Reducing Conservativeness in Predictive Control of Constrained Systems with Disturbances

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Abstract

Predictive controllers which are able to guarantee constraint fulfillment in the presence of input disturbances, typically based on min-max formulations, often suffer excessive conservativeness. One of the main reasons for this is that the control action is based on the *open-loop* prediction of the evolution of the system, because the uncertainty due to the disturbance grows as time proceeds on the prediction horizon. On the other hand, such an effect can be moderated by adopting a *closed-loop* prediction. In this paper, closed-loop prediction is achieved by including a free feedback matrix gain in the set of optimization variables. This allows to well balance computational burden and reduction of conservativeness.

1 Introduction

In almost all industrial applications the design of feedback control systems is complicated by the presence of constraints, such as physical limits of actuators, safety margins, limited manufacture tolerances. In order to cope with this kind of control problems, in recent years *model predictive control* (MPC) has been investigated and successfully employed [1, 2, 3, 4, 5, 6]. The main idea of MPC is to use a *model* of the plant to *predict* the future evolution of the system and, accordingly, select the command input. Prediction are handled according to the so called *receding horizon* (RH) philosophy: a sequence a future control actions is chosen, by predicting the future evolution of the system, and applied to the plant until new measurements are available. Then, a new sequence is evaluated so as to replace the previous one. Each selected sequence is the result of an optimization procedure which takes into account two objectives: (i) maximize the tracking performance, and (ii) guarantee that the constraints are and will be fulfilled—i.e., no "blind-alley" is entered. Schemes developed for deterministic frameworks often lead to either intolerable constraint violation or over-conservative control action. In order to guarantee constraint fulfillment for every possible disturbance realization within a certain set, it is clear that the control action has to be chosen safe enough to cope with the effect of the worst disturbance realization [7, 8]. This effect is typically evaluated by predicting the *open-loop* evolution of the system driven by such a worst-case disturbance. As investigated in [9], this inevitably leads to over-conservative schemes, and the authors suggest to conveniently exploit the control moves to moderate the effect of the disturbance. This is achieved by performing *closed-loop* predictions. Because of the pursued rigorous min-max approach, the control scheme developed in [9] is computationally demanding. In this paper, the closed-loop predictive action is limited to include a constant feedback matrix gain in the set of optimization variables. The scheme therefore renounces some degrees of freedom which in principle are available within a general min-max formulation. On the other hand, it allows to well balance increased computational burden and reduction of conservativeness, as testified by the reported simulation results.

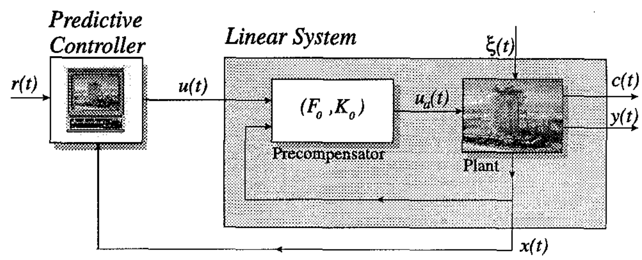

Figure 1: Problem formulation.

2 Problem Formulation

Consider the situation depicted in Fig. 1. The model of the plant under consideration is described by the linear difference equations

$$\begin{cases} x(t+1) &= A_0x(t) + B_0u_a(t) + H\xi(t) \\ y(t) &= C_0x(t) + D_0u_a(t) \\ c(t) &= E_0x(t) + G_0u_a(t) + L\xi(t) \end{cases} \quad (1)$$

where $x(t) \in \mathbb{R}^n$ is the state, $u_a(t) \in \mathbb{R}^{m_a}$ is the command input to the actuators, $y(t) \in \mathbb{R}^p$ the output which should track a desired reference $r(t) \in \mathbb{R}^p$, $c(t) \in \mathbb{R}^l$ is a vector to be constrained within the con-

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vex polyhedral set

$$\mathcal{C} = \left\{c \in \mathbb{R}^l : A_c c \leq B_c\right\}, \ B_c \in \mathbb{R}^q, \quad (2)$$

