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Nonlinear Predictive Reference Governor For Constrained Control Systems

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Abstract

This paper presents a new methodology for solving control problems where hard contraints on the state and/or the inputs of the system are present. This is achieved by adding to the control architecture a command governor which prefilters the reference to be tracked, taking into account the current value of the state and aiming at optimizing a tracking performance index. The overall system is proved to be asymptotically stable, and feasibility is ensured by a weak condition on the initial state. linear loops, a complete solution is developed for the latter. The resulting on-line computational burden turns out to be moderate and the related operations executable with current low-priced computing hardware.

1. Introduction

In recent years there have been substantial theoretical advancements in the area of feedback control of dynamic systems with input and/or state-related constraints. For an account of related results see [1] which also includes relevant references. Amongst the various approaches, the developments of this paper are more akin to the receding-horizon or predictive control methodology [2]-[6]. Predictive control, wherein the receding horizon philosophy is used, selects the control action by also taking into account the future evolution of the reference. Such an evolution can be: known in advance as in applications where repetitive tasks are executed, e.g. industrial robots; predicted if a dynamic model for the reference is given; or designed in real time. Indeed, the last instance is a peculiar and important potential feature of predictive control. In fact, taking into account the current value of both the state vector and the reference, a *virtual* reference evolution can be designed on line so as to ensure that the corresponding input and state responses be admissible. We point out the relevance of such an approach, being the feasibility issue one of the most important problems in predictive control. In most cases, predictive control computations require the numerical solution of a convex quadratic programming (QP) problem, which is computationally formidable if, as in predictive control, on-line solutions are required. In order to lighten computations, it would be thus important to know whether and when it is possible to borrow from predictive control the concept of on-line reference management so as to tackle constrained control problems without the computational burden intrinsic to predictive control. The main goal of the present paper is to address this issue. As anticipated, in this direction there are no contributions with the only exception of [4]-[6]. However, the problem of on-line modifying the reference in such a way that a compensated control system can operate within its linear dynamic range with no constraint violation has been recently addressed outside the predictive control realm [7]-[9].

2. Problem Formulation

Consider the discrete-time linear time-invariant system

$$\left\{ \begin{array}{rcl} x(t+1) & = & \Phi x(t) + Gv(t) \\ y(t) & = & Hx(t) \\ c(t) & = & H_c x(t) + Dv(t) \end{array} \right. \tag{1}$$

where: $t \in \mathbb{Z}_+ := \{0, 1, \ldots\}$; $x(t) \in \mathbb{R}^n$ is the state-vector; $y(t) \in \mathbb{R}^p$ the output that is desired to be close to the *set-point* trajectory $w(t) \in \mathbb{R}^p$; $v(t) \in \mathbb{R}^p$ the *command input*; and $c(t) \in \mathbb{R}^q$ a vector which is required to belong to a specified *constraint* set $\mathcal{C} \subset \mathbb{R}^q$

$$c(t) \in \mathcal{C}, \ \forall t \in \mathbb{Z}_+ \tag{2}$$

The problem that we wish to study is how to choose the command sequence $v(\cdot) := \{v(t)\}_{t=0}^{\infty}$ with

$$v(t) = \underline{v}(x(t), w(t)) \tag{3}$$

so that $y(\cdot)$ can be possibly close to the set-point sequence $w(\cdot)$ while $c(\cdot)$ fulfills the constraints (2). The transformation (3) is referred to as the *command governor* (CG). Eq. (1) can represent a linear time-invariant system under state-feedback. We shall assume that (1) is asymptotically stable and that there

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exists a non empty bounded set $W \subset \mathbb{R}^p$ such that, $\forall w \in W$, the equilibrium state $x_w := (I - \Phi)^{-1} G w$ fulfills the constraints $c_w := H_c x_w + D w \in \mathcal{C}$, and yields zero offset, $y_w := H x_w = w$.

