of $\boldsymbol{\delta}$ are relaxed. Therefore, $P(root)$ is solved, meaning to solve a Quadratic Programming (QP) problem. The cost $V_N(P(root))$ is a lower bound on the optimal cost of the original problem P because $P(root)$ is less constrained since the binary variables are relaxed. The next level of the tree is composed with the nodes generated by imposing to each branch one of the possible s modes of $\boldsymbol{\delta}_t^0 = \{\delta(0|t)\}$. At each node, the problem $P(\Delta_0^0)$ is solved, i.e. $\delta(0|t)$ is fixed equal to Δ_0^0 and sequence $\{\delta(1|t), \ldots, \delta(N-1|t)\}$ is relaxed. These problems define again QP problems. The optimal cost of each one of the s $P(\Delta_0^j)$ problems, i.e. $V_N(P(\Delta_0^j))$, is computed and compared with the cost of the best existing feasible solution (which is an upper bound of the optimal cost of the original problem). If no solution is found or the computed cost is higher or equal to the existing feasible one, then the associated subtree is discarded, since the obtained optimal cost is a lower bound of $V_N(P)$ *on that path*. If the computed optimal cost is lower than the cost of the best existing feasible solution then another set of s branches are generated from that node. If the optimal solution of an intermediate node is binary, the node is labelled as "fathomed", and no more nodes are generated from this one, and if adequate its solution substitutes the current upper bound of the optimal cost of the original problem. The algorithm proceeds choosing the node with lower cost, until all nodes have been investigated/discarded.

The presented algorithm is based on a problem formulation that explicitly considers the state trajectories associated with all possible disturbance vertices. Therefore, this formulation substantially increases the number of constraints and variables of the associated optimal control problem comparing to the nominal problem. In fact, the cardinality of $\mathcal{L}_v$ is exponential on the prediction horizon N, specifically if q is the number of vertices of $\mathbb{V}$ then the number of constraints and variables increase by q^N. To circumvent this problem a different methodology is presented in (Silva *et al.*, 2003), which main idea is to compute the maximum effect of the disturbance for each component of the constraints by using linear programming.

4. ILLUSTRATIVE EXAMPLE

We will now apply the presented methodology to the two tank system depicted in Figure 1, which is based on a benchmark problem for hybrid systems (see e.g. (Bemporad *et al.*, 1999)). The variables have the following meaning: h_i is the liquid level in tank T_i and $Q_{\{1,2,3\}}$ represents the volumetric flow across valve $V_{\{1,2,3\}}$. Valve V_1 is unidirectional, allowing flow from tank T_1 to tank T_2; valve V_2 is bidirectional and on/off, controlled by an electric signal C_2 with the following characteristic: $C_2 \in [0, 0.5[$ – Off, $C_2 \in [0.5, 1]$ – On; valve V_3 is also bidirectional but not controlled. The control objective is to regulate the flow Q_3, which in practice is equivalent to regulate the level h_2, by using signal C_2 and also controlling flow Q_4, which is limited between $[0, Q_{4M}]$. The system is perturbed by a bounded disturbance flow $Q_5 \in [Q_{5m}, Q_{5M}]$. System nonlinearities are due to the constitutive nonlinear characteristic of the valves, to the position of valve V_1 above the "ground" level that enforces a change of dynamics dependent on the levels of liquid on both tanks, unidirectionality of valve V_1, and on/off characteristic of valve V_2. The PWA model is obtained by first considering the partition of the state-space (h_1, h_2) into four different regions, which have different dynamics dependent on the height of h_1, h_2, and h_v (see Figure 2, where $(h_1, h_2) = (x_1, x_2)$). The inherent nonlinearity in the flow across valve l, from tank i to tank j, when the levels (measured from the valve) are h_i and h_j respectively, will be linearized, for each partition, using the same procedure as in (Bemporad *et al.*, 1999): $Q_{l(i \to j)} = C_l\, k_l \mathrm{sign}(h_i - h_j)\sqrt{|h_i - h_j|} \approx C_l \frac{1}{R_l}(h_i - h_j)$, where C_l is the On/Off control of the valve (if valve V_l is not controlled then $C_l = 1$), k_l is a coefficient of resistance for the valve and R_l is the inverse of the coefficient resulting from the linearization. Each one of the four state dependent partitions has different dynamics depending if valve V_2 is on or off, so the system is characterized by eight modes. At modes 1–4 valve V_2 is fully open, while at modes 5–8 it is completely closed (see Figure 2). To obtain a discrete-time PWA model, continuous dynamic equations are approximated by the forward Euler integration method, i.e. $\dot{y}(k) \approx \frac{y(k+1)-y(k)}{T_s}$. Table 1 presents the numerical values of system parameters, and

