Proceedings of the 38th
Conference on Decision & Control
Phoenix, Arizona USA • December 1999

Observability and Controllability of Piecewise Affine and Hybrid Systems

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Abstract

In this paper we prove, in a constructive way, the equivalence between hybrid and piecewise affine systems. By focusing our investigation on the latter class, we show through counterexamples that observability and controllability properties cannot be easily deduced from those of the component linear subsystems. Instead, we propose practical numerical tests based on mixed-integer linear programming.

Key Words: Hybrid systems, controllability, observability, piecewise linear systems, piecewise affine systems, mixed-integer linear programming

1 Introduction

In recent years both control and computer science have been attracted by *hybrid systems* [1, 2, 13, 15, 16], because they provide a unified framework for describing processes evolving according to continuous dynamics, discrete dynamics, and logic rules. The interest is mainly motivated by the large variety of practical situations, for instance real-time systems, where physical processes interact with digital controllers.

Several modeling formalisms have been developed to describe hybrid systems, as reviewed in [14]. It is apparent that the tools for the analysis of hybrid systems strongly depend on the adopted mathematical description.

Recently, Bemporad and Morari [5] introduced a new class of discrete-time hybrid systems called Mixed Logical Dynamical (MLD) systems. The justification for the MLD form is that it is capable to model a broad class of systems arising in many applications: PieceWise Affine (PWA) systems, linear hybrid dynamical systems, hybrid automata, some classes of discrete event systems, linear systems with constraints, etc. Examples of real-world applications that can be naturally modeled within the MLD framework are reported in [4, 5, 6].

The first result of this paper is to prove, in a constructive way, that MLD systems are formally equivalent to PWA systems. This result allows extending all the techniques developed for PWA models to the general MLD description of hybrid systems, therefore rendering the PWA framework a useful companion for investigating properties and designing algorithms. Although based on different arguments, this importance has also been pointed out by Sontag [20], who highlights the equivalence between Piecewise Linear (PWL) systems and interconnections of linear systems and finite automata.

Piecewise affine systems are described by the state-space equations

$$\begin{array}{l} x(t+1) = A_i x(t) + B_i u(t) + f_i \\ y(t) = C_i x(t) + g_i \end{array} , \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} \in \mathcal{X}_i \tag{1}$$

where $\{\mathcal{X}_i\}_{i=1}^s$ is a partition of the state+input set and f_i, g_i are suitable constant vectors. Each subsystem defined by the 5-tuple $(A_i, B_i, f_i, C_i, g_i)$, $i \in \{1, 2, \ldots, s\}$ is termed a *component* of the PWA system (1). If f_i and g_i are null, system (1) is referred to as piecewise linear.

Despite the fact that PWA models are just a composition of linear time-invariant dynamic systems, their structural properties such as observability, controllability, and stability are complex and articulated, as typical of nonlinear systems.

The research into stability criteria for PWL systems has been motivated by the fact that the stability of each component subsystem is not enough to guarantee stability of a PWL system [7] (and vice versa [23]). Very little research focused on observability and controllability properties of hybrid systems, apart from contributions limited to the field of timed automata [1, 11] and the pioneering works of Sontag [19, 20] for PWL systems. Needless to say, these concepts are fundamental for understanding *if* and *how well* a state observer and a controller for a hybrid system can be designed.

Controllability and observability properties have been investigated in [8, 10] for linear time-varying systems, and in particular for the so-called class of piecewise constant systems (where the matrices in the state-space representation are piecewise constant functions of time). Although in principle applicable, these results do not allow to catch the peculiarities of PWA systems.

In this paper we provide two main contributions to the analysis of controllability and observability of hybrid and PWA systems: (i) we show the reader that observability and controllability properties can be very complex; we present a number of counterexamples that rule out obvious conjectures about inheriting observability/controllability properties from the composing linear

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subsystems; (ii) we provide observability and controllability tests based on *Linear* and *Mixed-Integer Linear Programs* (MILP).