$\xi(t) \in \mathbb{R}^s$ the unknown disturbance input acting on the system, and $t \in \mathbb{Z}_+ = \{0, 1, \ldots\}$. We assume that $\xi(t)$

$$\xi(t) \in \Xi, \ \forall t \in \mathbb{Z}_+ \quad (3)$$

where Ξ is the hyperrectangle

$$\Xi \triangleq \{\xi \in \mathbb{R}^s : \ \xi_i^- \leq \xi_i \leq \xi_i^+, \ \xi_i^- \leq 0 \leq \xi_i^+, \ i = 1, \ldots, s\}$$

and ξ_i^-, ξ_i^+ are given bounds. For the linear plant (1), we assume that the linear precompensator

$$u_a(t) = F_0 x(t) + K_0 u(t) \quad (4)$$

has been already designed *without taking care of the constraints*, for instance through standard control techniques such as LQG or PID control, to provide stability and noise attenuation properties. By considering $u(t) \in \mathbb{R}^m$ as a new input, the overall closed-loop system can be rewritten as

$$\left\{ \begin{array}{rcl} x(t+1) & = & Ax(t) + Bu(t) + H\xi(t) \\ y(t) & = & Cx(t) + Du(t) \\ c(t) & = & Ex(t) + Gu(t) + L\xi(t) \end{array} \right. \quad (5)$$

where $A \triangleq A_0 + B_0 F_0$, $B \triangleq B_0 K_0$, $C \triangleq C_0 + D_0 F_0$, $D \triangleq D_0 K_0$, $E \triangleq E_0 + G_0 F_0$, $G \triangleq G_0 K_0$. Without loss of generality, we can therefore assume that the linear system (5) satisfies the following

Assumption 1 *A is asymptotically stable.*

Note that the generic expression for $c(t)$ allows to take into account possible actuator saturations, i.e. constraints on the original input $u_a(t)$, which, by (4), is now a state-dependent vector.

The goal of this paper is to design a feedback control law $u(t) = f(x(t), r(t))$ such that the output $y(t)$ tracks the desired reference $r(t) \in \mathbb{R}^p$ while the vector $c(t)$ fulfils the constraints

$$c(t) \in \mathcal{C}, \quad (6)$$

for all possible disturbance realizations $\xi(t) \in \Xi$. Hereafter, we shall assume that

Assumption 2 *$\mathcal{C}$ is a polytope*

Note that assuming that $\mathcal{C}$ is bounded is not restrictive in practice, since usually inputs and states are often bounded for physical reasons. The following developments will be meaningful if, in addition, $\mathcal{C}$ has a nonempty interior. Moreover, we shall assume that

Assumption 3 *The matrix dc-gain $H_{yu} \triangleq C(I - A)^{-1}B$ has full rank,* $\operatorname{rank} H_{yu} = \min\{m, p\}$.

We adopt the following receding horizon control law. Let

$$\mathcal{V} \triangleq \begin{bmatrix} w \\ v(N_u - 2) \\ \vdots \\ v(0) \end{bmatrix} \in \mathbb{R}^{N_u m} \quad (7)$$

be a vector of N_u free parameters of $\mathbb{R}^m$, and let $F \in \mathcal{F}$ a free constant state-feedback gain, where $\mathcal{F}$ is the family of state-feedback gains

$$\mathcal{F} = \left\{ F \in \mathbb{R}^{n \times m} : \begin{array}{l} (i) \ (A + BF) \text{ asymptotically stable,} \\ (ii) \ \operatorname{rank} K(F) = \min\{n, p\} \end{array} \right\} \quad (8)$$

and $K(F)$ is the closed-loop DC gain

$$K(F) \triangleq C(I - A - BF)^{-1}B + D.$$

Notice that by Assumption 3, $0 \in \mathcal{F}$. Hereafter, $x(k, t)$ will denote the state vector at time $t + k$, predicted at time t, according to model (5), initial state $x(t)$, disturbance input sequence $\{\xi(j)\}_{j=0}^{k-1}$, and by setting

$$u(k, t) = \left\{ \begin{array}{ll} v(k) + Fx(k, t) & \text{if} \quad 0 \leq k \leq N_u - 2 \\ w + Fx(k, t) & \text{if} \quad k \geq N_u - 1 \end{array} \right. \quad (9)$$

(similarly, the notation $c(k, t)$ will be used for the predicted evolution of the constrained vector).