3. Command Governor

Consider the pair state/set point $(x(t), w(t))$ at time t. Introduce a *virtual command*

$$v(t+k|t,\mu) := \gamma^k \mu + w(t), \quad k \in \mathbb{Z}_+, \tag{4}$$

where $\gamma \in (0,1)$, and $\mu \in \mathbb{R}^p$ is a vector to be suitably chosen in order to possibly drive with no constraint violation the system state to $x_{w(t)}$, as $k \to \infty$. The idea is that if $x(t) = x_{\bar{w}}$, $\bar{w}$, $w(t) \in W$, $\mu = \bar{w} - w(t)$, and $\gamma \approx 1$, (4) defines a monotonically slowly-varying sequence exponentially approaching $w(t)$ from $\bar{w}$. In this way, one can compress the dynamic range of $c(t)$ in order to possibly satisfy the prescribed constraints. In fact, taking an arbitrarily small $\delta > 0$ and defining

$$W_\delta := \{w \in W \mid c_w + \tilde{c} \in \mathcal{C}, \ \forall \|\tilde{c}\| \leq \delta\} \tag{5}$$

which will be assumed non empty, the following result can be stated:

Lemma 1 *Consider the system (1) with $\mathcal{C}$ convex and W bounded. Then, given any pair of set points $\bar{w}$ and $w(t)$, $\bar{w}, w(t) \in W_\delta$, (4) drives (1) from an equilibrium state $x(t) = x_{\bar{w}}$ to the equilibrium state $x_{w(t)}$ with no constraint violation by setting $\mu = \bar{w} - w(t)$ and $\gamma \in (\gamma_\delta, 1)$, $\forall i = 0, \ldots, p$, where $\gamma_\delta > 0$ depends on W, δ, and the system (1).*

Proof. Suppose for now on that the plant is SISO ($p = 1$). By setting $\mu = \bar{w} - w$, then $v(t) = \gamma^t \bar{w} + (1-\gamma^t) w$, and $c(t) = \bar{c}(t) + H_c(1-\gamma) L(t) G \mu$, where $L(t) := \gamma^{t-1} \sum_{i=0}^{t-1} \gamma^{-i} \Phi^i$ and $\bar{c}(t) = \gamma^t c_{\bar{w}} + (1-\gamma^t) c_w$. Now, if $\mathcal{C}$ is convex, then W_δ is convex, $\forall \delta > 0$. Because the system (1) is asymptotically stable, there exist two positive reals m and λ, $\lambda < 1$, such that, for every $x \in \mathbb{R}^n$, $\|\Phi^t x\| \leq m \lambda^t \|x\|$, $\forall t \in \mathbb{Z}_+$. In particular, if λ_M, $|\lambda_M| < 1$, denotes the eigenvalue of Φ with maximum modulus and ϵ an arbitrarily small positive real, we can set $\lambda := |\lambda_M| + \epsilon < 1$. Assume now $p > 1$, and $\lambda < \gamma < 1$. By superimposing the effect of each component of r, $c(t) = \bar{c}(t) + \sum_{i=1}^p (1-\gamma) H_c (\Phi^t - \gamma^t I)(\gamma I - \Phi)^{-1} (I - \Phi)^{-1} G_i (\bar{w}_i - w_i)$, where G_i is the i-th column of G and w_i the i-th component of the vector w. If $\mathcal{C}$ is convex, then $\bar{c}(t) \in \mathcal{C}$, $\forall t \in \mathbb{Z}_+$, provided that $\gamma \in [0,1]$. Take $\gamma > \lambda$. Then, by asymptotical stability of Φ, $H_c(\Phi^t - \gamma^t I)(\gamma I - \Phi)^{-1}(I - \Phi)^{-1} G_i$ is uniformly bounded with respect to γ and t. Moreover boundness of W implies

$$|W| := \sup_{\bar{w}, w \in W} \|\bar{w} - w\| < \infty$$

Thus $\forall t \in \mathbb{Z}_+$ exists a scalar γ_δ, $\lambda < \gamma_\delta < 1$, such that $\|\tilde{c}(t)\| = \|c(t) - \bar{c}(t)\| \leq \| \sum_{i=1}^p (1-\gamma) H_c (\Phi^t - \gamma^t I)(\gamma I - \Phi)^{-1} (I - \Phi)^{-1} G_i \| \, |W| \leq \delta$ when $\gamma_\delta < \gamma < 1$. Because $\bar{w}, w \in W_\delta$, this ensures $c(t) \in \mathcal{C}$ $\forall t \in \mathbb{Z}_+$. Notice that when $\Phi = P^{-1} \Lambda P$, Λ diagonal, it is easy to check that