2 Mixed Logical Dynamical (MLD) Systems

The mixed logic dynamical (MLD) form was introduced in [5], based on the idea of transforming logic relations into mixed-integer linear inequalities [22, 17]. The general MLD form is:

$$x(t+1) = Ax(t) + B_1u(t) + B_2\delta(t) + B_3z(t) \quad \text{(2a)}$$

$$y(t) = Cx(t) + D_1u(t) + D_2\delta(t) + D_3z(t) \quad \text{(2b)}$$

$$E_2\delta(t) + E_3z(t) \leq E_1u(t) + E_4x(t) + E_5 \quad \text{(2c)}$$

where $x \in \mathbb{R}^{n_c} \times \{0,1\}^{n_\ell}$ are the continuous and binary states, $u \in \mathbb{R}^{m_c} \times \{0,1\}^{m_\ell}$ are the inputs, $y \in \mathbb{R}^{p_c} \times \{0,1\}^{p_\ell}$ the outputs, and $\delta \in \{0,1\}^{r_\ell}$, $z \in \mathbb{R}^{r_c}$ represent auxiliary binary and continuous variables respectively. All constraints on state, input, z and δ variables are summarized in the inequality (2c). Although the description (2) seems to be linear, nonlinearity is concentrated and hidden in the integrality constraints over binary variables.
We assume that system (2) is *completely well-posed* [5], which in words means that for all x, u within a bounded set the variables δ, z are uniquely determined, i.e. there exist functions F, G such that, at each time t, $\delta(t) = F(x(t), u(t))$, $z(t) = G(x(t), u(t))$ [1]. This allows assuming that $x(t+1)$ and $y(t)$ are a uniquely defined once $x(t)$, $u(t)$ are given, and therefore that x- and y-trajectories exist and are uniquely determined by the initial state $x(0)$ and input trajectory u.
The auxiliary variables are introduced when transforming propositional logic into linear inequalities. We refer the reader to [5] for a detailed exposition.

3 Equivalence Between Hybrid and PWA Systems

Consider a *Piecewise Affine* (PWA) time-invariant dynamic system of the form (1), where $x \in \mathbb{R}^n$, $y \in \mathbb{R}^p$ and $u \in \mathbb{R}^m$. We take into account constraints on the state and the input assuming that the state+input admissible set $\mathcal{X} \subseteq \mathbb{R}^{n+m}$ is a convex and bounded polyhedron. Moreover we suppose that $\mathcal{X}_i$, $i = 1, 2, \ldots, s$ form a polyhedral partition[2] of $\mathcal{X}$.
PWA systems can be represented in the MLD form (2). The translation consists of defining logical δ_i variables $[\delta_i = 1] \leftrightarrow [[\begin{smallmatrix} x \\ u \end{smallmatrix}] \in \mathcal{X}_i]$ and imposing the exclusive-or condition $\bigoplus_{i=1}^{s} [\delta_i = 1]$. For details, the reader is referred to [5].
Conversely, we will show in Proposition 1 that every MLD model (2) is equivalent to a PWA system.

[1] A more general definition of well-posedness where only the components of δ and z entering (2a)–(2b) are required to be unique is given in [5].

[2] Each set $\mathcal{X}_i$ is a (not necessarily closed) convex polyhedron s.t. $\mathcal{X}_i \bigcap \mathcal{X}_j = \emptyset$, $\forall i \neq j$, $\bigcup_{i=1}^{s} \mathcal{X}_i = \mathcal{X}$.

Proposition 1 *Consider generic trajectories $x(t)$, $u(t)$, $y(t)$ of a MLD system (2). Then there exist a polyhedral partition $\{\mathcal{X}_i\}_{i=1}^{s}$ of the state+input space $\mathcal{X}$ and 5-tuples $(A_i, B_i, C_i, f_i, g_i)$, $i = 1, \ldots, s$, such that $x(t)$, $u(t)$, $y(t)$ satisfy (1).*

Proof: In order to simplify the proof, without loss of generality we assume that the logical components $x_{\ell i}$ of x_ℓ are also auxiliary variables, i.e. $\forall i = 1, \ldots, n_\ell$ $\exists j$ such that $x_{\ell i} = \delta_j$. This is not a restrictive assumption, as typically the state transition of logical states derives from a logic predicate involving literals associated with components of $\delta(t)$ and $x_\ell(t)$, and the latter can be expressed again as additional auxiliary variables by simply adding the constraints $\delta_j(t) \leq x_{\ell i}(t)$, $-\delta_j(t) \leq -x_{\ell i}(t)$ in (2c).
By well-posedness of system (2), given $x(t)$, $u(t)$ the vector $\delta(t)$ is uniquely defined, namely $\delta(t) = F(x(t), u(t))$. Moreover, it only takes a value δ_i within a set of (at most) 2^{r_ℓ} values (corresponding to all possible 0–1 combinations). Let s be the number of valid combinations, i.e. the number of all different vectors $\delta \in \{0,1\}^{r_\ell}$ satisfying constraints (2c) for some $x(t)$, $u(t)$, $z(t)$. The idea is to partition the state+input space by grouping in regions $\mathcal{X}_i$ all $[\begin{smallmatrix} x(t) \\ u(t) \end{smallmatrix}]$ corresponding to the same binary vector $\delta_i = F(x(t), u(t))$. Let us fix $\delta(t) \equiv \delta_i$. The inequalities (2c) define a polyhedron $\mathcal{P}$ in $\mathbb{R}^{n+m+r_c}$. By well-posedness of $z(t)$, given a pair $x(t)$, $u(t)$ there exists only one value $z(t) \in \mathbb{R}^{r_c}$ satisfying (2c), namely $z(t) = G(x(t), u(t))$. As all the inequalities (2c) are linear, G is an affine function, namely