The use of *closed-loop* prediction (9) instead of the more classical *open-loop* form produces benefits when min-max approaches to cope with disturbances are adopted [9]. The main reason is that closed-loop prediction attempts to reducing the effect of disturbances, while in open-loop prediction this effect is passively suffered. As a result, open-loop schemes are conservative, because the uncertainty produced by the disturbances grows more and more as time proceeds on the prediction horizon. In order to better underline this key aspect, consider the evolution of the constrained vector due to (9)

$$c(k, t) = E(A + BF)^k x(t) + \sum_{k=0}^{t-1} E(A + BF)^k v(k) + Gv(t) + \sum_{k=0}^{t-1} (A + BF)^k H\xi(t - 1 - k) + L\xi(t) \quad (10)$$

It is clear that F offers some degrees of freedom to contrast the effect of $\xi(t)$, by modifying the multiplicative term $(A + BF)^k$. For instance, if F renders $(A + BF)$ nilpotent, $c(k, t)$ is only affected by the last n disturbance inputs $\xi(k - n + 1), \ldots, \xi(k)$, with consequent no uncertainty accumulation. On the other hand, if F is set to 0 (open-loop prediction) and A has eigenvalues close to the unitary circle, the disturbance action leads to very conservative constraints, and consequently to poor performance. Fig. 2 shows this effect for different gains F, selected by solving LQ problems with input weights $\rho = 0$, $\rho = 1$, and $\rho = +\infty$. The last one corresponds to open-loop prediction ($F = 0$).

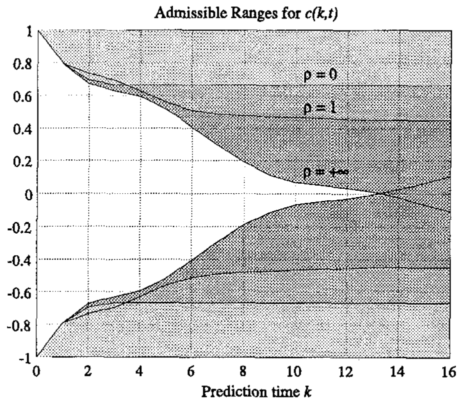

Figure 2: Admissible ranges for the constrained vector $c(k,t)$ for different feedback LQ gains F (input weight $\rho = 0, 1, +\infty$).

In order to select the future input sequence (9), consider the performance function

$$J(\mathcal{V},F,r) \triangleq \sum_{k=0}^{N_u-1} \|v(k)-w\|^2_{\psi_u} + \|K(F)w-r\|^2_{\psi_r} + \gamma\|F\|^2 \quad (11)$$

where $\psi_r > 0, \psi_u > 0$ are symmetric weight matrices, $\|u\|^2_\psi \triangleq u'\psi u$, and $\gamma > 0$ is a fixed scalar. Since w represents the final constant input on the prediction horizon, ψ_r penalizes the predicted steady-state tracking error, while ψ_u penalizes deviations from such a steady-state input of the control moves. The weight γ reflects the desire of maintaining unchanged the original closed-loop dynamics, for instance to preserve bandwidth and sensitivity characteristics provided by the precompensator. The performance function (11) is minimized in the presence of the constraints

$$c(k,t) \in \mathcal{C}, \ \forall k \in \mathbb{Z} \quad (12)$$

Notice that no feedback term is present in (11). Since, by considering (10), the constraints (12) involved in the minimization of (11) depend on the current state $x(t)$, feedback will be present only when the constraints are active. This should not be considered as a drawback, since noise and unmodeled dynamics effects rejection are taken into account by the precompensator.