$$\gamma_\delta := 1 - \frac{\underline{\sigma}(\gamma I - \Phi)}{\overline{\sigma}(H_c) \|(I - \Phi)^{-1} G\| \, |W| \sqrt{1 + \frac{\overline{\sigma}^2(P)}{\underline{\sigma}^2(P)}}} \delta$$

where $\underline{\sigma}$ denotes the minimum singular value. □

From now on, we will take

$$\gamma \in (\gamma_\delta, 1). \tag{6}$$

Define now the quadratic *command selection index*

$$J(x(t), w(t), v(\cdot|t,\mu)) := \|v(t|t,\mu) - w(t)\|^2_{\psi_v} + \sum_{k=0}^{\infty} \|y(t+k|t,\mu) - w(t)\|^2_{\psi_y} \tag{7}$$

with $\|x\|^2_\psi := x' \psi x$, $\psi_v > 0$ diagonal, $\psi_y = \psi_y' > 0$, and $y(t+k|t,\mu)$ the output response of (1) at $t+k$ from the state $x(t)$ at t to the virtual command (4). The sum in (7) accounts for the tracking performance, the first term guarantees internal stability, as will be proved later. Assuming that the minimizer exist, set

$$\mu_t := \arg \min_{\mu \in \mathbf{R}^p} \{J(x(t), w(t), v(\cdot|t,\mu)) \mid c(\cdot|t,\mu) \subset \mathcal{C}\} \tag{8}$$

Under such circumstances, we say that $(x(t), w(t))$ is *admissible*, and the problem *feasible*. We next study the consequences of (8) under the assumption that the set point is kept constant, $w(t) \equiv w$ and the problem is initially feasible. To this end we resort to a Lyapunov function argument. Look first at

$$V(x(t)) := \min_{\mu \in \mathbf{R}^p} \{J(x(t), w, v(\cdot|t,\mu)) \mid c(\cdot|t,\mu) \subset \mathcal{C}\} \tag{9}$$

Let $v(t) := v(t|t,\mu_t) = \mu_t + w$ be the *actual* command used in (1). Noting that by (4) $v(t+1+k|t,\mu) = v(t+1+k|t+1,\gamma\mu)$, $\forall k \in \mathbb{Z}_+$, Assuming that the minimizer exist and equal μ_t, (9) becomes

$$V(x(t)) = \|\mu_t\|^2_{\bar{\psi}_v} + \|y(t) - w\|^2_{\psi_y} + J(x(t+1), w, v(\cdot|t+1, \gamma\mu_t)) \tag{10}$$

where $\bar{\psi}_v := (1-\gamma^2)\psi_v$. Taking into account that

$$\begin{aligned} &J(x(t+1), w, v(\cdot|t+1,\gamma\mu_t)) \geq \\ &\quad \min_{\mu \in \mathbf{R}^p} \{J(x(t+1), w, v(\cdot|t+1,\mu)) | c(\cdot|t+1,\mu) \subset \mathcal{C}\} \\ &\quad = V(x(t+1)) \end{aligned}$$

we find that along the trajectories of the system

$$V(x(t)) - V(x(t+1)) \geq \|\mu_t\|^2_{\bar{\psi}_v} + \|y(t) - w\|^2_{\psi_y} \tag{11}$$

Hence $V(x(t))$, being positive and monotonically non-increasing, converges as $t \to \infty$. Thus, summing both sides of (11) from t to ∞, $\sum_{i=t}^{\infty} [\|\mu_i\|^2_{\bar{\psi}_v} + \|y(i) - w\|^2_{\psi_y}] < \infty$.

Proposition 1 *Consider the system (1) along with the command governor (4), (5), (6), (8). Let the set point at and after time t be constant and equal to $w \in W$, and (1) be fed by $v(i) = \mu_i + w$, $\forall i \geq t$. Suppose that the pair $(x(t), w)$ be admissible, the minimizer μ_i, $i \geq t$, exist, and ψ_v, ψ_y be positive definite. Then the overall system results in an asymptotically stable behaviour in that*

$$c(i) \in \mathcal{C}, \ \forall i \geq t \tag{12}$$

and

$$\lim_{i\to\infty} y(i) = \lim_{i\to\infty} v(i) = w \tag{13}$$

at a rate faster than $1/i^{\frac{1}{2}}$.