$$z(t) = K_{4i}x(t) + K_{1i}u(t) + K_{5i}, \ \forall x(t), u(t) \quad (3)$$
$$\text{s.t. } F(x(t), u(t)) = \delta_i$$

and $\mathcal{P} \subset \mathbb{R}^{n+m+r_c}$ is a polyhedral set of dimension less than or equal to $n+m$ (for instance if $n = 1$, $m = 0$, $r_c = 1$, $\mathcal{P}$ would be a segment in $\mathbb{R}^2$). By substituting (3) in (2a)–(2b), we obtain $x(t+1) = (A + B_3K_{4i})x(t) + (B_1 + B_3K_{1i})u(t) + (B_2\delta_i + B_3K_{5i})$, $y(t) = (C + D_3K_{4i})x(t) + (D_1 + D_3K_{4i})u(t) + (D_3K_{5i} + D_2\delta_i)$, which, by suitable choice of A_i, B_i, C_i, f_i, g_i, $i = 1, \ldots, s$, corresponds to (1) for $\mathcal{X}_i = \{[\begin{smallmatrix} x \\ u \end{smallmatrix}] : (E_3K_{4i} - E_4)x + (E_3K_{1i} - E_1)u \leq (E_5 - E_3K_{5i} - E_2\delta_i)\}$ ■

Remark 1 We stress the fact that the proof is based on a constructive argument. In particular, the matrices K_{1i}, K_{4i} and K_{4i} involved in formula (3), can be derived either from direct insight or automatically from the inequalities (2c). Further details are reported in [3].

4 Observability

In this section, we consider observability of MLD systems (2), or equivalently PWA systems in view of Proposition 1.
Denote by $y(t, x, u)$ the output evolution at time t starting from the initial condition $x(0) = x$ and driven by the input $u(t)$, $t = 0, 1, \ldots$. We extend the definition

of observability given in [12, 18] to non-autonomous hybrid systems of the form (2)

Definition 1 *Let $\mathcal{X}(0) \subseteq \mathbb{R}^{n_c} \times \{0,1\}^{n_\ell}$ be a set of initial states, and $\mathcal{U} \subseteq \mathbb{R}^{m_c} \times \{0,1\}^{m_\ell}$ a set of inputs. The MLD system (2) is* incrementally observable in T steps on $\mathcal{X}(0)$ and $\mathcal{U}$ *or simply* incrementally observable *if there exist two norms $\|\cdot\|_a$ (on $\mathbb{R}^{n_c+n_\ell}$) and $\|\cdot\|_b$ (on $\mathbb{R}^{p_c+p_\ell}$) and a positive scalar w such that $\forall x_1, x_2 \in \mathcal{X}(0)$ and $\forall$ input sequences $\{u(t)\}_{t=0}^{T-1} \subseteq \mathcal{U}$*

$$\sum_{t=0}^{T-1} \|y(t,x_1,u) - y(t,x_2,u)\|_b \geq w\|x_1 - x_2\|_a \quad (4)$$

Remark 2 The parameters T and w appearing in Definition 1 admit a practical interpretation. The scalar w can be viewed as an observability measure[3] for an incrementally observable system. For fixed initial states x_1 and x_2, the larger w, the more different the trajectories $y(t,x_1,u)$, $y(t,x_2,u)$ (from now on, we will write in short $y_1(t)$, $y_2(t)$). Hence, in practice, one would fix a minimum observability level w_{min} and require that $w \geq w_{min}$. If this condition is not fulfilled, we classify the system as *practically unobservable*. Practical unobservability also arises if Definition 1 is satisfied only for large T. Therefore, it is sensible to fix an upper bound T_{max} on T and define an MLD system as practically observable when it satisfies Definition 1 with $T < T_{max}$. ■

4.1 Observability Counterexamples for PWA Systems

Definition 1 was formulated for the general class of hybrid systems described by the MLD form (2), or equivalently the PWA form (1). One might expect to exploit the structure of PWA systems to derive results about observability similar to those holding for linear systems. Below we show some counterexamples which undermine these hopes, even in the simpler case of autonomous PWL systems.