At each time t, the selection of the input $u(t)$ proceeds as follows.

$$(\mathcal{V}(t),F(t)) \triangleq \left\{ \begin{array}{l} \arg\min_{(\mathcal{V},F)} J(\mathcal{V},F,r(t)) \\ \text{subject to} \left\{ \begin{array}{l} F \in \mathcal{F} \\ c(k,t) \in \mathcal{C}, \ \forall k \in \mathbb{Z}, \\ \quad \forall \{\xi(k)\}_{k=0}^\infty \subseteq \Xi \end{array} \right. \end{array} \right. \quad (13)$$

Then, according to the receding horizon strategy described above, we set

$$u(t) = u(0,t). \quad (14)$$

3 Constraint Horizons

In this section we show that in the constrained optimization problem (13) only a finite number of linear constraints suffices to ensure constraint fulfillment over a seminfinite prediction horizon. To this end, we extend here results developed in [10] for the disturbance-free case and two-degree-of-freedom input parameterizations, by referring to the *output admissible set* theory developed in [11, 7] for the regulation problem.

Consider the problem of fulfilling constraints (6) when system (5) evolves from initial state $x(0)$ and is fed by the input sequence

$$u(t) = \left\{ \begin{array}{ll} v(t) & \text{if } \ 0 \le t \le N_u - 2 \\ w & \text{if } \ t \ge N_u - 1 \end{array} \right. \quad (15)$$

The corresponding evolution of the constrained vector is

$$c(t) = EA^t x(0) + \sum_{k=0}^{t-1} EA^k Bu(t-1-k) + Gu(t) + \sum_{k=0}^{t-1} EA^k H\xi(t-1-k) + L\xi(t) \quad (16)$$

Since (6) must be fulfilled for all possible $\xi(t) \in \Xi$, we can replace B_c in (2) by $B_c - \max_{\xi\in\Xi} L\xi$ and, without loss of generality, set $L = 0$.

In order to obtain finite constraint representation, we impose that the steady-state input vector w satisfies

$$w \in \mathcal{W}^\delta_\infty \triangleq \left\{ w \in \mathbb{R}^m : \ A_c \left[H_{cu} w + \sum_{k=0}^\infty EA^k H\xi(k) \right] \le B_c - \delta, \ \forall \{\xi(k)\}_{k=0}^\infty \subseteq \Xi \right\} \quad (17)$$

where $H_{cu} \triangleq E(I-A)^{-1}B + G$, and $\delta > 0$ is a fixed arbitrarily small scalar.

Remark 1 The set $\mathcal{W}^\delta_\infty$ might be empty for large δ and/or large disturbance set Ξ. When $\mathcal{W}^\delta_\infty = \emptyset$ for all positive δ, no feasible steady state exists. In this case, for every constant input level w there exists a sequence of disturbance inputs which produces constraint violation. The results developed below are still valid for the degenerate case $\mathcal{W}^\delta_\infty = \emptyset$.

By letting

$$z(t) \triangleq \begin{bmatrix} x(t) \\ \tilde{\mathcal{V}}(t) \\ w \end{bmatrix}, \ \tilde{\mathcal{V}}(t) = \begin{bmatrix} u(t+N_v-2) - w \\ \vdots \\ u(t) - w \end{bmatrix}$$

the evolution $c(t)$ in (16) of system (5) with input (15) can be rewritten as a free evolution of the following system

$$\left\{ \begin{array}{lcl} z(t+1) & = & \bar{A}z(t) + \bar{H}\xi(t) \\ c(t) & = & \bar{E}z(t) \end{array} \right. \quad (18)$$

where

$$\bar{A} \triangleq \left[\begin{array}{c|c|c} A & 0 \;\ldots\; 0\; B & B \\ \hline 0 & 0 \quad I_{N_u-2} & 0 \\ 0 & 0 \quad 0 & 0 \\ \hline 0 & 0 \quad 0 & I_m \end{array}\right], \quad \bar{H} \triangleq \left[\begin{array}{c} H \\ \hline 0 \\ \hline 0 \end{array}\right]$$

$$\bar{E} \triangleq \left[\begin{array}{c|c|c} E & 0 \ldots 0\, G & G \end{array}\right]$$