Notice that (13) implies $\lim_{i\to\infty} x(i) = x_w$ and $\lim_{i\to\infty} \mu_i = 0$. Moreover, we point out that the argument used to prove Prop. 1 does not involve the linearity of (1).

In order to proceed further, we introduce some extra notation. We denote by $c(\cdot, x, \mu, w)$ the c-variable response from state x and command $v(k) = \gamma^k \mu + w$. Then

$$\mathcal{M}(t) := \{\mu \in \mathbb{R}^p : \ c(\cdot, x(t), \mu, w(t)) \subset \mathcal{C}\} \tag{14}$$

will be referred to as the *admissible set*. We next specify the command governor that will be considered from now on:

Command Governor (CG). Let γ be as in (6). Then at each $t \in \mathbb{Z}_+$ define a *virtual command* of the form

$$v(t+k|t) = \begin{cases} \gamma^k \mu_t + w(t), & \mathcal{M}(t) \text{ is non empty} \\ v(t+k|t-1, \mu_{t-1}), & \text{otherwise} \end{cases} \tag{15a}$$

with μ_t chosen in accordance with (8), and set

$$v(t) = v(x(t), w(t)) = v(t|t) \tag{15b}$$

The rationale for using the CG logic (15) stems from Proposition 1 along with the following considerations. Suppose that $(x(0), w(0))$ be admissible. Hence $\mu_0 \in \mathcal{M}(0)$ is determined. Then, $v(0) = \mu_0 + w(0)$ is applied to system (1). At $t = 1$, if $\mathcal{M}(1)$ is non empty, we compute and apply $v(1) = \mu_1 + w(1)$. On the contrary, by definition of $v(k|0, \mu_0)$ $v(1) = v(1|0, \mu_0)$ results in an admissible command input in that the constraint $c(1) \in \mathcal{C}$ is not violated. Moreover, $v(1|0, \mu_0)$ brings the state to $x(2)$ for which $v(2) = v(2|0, \mu_0)$ is an admissible command input. Thus, if we adopt the CG logic (15), the condition $(x(0), w(0))$ admissible ensures constraint satisfaction at all future times. The other important issue is the tracking performance achievable by (15). Next theorem, whose proof exploits Lemma 1 and Proposition 1, shows that (15) yields desirable asymptotic performance properties provided that the set-points be restricted to W_δ.

Theorem 1 (Conditional stability and offset-free behaviour) *Consider the system (1) along with the command governor (15) with γ_δ replaced by $\gamma^- := \gamma_{\delta-\eta}$, η being a arbitrarily small positive number. Let: the initial state $x(0)$ at time 0 be admissible for some virtual command sequence $v(k|-1, \mu_{-1}) = \gamma^k \mu_{-1} + w(-1)$, $w(-1) \in W_\delta$; the set-point sequence be such that $w(t) \in W_\delta$, $\forall t \in \mathbb{Z}_+$; and $w(t) = w$, $\forall t \geq \bar{t} \geq 0$. Then, the overall system results in an offset-free asymptotically stable behaviour in that*

$$c(i) \in \mathcal{C}, \ \forall i \geq t \tag{16}$$

and

$$\lim_{t\to\infty} y(t) = \lim_{t\to\infty} v(t) = w \tag{17}$$

at a rate faster that $1/t^{\frac{1}{2}}$.