We first show that for PWL systems the time of observability T is not related to the order n of each subsystem, and therefore that if a PWL system is incrementally observable nothing can be said, in general, about the minimum T such that Definition 1 holds.

Then, we show examples where observability properties of a PWL system cannot be directly inferred from the observability properties of its linear subsystems. In fact, we will show that unobservable subsystems can be composed to build an observable PWL system, and vice versa that the composition of observable subsystems can become unobservable.

4.1.1 A PWL system incrementally observable with T arbitrarily large.: Consider the following system

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}(t+1) = \begin{cases} \begin{bmatrix} 1.1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}(t), & x_1(t) \in [\epsilon, 1), \\ \begin{bmatrix} 0 & 0.9 \\ 0.9 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}(t), & \text{otherwise} \end{cases} \quad (5)$$
$$y(t) = x_1(t),$$

[3] More precisely, one should use $\tilde{w} = \sup\{w > 0 \text{ s.t. (4) holds}\}$ as observability measure.

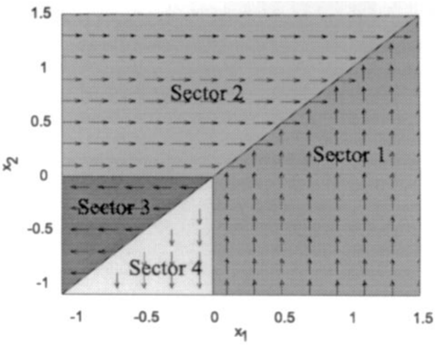

Figure 1: State space plane: $x(t+1) - x(t)$ normalized vector field

where $\epsilon > 0$ is fixed and set

$$\mathcal{X}(0) = \left\{ \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} \in \mathbb{R}^2 : \epsilon < x_1(0) < 1 \right\} \quad (6)$$

Then $y(t) = 1.1^t x_1(0)$, $\forall t \leq \bar{T}$ where $\bar{T} \triangleq \left\lceil \frac{\log \frac{1}{\epsilon}}{\log 1.1} \right\rceil$, and $\lceil\cdot\rceil$ denotes the least upper integer. Moreover, $y(\bar{T}+1) = 0.9x_2(0)$ and therefore two initial states $x_1 = \begin{bmatrix} x_{11} \\ x_{12} \end{bmatrix}$, $x_2 = \begin{bmatrix} x_{11} \\ x_{22} \end{bmatrix}$, with $x_{12} \neq x_{22}$, are indistinguishable for $T \leq \bar{T}$. By Definition 1, system (5) is incrementally observable in $\bar{T}+2$ steps. We can render $\bar{T}$ arbitrarily large by choosing smaller and smaller values of ϵ (intuitively, the smaller the initial condition $x_1(0)$, the longer the time required for the output to overpass 1 and switch dynamics). By setting $\epsilon = 0$ in (5) and (6), it follows that the system (5) becomes incrementally observable on $\mathcal{X}(0)$ only in infinite steps, in the sense that for each $\tilde{T}$ there exist initial states in $\mathcal{X}(0)$ that can be observed only after $T > \tilde{T}$ steps.

4.1.2 An incrementally observable PWL system whose components are unobservable: Consider the system

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}(t+1) = \begin{cases} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}(t), & x_1(t) > x_2(t) \\ \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}(t), & x_1(t) \leq x_2(t) \end{cases} \quad (7a)$$

$$y(t) = \begin{cases} x_1(t) & \text{if} \quad x_1(t) > x_2(t) \\ x_2(t) & \text{if} \quad x_1(t) \leq x_2(t) \end{cases} \quad (7b)$$

whose component subsystems are unobservable. The evolutions of the state-space trajectories are depicted in Fig. 1.