Consider the following sets

$$\mathcal{Z}_t \triangleq \left\{ z \in \mathbb{R}^{n+N_u m} : \left\{ \begin{array}{l} w \in \mathcal{W}_\infty^\delta,\ A_c z(k) \le B_c, \\ \forall k = 0,\ldots,t,\ \forall \{\xi(k)\}_{k=0}^{t-1} \subset \Xi \end{array} \right. \right\} \tag{19}$$

and

$$\mathcal{Z}_\infty \triangleq \bigcap_{t=0}^{\infty} \mathcal{Z}_t, \tag{20}$$

where $\mathcal{Z}_t$ is the set of initial states z which gives rise to evolutions which fulfil the constraints up to time t for all possible disturbance realizations of ξ, while $\mathcal{Z}_\infty$ characterizes the pairs (initial state, input sequence) fulfilling the same constraints over a semifinite horizon. By proceeding as in [7, Theorem 4.1], it is easy to prove by recursion the following lemma:

Lemma 1 *Let $\mathcal{Z}_t = \mathcal{Z}_{t+1}$. Then $\mathcal{Z}_\infty = \mathcal{Z}_t$.*

Next theorem shows that the situation depicted in Lemma 1 holds, i.e. $\mathcal{Z}_\infty$ is *finitely determined.* The proof is reported in [12], and is an extension of a result in [10].

Theorem 1 *There exists an index t^* such that $\mathcal{Z}_\infty = \mathcal{Z}_{t^*}$.*

Theorem 1 requires that the extra constraint (17) is satisfied, in order to reduce the number of constraints involved in the optimization problem to a finite number, as required by standard optimization tools. However, this effort would be nullified if the set $\mathcal{W}_\infty^\delta$ were described by an infinite number of linear inequalities. In order to sidestep this difficulty, with desired accuracy we define an inner approximation of $\mathcal{W}_\infty$ which is finitely generated. To this end, fix an arbitrarily small $\epsilon > 0$ (e.g. $\epsilon = \delta$), and define $\bar{t} \ge n + N_u m$ such that, $\forall t \ge t_2^*$,

$$\max_{\{\xi(k)\}_{k=t}^\infty} \Big| \sum_{k=t}^{\infty} [A_c]_i E A^k H \xi(k) \Big| \le \epsilon.$$

Then,

$$\begin{aligned} \mathcal{W}_{\bar{t}}^{\delta+\epsilon} &\triangleq \Big\{ w \in \mathbb{R}^m : A_c H_{cu} w \le B_c - \max_{\{\xi(k)\}_{k=0}^{\bar{t}-1}} \Big[\sum_{k=0}^{\bar{t}-1} A_c \cdot \\ &\quad E A^k H \xi(k) \Big] - \delta - \epsilon \Big\} \\ &\subseteq \Big\{ w \in \mathbb{R}^m : A_c H_{cu} w \le B_c - \max_{\{\xi(k)\}_{k=0}^{\bar{t}-1}} \Big[\sum_{k=0}^{\bar{t}-1} A_c \cdot \\ &\quad E A^k H \xi(k) \Big] - \delta \max_{\{\xi(k)\}_{k=\bar{t}}^{\infty}} \Big[\sum_{k=\bar{t}}^{\infty} A_c E A^k H \xi(k) \Big] \Big\} \\ &\subseteq \Big\{ w \in \mathbb{R}^m : A_c H_{cu} w \le B_c - \max_{\{\xi(k)\}_{k=0}^{\infty}} \Big[\sum_{k=0}^{\infty} A_c \cdot \\ &\quad E A^k H \xi(k) \Big] - \delta \Big\} = \mathcal{W}_\infty^\delta \end{aligned}$$

is the desired inner approximation. In the sequel, constraint (17) will be replaced by the following

$$w \in \mathcal{W}_{\bar{t}}^{\delta+\epsilon} \tag{21}$$

In order to efficiently perform on-line optimization, we are interested in finding the smallest integer t_o such that $\mathcal{Z}_\infty = \mathcal{Z}_{t_o}$. Such an integer can be determined off-line by an algorithm, adapted by [11], which is reported in [12].