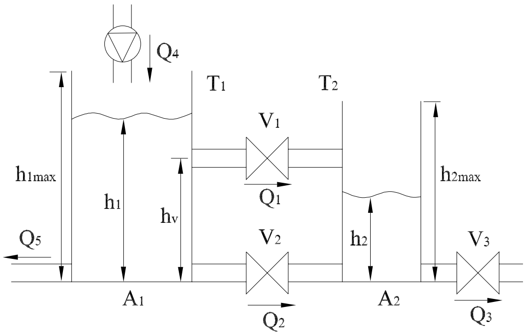

Fig. 1. Two tank system

Table 1. System parameters. Height is measured in meters and flow in cubic meters by second.

Symbol	Meaning	Value
h_v	Height of V_1	5
h_{1M}	Max. liquid level in T_1	10
h_{2M}	Max. liquid level in T_2	10
A_1	Cross-sectional area of T_1	20
A_2	Cross-sectional area of T_2	10
Q_{4M}	Max. input flow into T_1	0.05
Q_{5m}	Min. disturbance flow into T_1	-0.001
Q_{5M}	Max. disturbance flow out of T_1	0.001
R_1	V_1 linearized resistance coef.	200
R_{2a}	V_2 linearized resistance coef. (Used in modes {1,2,3})	350
R_{2b}	V_2 linearized resistance coef. (Used in mode {4})	500
R_{3a}	V_3 linearized resistance coef. (Used in modes {2,6})	700
R_{3b}	V_3 linearized resistance coef. (Used in modes {1,3,4,5,7,8})	1000
T_s	Sampling interval	1

Table 2. Numerical parameters of the PWA system model.

mode	A_i	B_i	e_i	W_i
1	$\begin{bmatrix} 0.6071 & 0.3929 \\ 0.7857 & 0.1143 \end{bmatrix}$	$\begin{bmatrix} 50 & 0 \\ 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 50 \\ 0 \end{bmatrix}$
2	$\begin{bmatrix} 0.6071 & 0.1429 \\ 0.7857 & 0.5714 \end{bmatrix}$	$\begin{bmatrix} 50 & 0 \\ 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 1.25 \\ -2.5 \end{bmatrix}$	$\begin{bmatrix} 50 \\ 0 \end{bmatrix}$
3	$\begin{bmatrix} 0.8571 & 0.1429 \\ 0.2857 & 0.6143 \end{bmatrix}$	$\begin{bmatrix} 50 & 0 \\ 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 50 \\ 0 \end{bmatrix}$
4	$\begin{bmatrix} 0.9000 & 0.1000 \\ 0.2000 & 0.7000 \end{bmatrix}$	$\begin{bmatrix} 50 & 0 \\ 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 50 \\ 0 \end{bmatrix}$
5	$\begin{bmatrix} 0.7500 & 0.2500 \\ 0.5000 & 0.4000 \end{bmatrix}$	$\begin{bmatrix} 50 & 0 \\ 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 50 \\ 0 \end{bmatrix}$
6	$\begin{bmatrix} 0.7500 & 0.0000 \\ 0.5000 & 0.8571 \end{bmatrix}$	$\begin{bmatrix} 50 & 0 \\ 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 1.25 \\ -2.5 \end{bmatrix}$	$\begin{bmatrix} 50 \\ 0 \end{bmatrix}$
7	$\begin{bmatrix} 1.0000 & 0.0000 \\ 0.0000 & 0.9000 \end{bmatrix}$	$\begin{bmatrix} 50 & 0 \\ 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 50 \\ 0 \end{bmatrix}$
8	$\begin{bmatrix} 1.0000 & 0.0000 \\ 0.0000 & 0.9000 \end{bmatrix}$	$\begin{bmatrix} 50 & 0 \\ 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 50 \\ 0 \end{bmatrix}$