Proof. We show that there exists a finite time $\hat{t}$, $\hat{t} \geq \bar{t}$, at which $\mathcal{M}(\hat{t})$ is non empty, and, hence, (16)-(17) follow from Proposition 1. Suppose, by contradiction, that such a $\hat{t}$ does not exist. Then, $\forall t \geq \bar{t}$ $\mathcal{M}(t)$ is empty and $v(t) = v(t|\tau, \mu_\tau)$, where τ, $-1 \leq \tau < \bar{t}$, is the greatest integer at which an admissible virtual reference was determined. Now, $v(t|\tau, \mu_\tau) = \gamma^{t-\tau}\mu_\tau + w(\tau)$, and consequently, as t increases, $v(t) \to w(\tau)$. Thus $\lim_{t\to\infty} x(t) = x_{w(\tau)}$. Consider the pair $(x_{w(\tau)}, w)$. By Lemma 1, there exists a virtual command which asymptotically drives the system state from $x_{w(\tau)}$ to x_w. This reference gives rise to an evolution for the c-variable of the form $c(k) = \bar{c}(k) + \tilde{c}_1(k)$ with $\bar{c}(k)$ corresponding to the steady-state c-response to the set point $\gamma^k w(\tau) + (1-\gamma^k)w \in W_\delta$, $\gamma^- < \gamma < 1$, and $\|\tilde{c}_1(k)\| \leq \delta - \eta$. Look now at the perturbed pair $(x_{w(\tau)} + \tilde{x}, w)$. The evolution of the c-variable from the perturbed state due to the previous virtual reference is given by $c(k) = \bar{c}(k) + \tilde{c}_1(k) + \tilde{c}_2(k)$, where $\tilde{c}_2(t)$ depends linearly on $\tilde{x}$. Then, there exists a positive ϵ such that $\|\tilde{c}_2(k)\| \leq \eta$ for all $\|\tilde{x}\| < \epsilon$, and hence $\|\tilde{c}_1(k) + \tilde{c}_2(k)\| \leq \delta$. Consequently, $(x_{w(\tau)} + \tilde{x}, w)$ is admissible for some $\tilde{x}$. Therefore, there exists a finite time at which $(x(t), w)$ is admissible, and this contradicts the assumption. □

Notice that the hypothesis of Theorem 1 are fulfilled when $x(0) = x_{w(-1)}$ is an equilibrium state and $\mu(-1) = 0$.

4. Command selection index computation

We now concentrate on finding the analytical form of the command selection index (7) in terms of the vector μ for the system (1). Let $\xi_v(k)$ be the state of a linear system which generates $v(k)$, $\xi_v(k+1) = \gamma\xi_v(k)$, $\xi_v(0) = \mu$, $v(k) = \xi_v(k) + w(t)$. Define $\tilde{x}(k) := x(t+k|t, \mu) - x_{w(t)}$. Recalling that $x_{w(t)} = (I - \Phi)^{-1}Gw(t)$, then $\tilde{x}(k+1) = \Phi\tilde{x}(k) + G\xi_v(k)$, and $\epsilon(t+k|t, \mu) := y(t+k|t, \mu) - w(t) = H\tilde{x}(k)$. Defining $\xi(k) := [\xi_v'(k) \ \tilde{x}'(k)]'$ and

$$A := \begin{bmatrix} \gamma I_p & 0 \\ G & \Phi \end{bmatrix}, \ C := \begin{bmatrix} 0_{p\times p} & H \end{bmatrix}$$

one has $\xi(k+1) = A\xi(k)$, $\epsilon(t+k|t,\mu) = C\xi(k)$, $\xi(0) = [\mu' \; \tilde{x}(0)']'$. Then, setting $J(\mu) := J(x(t), w(t), v(\cdot|t,\mu))$,

$$\begin{aligned} J(\mu) &= \|\mu\|^2_{\psi_v} + \sum_{k=0}^{\infty} \|CA^k\xi(0)\|^2_{\psi_y} \\ &= \mu'\psi_v\mu + \xi(0)'\mathcal{L}\xi(0) \end{aligned}$$

where $\mathcal{L} = \mathcal{L}'$ solves the Lyapunov equation

$$\mathcal{L} = A'\mathcal{L}A + C'\psi_y C \tag{18}$$

Then, denoting by $\mathcal{L}_{(i_1:i_2,j_1:j_2)}$ the submatrix of $\mathcal{L}$ obtained by extracting the entries $\mathcal{L}_{i,j}$ for $i_1 \le i \le i_2$ and $j_1 \le j \le j_2$, and setting $n := \dim x(t)$, we have

$$J(\mu) = \mu' A_J \mu + 2B_J'\mu + C_J$$

where

$$\begin{aligned} A_J &= \psi_v + \mathcal{L}_{(1:p,1:p)} \ge \psi_v > 0 \\ B_J &= \mathcal{L}_{(1:p,p+1:p+n)}[x(t) - x_{w(t)}] \\ C_J &= [x(t) - x_{w(t)}]'\mathcal{L}_{(p+1:p+n,p+1:p+n)}[x(t) - x_{w(t)}] \end{aligned}$$

Notice that if the constraints are non active, the minimizer equals $\mu_t = -A_J^{-1}B_J$. In this case the CG builds $v(t)$ as a linear combination of the desired trajectory $w(t)$ and the state.