4 On-Line Optimization Scheme

The closed-loop predictive control law developed in the previous sections amounts to solving the nonlinear optimization problem (13)–(21). In this section, we attempt to simplifying this for on-line implementation. By exploiting results from Section 3, the constraints $c(k,t) \in \mathcal{C}$, $\forall k \in \mathbb{Z}$, are reduced to a finite number, according to the following corollary of Theorem 1

Corollary 1 *For all $F \in \mathcal{F}$ there exist a finite index $t_o(F)$ such that fulfillment of constraints $c(k,t) \in \mathcal{C}$, $\forall k = 0,\ldots,t_o(F)$, implies constraint fulfillment $\forall k \in \mathbb{Z}$.*

Moreover, we add some extra structure to the optimization problem, by rewriting it in the cascaded form

$$F(t) \triangleq \left\{ \begin{array}{l} \arg\min_F J(\mathcal{V}(F), F, r(t)) \\ \text{subject to } \left\{ \begin{array}{l} \mathcal{W}_\infty^\delta(F) \neq \emptyset, \\ F \in \mathcal{F}, \end{array} \right. \\ \text{where } \mathcal{V}(F) \triangleq \left\{ \begin{array}{l} \arg\min_{\mathcal{V}} J(\mathcal{V}, F, r(t)) \\ \text{subject to } \left\{ \begin{array}{l} c(k,t) \in \mathcal{C}, \\ \forall k \in \mathbb{Z}, \\ \forall \{\xi(k)\}_{k=0}^\infty \subseteq \Xi \\ w \in \mathcal{W}_\infty^\delta(F) \end{array} \right. \end{array} \right. \end{array} \right. \tag{22}$$

where the inner minimization with respect to $\mathcal{V}$ is a *quadratic program* (QP). Notice that, by recalling (10), the structure (22) is computationally more convenient than a structure in which the inner minimization is performed with respect to F. Moreover, in (22) the outer minimization with respect to F is reduced to a scalar optimization, by parameterizing the F by means of a scalar parameter $\rho \ge 0$. An effective way is to define $F(\rho)$ as the solution of the LQR problem for the pair (A, B) and weight matrices R, ρS, where $R \in \mathbb{R}^{n\times n}$, $S \in \mathbb{R}^{m\times m}$ are fixed matrices, (e.g. $R = I_n$, $S = I_m$). Since, by Assumption 1, A is asymptotically stable, the pair (A, B) is stabilizable, and hence a unique solution $P > 0$ exists of the following algebraic Riccati equation

$$P - A'PA + A'PB(\rho S + B'PB)^{-1}BP'A - R = 0.$$

Therefore,

$$F(\rho) \triangleq -(\rho S + B'PB)^{-1}B'PA \tag{23}$$

is well defined. The parameterization (23) enjoys the following properties

$$\begin{array}{llll} (i) & \rho \to 0 & \Rightarrow & \text{sp}(A+BF(\rho)) \to \{0\} \\ (ii) & \rho \to +\infty & \Rightarrow & F \to 0 \end{array}$$

where $\text{sp}(A)$ denotes the set of the eigenvalues of matrix A. Property (i) offers the way to reduce uncertainty accumulation by moving the eigenvalues of $(A+BF)$ towards the origin, while (ii) consents to obtain small feedback gains F, which leave the original system dynamics unchanged. Problem (22) can be then solved by letting $\rho \triangleq 10^a$, and by performing a bisection over a in a given interval. This amounts to solving N quadratic programming (QP) problems, where N is the number of bisection iterates. However, since the feedback gains $F(\rho)$ are determined on-line, for each selected gain F the approximated set $\mathcal{W}_t^{\delta+\epsilon}(F)$ and Algorithm 1 should be evaluated on-line. Moreover, since the constraint horizon $t_o(F(\rho))$ could not be arbitrarily bounded a priori, the overall on-line computational complexity could not be bounded a priori as well, which renders difficult to implement the proposed scheme in real-time applications. In alternative, we suggest a look-up table for F of the form

$$\begin{aligned} F &\in \mathcal{F}_N \triangleq \{0, F(\rho_1), \ldots, F(\rho_{N-1})\}, \\ \rho_k &\triangleq 10^{[a^- + \frac{k}{N-1}(a^+ - a^-)]}, \; k = 1, \ldots, N-1 \end{aligned} \tag{24}$$

where a^-, a^+ are given bounds. The resulting on-line computations amount to solving N QP problems of order $N_u m$ with at most $q \max_k\{t_o(F(\rho_k))\}$ linear constraints.

Note that different look-up tables could be set up, for instance by sampling Youla-Kucera parameterizations.