Table 2 presents the numerical values of the PWA model, where $\left[\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right] = \left[\begin{smallmatrix} h_1 \\ h_2 \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} u_1 \\ u_2 \end{smallmatrix}\right] = \begin{bmatrix} Q_4 \\ C_2 \end{bmatrix}$. The optimal control problem to be solved is defined by the following parameters: $N = 4$, $Q = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 5 \end{smallmatrix}\right]$, $R = \left[\begin{smallmatrix} 2 & 0 \\ 0 & 1 \end{smallmatrix}\right]$, $P = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 10 \end{smallmatrix}\right]$, $x_0 = \left[\begin{smallmatrix} 2.5 \\ 2.5 \end{smallmatrix}\right]$, $x_f = \left[\begin{smallmatrix} 7.1733 \\ 5.9777 \end{smallmatrix}\right]$, $u_f = \left[\begin{smallmatrix} 0.0060 \\ 0.0000 \end{smallmatrix}\right]$, where (x_f, u_f) is an equilibrium pair "near" to a desired steady-state $x_d = \left[\begin{smallmatrix} 7 \\ 6 \end{smallmatrix}\right]$. In addition to constraints related with tanks dimensions, a terminal constraint $\mathcal{C}(4)$ on state variables must be also satisfied (see Figure 2). The robust mode optimal input trajectory was computed by Algorithm 1 and the optimal state trajectory obtained is presented in Figure 2. Associated with each computed $x(k)$, it is presented the associated uncertainty set. Notice that all constraints are respected, as well as the robust mode restriction. The optimal solution took 895 sec., while the

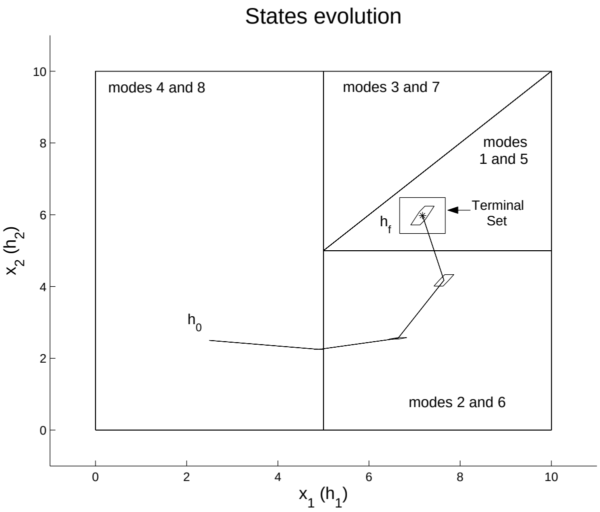

Fig. 2. State partition and optimal trajectory

first feasible solution was found after 241 sec., using interpreted MATLAB® code on a 1.6 GHz PENTIUM® 4. A more efficient code implementation, as well as adequate search heuristics, could improve these results.

5. CONCLUSIONS

An illustrative example shows the applicability of solving robust finite-horizon optimal control problems, within the MLD framework, for constrained PWA systems subject to bounded additive disturbances. The presented approach computes a control sequence that minimizes a nominal quadratic performance index, and guarantees that the mode of the dynamics, at each time instant, is independent of the disturbances and that all safety/performance constraints are verified.

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