Let $\mathcal{X}(0) \subset$ Sector 1 $\cup$ Sector 2 depicted in Fig. 1 be a bounded set of admissible initial states. If $x(0)$ lies in Sector 1, we have $y(0) = x_1(0)$ and the first component of the initial state is immediately observed. However, since $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}^t \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} x_1(0) \\ tx_1(0)+x_2(0) \end{bmatrix}$ and $\mathcal{X}(0)$ is bounded, there exists a finite time $\bar{T} \geq 1$ such that the state enters Sector 2. Then, $y(\bar{T}) = \bar{T}x_1(0) + x_2(0)$ and the second component $x_2(0)$ can be determined as well from the output knowledge. *Mutatis mutandis*, the same rationale applies when the initial state lies in Sector 2. Then the system is incrementally observable in $\bar{T}$ steps on $\mathcal{X}(0)$. Note however that the system is not incrementally observable on initial sets $\mathcal{X}(0)$ intersecting Sectors 3 or 4. Consider in fact an initial state that lies in Sector 3 (or 4). From Fig. 1, it is clear that the state trajectory never crosses the line $x_1 = x_2$. Therefore the evolutions will be governed by the first (the second) component of (7), thus implying the unobservability of the first (second) coordinate of the initial state.

4.1.3 An unobservable PWL system whose components are observable.: Consider the system

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}(t+1) = \begin{cases} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}(t), & x_1(t) > x_2(t) \\ \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}(t), & x_1(t) \leq x_2(t) \end{cases}$$

$$y(t) = \begin{cases} x_1(t) & \text{if} \quad x_1(t) > x_2(t) \\ x_2(t) & \text{if} \quad x_1(t) \leq x_2(t) \end{cases}$$

whose components are observable. We partition again the state space as in Fig. 1. If the initial state lies in Sector 3, by direct calculation one has $y(0) = x_2(0)$ and $y(t) = 0$, $\forall t > 0$. Indeed, the state evolution for $t > 0$ is $x(t) = \left[\begin{smallmatrix} x_1(0) \\ 0 \end{smallmatrix}\right]$ if t even, $\left[\begin{smallmatrix} 0 \\ x_1(0) \end{smallmatrix}\right]$ if t odd, and $x_1(0) < 0$. Since the same rationale can be applied for initial states lying in Sector 4, it can be concluded that the system is not incrementally observable on $\mathcal{X}(0) =$ Sector 3 $\cup$ Sector 4 (although it is easy to verify that the system is still incrementally observable on $\mathcal{X}(0) =$ Sector 1 $\cup$ Sector 2).

4.2 An Observability Test for Hybrid Systems

The purpose of this section is to derive an observability test for hybrid systems in the MLD form (2). In fact, the observability condition formulated in Definition 1 can be difficult to check, and thus one needs computationally tractable tests. Before stating Theorem 1, where we show that for MLD systems the incremental observability in T steps on $\mathcal{X}(0)$ and $\mathcal{U}$ is reduced to the solution of a Mixed Integer Linear Program (MILP), we need some preliminary results.

Proposition 2 *The MLD system (2) is incrementally observable if and only if there exists a scalar $w > 0$ such that*

$$\min_{\substack{x_1 \in \mathcal{X}(0),\ x_2 \in \mathcal{X}(0) \\ u(t) \in \mathcal{U},\ t=0,\dots,T-1}} J_1 \geq 0 \tag{8}$$

$$J_1 \triangleq \sum_{t=0}^{T-1} \|y_1(t) - y_2(t)\|_\infty - w\|x_1 - x_2\|_1. \tag{9}$$

Proof: The proof easily follows from the fact that all the norms in finite-dimensional Euclidean spaces are equivalent. ■

The minimization problem (8) is in general nonconvex. Anyway, the use of the norms $\|\cdot\|_\infty$ and $\|\cdot\|_1$ allows us to formulate it as an MILP problem. To this purpose we need a technical lemma. In the sequel $[x]_i$ will denote the i-th element of vector x.

Lemma 1 *Let $\mathcal{X}(0)$ be bounded. For two vectors x_1, x_2 in $\mathcal{X}(0)$, it holds $\|x_1 - x_2\|_1 = \sum_{i=1}^n [x_1 - x_2]_i - 2[s]_i$, along with the inequalities $x_1 - x_2 \leq (M-m)'(\mathbf{1}_n - \mu)$, $x_1 - x_2 \geq \sigma \mathbf{1}_n + (m - M - \sigma\mathbf{1}_n)'\mu$, $s \leq (M-m)'\mu$, $s \geq (m-M)'\mu$, $s \leq x_1 - x_2 - (m-M)'(\mathbf{1}_n - \mu)$, and $s \geq x_1 - x_2 - (M-m)'(\mathbf{1}_n - \mu)$, where $\mu \in \{0,1\}^n$, $[M]_i = \max_{x \in \mathcal{X}(0)} x_i$, $[m]_i = \min_{x \in \mathcal{X}(0)} x_i$, $i = 1, \dots, n$ and σ is a small tolerance (e.g. the machine precision).*