Remark 2 The proposed control scheme amounts to a *multicontroller* configuration, where a finite set of controllers is *switched* according to the optimization criterion (22). From this point of view, the resulting switching control law has connections with the scheme proposed in [13].

5 Main Results

Assumption 4 *For the initial state $x(0)$ problem (22) is feasible, in that there exists at least one pair $(\mathcal{V}, F)$ satisfying the constraints in (22).*

The following results are obtained for set-point tracking problems, i.e. for references $r(t)$ which are constant, or become constant, after a finite time. Lemma 2 proves that the value of the performance index (11) attained at the minimizer $(\mathcal{V}(t), F(t))$ is a Lyapunov function, Lemma 3 shows convergence properties of the optimal input sequences, and Theorem 2 characterizes the asymptotic convergence of the command input $u(t)$, and consequently the steady-state behaviour of system (5).

Lemma 2 *Let $r(t) \equiv r$, $\forall t \geq t_r \in \mathbb{Z}_+$, and let $\mathcal{L}(t) \triangleq J(\mathcal{V}(t), F(t), r(t))$ the value attained by the performance index at the minimizer $(\mathcal{V}(t), F(t))$. Then, $\mathcal{L}(t)$ is a monotonically non-increasing nonnegative sequence, $\forall t \geq t_r$.*

Proof: Let $t \geq t_r$ and consider the one-step shifted version $\mathcal{V}_1(t)$ of $\mathcal{V}(t)$, defined as

$$\mathcal{V}_1(t) \triangleq [w'(t-1)\; w'(t-1)\; v_{N_v-2}(t-1)\; \ldots\; v_1(t-1)], \tag{25}$$

and let $F_1(t) \triangleq F(t-1)$. At time t, $\mathcal{V}_1(t)$ is feasible for the inner optimization problem in (22), and hence $J(\mathcal{V}(F_1(t)), F_1(t), r) \leq J(\mathcal{V}_1(t), F_1(t), r)$. Moreover, feasibility of $\mathcal{V}_1(t)$ implies feasibility of $F_1(t)$ for the outer optimization. Then, $J(\mathcal{V}(t), F(t), r) \leq J(\mathcal{V}_1(t), F_1(t), r) \leq J(V(t-1), F(t-1), r)$, or, equivalently, $\mathcal{L}(t) \leq \mathcal{L}(t-1)$. □

Note that the parameterization of F proposed in (23) and the quantization of F in (24) do not affect the proof of Lemma 2. In order to prove further results for set-point tracking problems, we assume that the gains $F(t)$ are indeed selected according to (24) and that the further constraint

$$\rho(t) \geq \rho(t-1), \quad \rho(-1) = 0 \tag{26}$$

is added in the optimization procedure (22), where $\rho(t) \in \{\rho_k\}_{k=1}^{N-1}$ is the parameter which characterizes the selected optimal gain $F(t)$. Under these assumptions, clearly there exists a finite time t_∞ such that

$$F(t) \triangleq F_\infty, \; \forall t \geq t_\infty$$

Lemma 3 *Under the hypothesis of Lemma 2 and (24), (26), $\mathcal{V}(t) - \mathcal{V}(t+1) \to 0$ and $v_i(t) - w(t) \to 0$, $\forall i = 1, \ldots, N_u$.*

Proof: See [12]. □

Theorem 2 *Under the same hypothesis of Lemma 3,*

$$\lim_{t\to\infty} u(t) = w_r \triangleq \arg \min_{w \in \mathcal{W}_\infty^\delta(F_\infty)} \|K(F_\infty)w - r\|_{\psi_r}^2$$

Proof: See [12]. □

Remark 3 The results presented here show that the input $u(t)$ generated by the proposed predictive controller converges to the vector w_r which provides the smallest set-point tracking error in steady-state. However, w_r depends on the asympotic gain F_∞, which is related to the set-point r, the constraint set $\mathcal{C}$, and the disturbance set Ξ.