Proof: The proof is omitted for brevity, and is reported in [3]. ■

Theorem 1 *Let $\mathcal{X}(0)$ be bounded and consider the following optimization problem:*

$$J^* = \min_{\substack{\{\epsilon_t\}_{t=0}^{T-1},\ \epsilon_t \in \mathbb{R} \\ x_1 \in \mathcal{X}(0),\ x_2 \in \mathcal{X}(0) \\ s \in \mathbb{R}^n,\ \mu \in \{0,1\}^n, \\ \{\delta(t)\}_{t=0}^{T-1},\ \delta(t) \in \{0,1\}^{r_\ell} \\ \{z(t)\}_{t=0}^{T-1},\ z(t) \in \mathbb{R}^{r_c} \\ \{u(t)\}_{t=0}^{T-1} \subseteq \mathcal{U}}} J \tag{10a}$$

$$J \triangleq \sum_{t=0}^{T-1} \epsilon_t - w\left(\sum_{i=1}^n [x_1 - x_2]_i - 2[s]_i\right)$$

subject to (2), the inequalities in Lemma 1, and

$$\mathbf{1}_p \epsilon_t \geq y_1(t) - y_2(t),\ t = 0, \dots, T-1 \tag{10b}$$

$$\mathbf{1}_p \epsilon_t \geq y_2(t) - y_1(t),\ t = 0, \dots, T-1 \tag{10c}$$

Then the MLD system (2) is incrementally observable in T steps on $\mathcal{X}(0)$ and $\mathcal{U}$ if and only if, for some $w > 0$, it holds $J^* \geq 0$.

Proof: We start by proving necessity. Inequalities (10b) and (10c) imply that $\epsilon_t \geq \max_{i=1,\dots,p} |[y_1(t) - y_2(t)]_i| = \|y_1(t) - y_2(t)\|_\infty$. By Lemma 1, $\|x_1 - x_2\|_1 = \sum_{i=1}^n [x_1 - x_2]_i - 2[s]_i$, and therefore $J^* \geq \sum_{t=0}^{T-1} \|y_1(t) - y_2(t)\|_\infty - w\|x_1 - x_2\|_1$. In view of Proposition 2, the condition $J^* \geq 0$ follows from the incremental observability of system (2).

To show sufficiency, let J_1 be defined as in Proposition 2. Then

$$J_1^* = \min_{\substack{x_1 \in \mathcal{X}(0),\ x_2 \in \mathcal{X}(0) \\ \{\delta(t)\}_{t=0}^{T-1},\ \delta(t) \in \{0,1\}^{r_\ell} \\ \{z(t)\}_{t=0}^{T-1},\ z(t) \in \mathbb{R}^{r_c} \\ \{u(t)\}_{t=0}^{T-1} \subseteq \mathcal{U}}} J_1 \tag{11}$$

subject to constraints (2) and let x_1^*, x_2^* denote the initial states that minimize (11). The variables $\{\epsilon_t\}_{t=0}^{T-1}$, μ and s are defined as $\epsilon_t \triangleq \|y(t, x_1^*) - y(t, x_2^*)\|_\infty$, $[s]_i \triangleq |[x_1^* - x_2^*]_i|$, $i = 1, \dots, n$, $[\mu]_i \triangleq 1$ if $[x_1^* - x_2^*]_i \leq 0$, or 0 if $[x_1^* - x_2^*]_i > 0$, $i = 1, \dots, n$, and are feasible for problem (10a). Thus, by optimality, $J_1^* \geq J^* > 0$, which proves incrementally observability. ■

Theorem 1 is also helpful for designing an algorithm that checks the practical observability of an MLD system (see Remark 2). The procedure is summarized in the following steps:

Algorithm 1

1. Choose w_{min} and T_{max} (see Remark 2);
2. Set $T = 1$ and $w = w_{min}$;
3. Solve the MILP (10a);
4. If $J^* > 0$, stop: The system is (practically) observable;
5. If $J^* \leq 0$, increase T;
6. If $T > T_{max}$, stop: The system is practically unobservable;
7. Go to step 3.

Remark 3 When the sets $\mathcal{X}(0)$, $\mathcal{U}$ are polytopes, the optimization problem (10) becomes a MILP in $T(1+r_c+m_c)+3n$ continuous variables and $T(r_\ell+m_\ell)+n$ integer variables. It is well known that, with the exception of particular structures, MILPs involving 0-1 variables are $\mathcal{NP}$-complete, which means that in the worst case the solution time grows exponentially with the number of integer variables [17]. Despite this combinatorial nature, several algorithmic approaches have been proposed and applied successfully to medium and large size application problems [9], and *branch and bound* methods were shown to be the most successful.

In case the observability horizon T becomes large, solving such an optimization can become computationally intractable. This has to be expected, because of the $\mathcal{NP}$-complete nature of the observability problem itself over finite horizon [20]. Consider for instance the autonomous case (no input). By looking more closely at the MILP (10a), the main reason for the complexity is the presence of integer variables $\delta(t)$. Indeed, determining the optimal sequence $\delta(0),\ldots,\delta(t)$ corresponds to finding the sequence of the switching of linear dynamics which leads to the worst case for observability. Nevertheless, by exploiting the equivalent PWA structure of hybrid systems, in [3] we propose an algorithm which, although still exponential in the worst case, checks the observability of PWA systems with considerably less computation burden than Algorithm 1. The idea is to adopt tools developed for *formal verification* of hybrid systems [6], where basically a set-reachability problem is solved through the exploration of all possible evolutions of the hybrid system from the set of initial states $\mathcal{X}(0)$. We remark that checking observability is simpler than verification, as the search does not proceed further as long as the system is found observable.

5 Controllability

We introduce the following definition of controllability for MLD systems.

Definition 2 *Let $\mathcal{X}(0)$ and $\mathcal{X}_f$ be nonempty sets of initial and final states, respectively. The MLD system (2) is controllable in T steps from $\mathcal{X}(0)$ to $\mathcal{X}_f$ if, $\forall x_0 \in \mathcal{X}(0)$, there exists an admissible input sequence $\{u(t)\}_{t=0}^{T-1}$ yielding*

$$x(T) \in \mathcal{X}_f \qquad (12)$$

If $\mathcal{X}(0)$ and $\mathcal{X}_f$ are singletons (i.e. $\mathcal{X}(0)=\{x_0\}$ and $\mathcal{X}_f=\{\bar{x}\}$), Definition 2 reduces to a classical controllability notion [21]. Anyway, letting $\mathcal{X}(0)$ be a general set, we take into account also the case of incompletely specified initial conditions. Moreover, in many situations, the control specifications demand to drive a system into a set of *safe* states $\mathcal{X}_f$ [6]. It is apparent that Definition 2 embraces also this scenario.

5.1 Controllability counterexamples for PWA systems

Analogously to the observability notion, we specialize the controllability definition to PWL systems. Again, through some counterexamples, we will show that this property cannot be inferred from the controllability of the component subsystems.

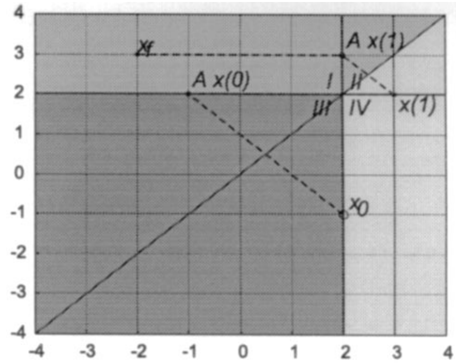

Figure 2: State-space for system (13), whose components are completely controllable: Region III is not reachable from x_0

5.1.1 An uncontrollable PWL system whose components are controllable.:

Consider the system

$$\begin{bmatrix}x_1\\x_2\end{bmatrix}(t+1)=\begin{bmatrix}0&1\\1&0\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}(t)+ \left\{\begin{array}{ll}\begin{bmatrix}0\\1\end{bmatrix}u(t), & x_1(t)>x_2(t)\\ \begin{bmatrix}1\\0\end{bmatrix}u(t), & x_1(t)\le x_2(t)\end{array}\right. \qquad (13)$$