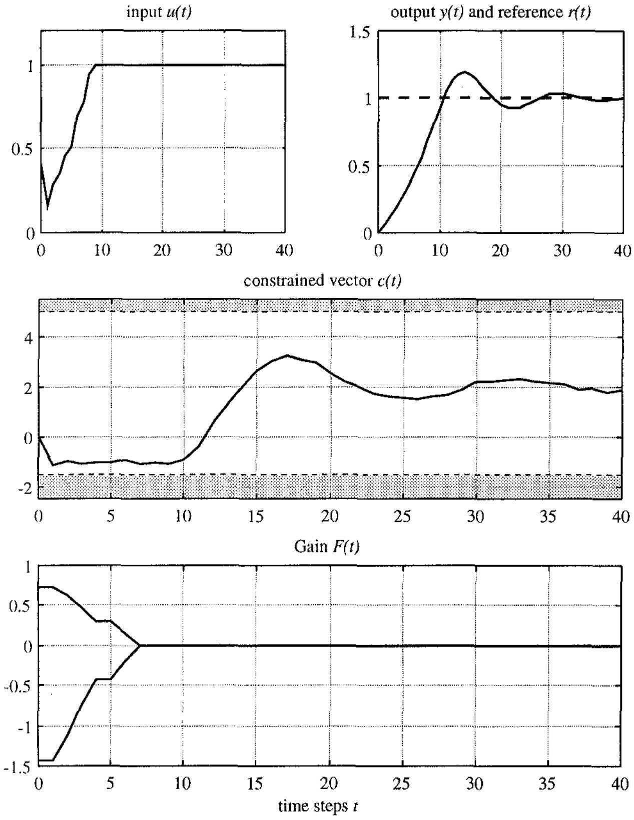

Figure 3: $\|\xi(t)\|_\infty \le 0.08$.

6 Simulation Tests

Consider the following second order linear system

$$\begin{aligned} x(t+1) &= \begin{bmatrix} 1.6463 & -0.7866 \\ 1 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t) + \xi(t) \\ y(t) &= \begin{bmatrix} 0.1404 & 0 \end{bmatrix} x(t) + \zeta(t) \\ c(t) &= \begin{bmatrix} -2.6238 & 2.9045 \end{bmatrix} x(t), \end{aligned} \tag{27}$$

The response of the constrained variable $c(t)$ is under-dumped and has an initial under-shoot. The predictive control law developed in the previous sections is applied in order to constrain $c(t)$ within the interval

$$\mathcal{C} = [-1.5, 4],$$

and make the output $y(t)$ track the constant reference $r(t) \equiv 1$. The parameters of the controller are $\psi_u = 1$, $\psi_r = 0.1$, $N_v = 4$, $\gamma = 0.001$, $\delta = 10^{-6}$, $\epsilon = 10^{-8}$, $t_o = 16$, with a matrix-gain look-up table (24) built by choosing $a^- = -4$, $a^+ = 4$, $N = 12$, $R = I_2$, $S = 1$.

Fig. 3 shows the resulting trajectories when system (27) is affected by independent randomly generated disturbances $\|\xi(t)\|_\infty \le 0.08$. In Fig. 4 are depicted the admissible ranges for the constrained vector $c(k,t)$ for the LQ gains $F(\rho_k)$ in the look-up table. Note that without the closed-loop mechanism (i.e. $F = 0$), no feasible sequence of control moves would have been found at $t = 0$ from the initial state $x(0) = [0\ 0]'$. For instance, from Fig. 4 it is clear that the zero input sequence is unfeasible, since it violates the lower constraint for $k \ge 7$.

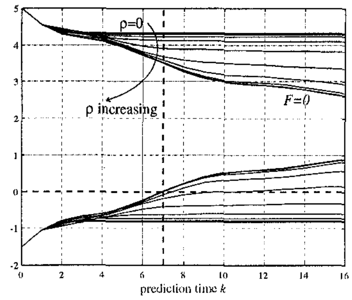

Figure 4: Admissible ranges of the constrained vector $c(k,t)$ for different feedback LQ gains $F(\rho_k)$, $\rho = 10^{-4}, \ldots, 10^4, +\infty$.

The simulation results reported hereafter were obtained on a Pentium 200 running Matlab 5.1 + Simulink 2.0, with no particular care of code optimization. The standard Matlab `QP.M` routine was used for quadratic optimization. The average time required to evaluate a control move $u(t)$ was 0.37 s.

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