whose components $\left(\begin{bmatrix}0&1\\1&0\end{bmatrix},\begin{bmatrix}0\\1\end{bmatrix}\right)$ $\left(\begin{bmatrix}0&1\\1&0\end{bmatrix},\begin{bmatrix}1\\0\end{bmatrix}\right)$ are completely controllable. Let $x_0=\begin{bmatrix}x_{10}\\x_{20}\end{bmatrix}$ be the initial state, and consider the partition of the state space depicted in Fig. 2. The Sectors I, II, III, IV are obtained by intersecting the lines $x_1=\max\{x_{10},x_{20}\}$, $x_2=\max\{x_{10},x_{20}\}$. It is easy to verify that only the Sectors I, II, IV are completely reachable from x_0, while III is not reachable. For instance, the point $x_f=(-2,3)$ can be reached from $x_0=(2,-1)$ by applying the input $u(0)=4$, $u(1)=-4$, but no input can steer x_0 to the origin. In general, the PWL system (13) is controllable to 0 from x_0 if and only if $0\in I\bigcup II\bigcup IV$, where we point out that Sectors I–IV depend on x_0. Therefore, x_0 is controllable to the origin if and only if $x_{01}<0, x_{02}<0$.

5.1.2 A controllable PWL system whose components are uncontrollable.:

Consider the system

$$\begin{bmatrix}x_1\\x_2\end{bmatrix}(t+1)=\begin{bmatrix}1&0\\0&1\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}(t)+ \left\{\begin{array}{lll}\begin{bmatrix}1\\0\end{bmatrix}u(t) & \text{if} & x_2(t)-1>x_1(t)>x_2(t)\\ \begin{bmatrix}0\\1\end{bmatrix}u(t) & \text{if} & x_1(t)\le x_2(t)\end{array}\right.$$

whose components $\left(\begin{bmatrix}1&0\\0&1\end{bmatrix},\begin{bmatrix}1\\0\end{bmatrix}\right)$ $\left(\begin{bmatrix}1&0\\0&1\end{bmatrix},\begin{bmatrix}0\\1\end{bmatrix}\right)$ are uncontrollable. The three regions in which the state space is partitioned are depicted in Fig. 3. It is easy to verify by inspection that every initial state in $\mathcal{X}(0)=\mathbb{R}^n$ can be controlled to any other state in at most three steps. This is, for instance, the situation depicted

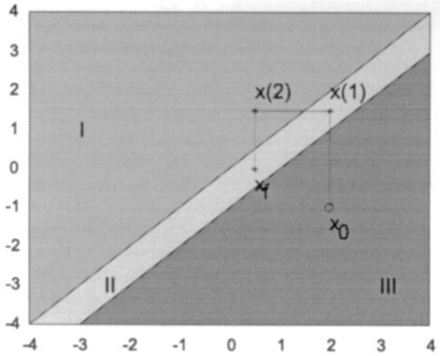

Figure 3: State-space for system (14), whose components are uncontrollable

in Fig. 3 where the point $x_f = (0.5, 0)$ is steered from $x_0 = (2, -1)$ by applying the input $u(0) = 2.5$, $u(1) = u(2) = -1.5$.

5.2 Controllability Tests for Hybrid Systems

In this section we discuss numerical tests for checking the controllability of an MLD system. We first notice that Definition 2 can be translated into the following *Mixed-Integer Feasibility Test* (MIFT)

$$\begin{cases} x(0) \in \mathcal{X}(0), \\ x(T) \in \mathcal{X}_f \\ x(t+1) = Ax(t) + B_1u(t) + B_2\delta(t) + B_3z(t) \\ E_2\delta(t) + E_3z(t) \leq E_1u(t) + E_4x(t) + E_5 \\ t = 0, 1, \dots, T \end{cases} \quad (14)$$

The feasibility test (14) is called a *verification* problem in the hybrid system literature. Unfortunately, solving the MIFT for large T becomes prohibitive. In fact, each problem (14) is $\mathcal{NP}$-complete which means that in the worst case the required computation time grows exponentially with T. Despite this strong theoretical limitation,a verification algorithm for the general class of MLD systems under the assumption that both $\mathcal{X}(0)$ and $\mathcal{X}_f$ are polyhedra was proposed [6]. This procedure is based on a sequence of linear and mixed-integer linear programs and can be adopted as a numerical controllability test. Various other verification techniques have been proposed in the literature [1, 2].

6 Conclusions

In this paper we illustrated, through a number of counterexamples, the complexity of observability and controllability properties of PWA and hybrid systems. After proving the equivalence between PWA and hybrid MLD systems, we exploited this equivalence to derive observability and controllability tests which are numerically appealing.

Acknowledgments

This research has been supported by the Swiss National Science Foundation.

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