5. Reduction to a finite constraint number

Because we require that $c(t+k|t,\mu) \in \mathcal{C}$, $\forall k \in \mathbb{Z}_+$, one has to minimize the quadratic functional $J(\mu)$ with an infinite number of constraints. We shall transform this into a finite constraint problem by adopting the approach in [8] by proving that $\mathcal{M}(t)$ can be determined by a finite number of constraints.

Let $c_{w(t)} = H_c x_{w(t)} + Dw(t)$ be the steady-state value taken on by the c-vector corresponding to a constant command $v(t+k|t) \equiv w(t)$. The evolution $c(\cdot|t,\mu)$ over the prediction horizon can be written as $\xi(k+1) = A\xi(k)$, $c(t+k|t,\mu) - c_{w(t)} = C_c\xi(k)$, $\xi(0) = [\; \mu' \;\; x(t)' - x'_{w(t)} \;]'$, where $C_c := [D \; H_c]$. In general $\xi(0)$ is not completely observable from $c(t+k|t,\mu) - c_{w(t)}$. In order to proceed further, we operate a canonical observability decomposition, getting a new (possibly reduced) state

$$\xi_o := S\xi, \; \xi_o \in \mathbb{R}^{n_o}$$

and correspondingly the system

$$\left\{ \begin{aligned} \xi_o(k+1) &= A_o\xi_o(k) \\ c(t+k|t,\mu) - c_{w(t)} &= C_o\xi_o(k) \\ \xi_o(0) &= S[\; \mu' \;\; x(t)' - x'_{w(t)} \;]' \end{aligned} \right. \tag{19}$$

Let us introduce the sets

$$\mathcal{X}_k(w) := \{\xi_o \in \mathbb{R}^{n_o} : \; c(h,\xi_o,w) \in \mathcal{C} \; \forall h \le k\} \tag{20}$$

$$\mathcal{X}_\infty(w) := \lim_{k\to\infty} \mathcal{X}_k. \tag{21}$$

where $c(h,\xi_o,w)$ is the c-variable response from state ξ_o. Note that $\mathcal{X}_\infty(w) \subseteq \mathcal{X}_{k+1}(w) \subseteq \mathcal{X}_k(w)$, and at least $0_{n_o} \in \mathcal{X}_\infty(w)$. Next lemmas are required to establish next Theorem 2.

Lemma 2 *$\mathcal{C}$ bounded implies that $\mathcal{X}_k(w)$ is uniformly bounded w.r.t. k, $k \ge n_o - 1$, and $w \in W_\delta$.*

Proof. Consider $\mathcal{X}_{n_o-1}(w)$ and a generic $\xi_o \in \mathcal{X}_{n_o-1}(w)$. By (19), denoting with Θ the observability matrix of the pair (A_o, C_o), $\Theta\xi_o = R$, where the vector $R \in \mathbb{R}^{qn_o}$ has the form

$$R = \left(\begin{bmatrix} c(0,\xi_o,w) \\ \vdots \\ c(n_o-1,\xi_o,w) \end{bmatrix} - \begin{bmatrix} I_q \\ \vdots \\ I_q \end{bmatrix} F_{cv}(1)w \right)$$

Consider n_o linearly independent rows of Θ. Form with these a new matrix $\bar{\Theta}$, and collect the respective n_o components of R to form a vector $\bar{R}$. Then $\xi_o = \bar{\Theta}^{-1}\bar{R}$. Being $\mathcal{C}$ and W_δ bounded, exists a bounded set $\mathcal{R}$ independent of w such that $\bar{R} \in \mathcal{R}$. Then $\bar{\Theta}^{-1}\mathcal{R}$ is bounded. Hence $\mathcal{X}_{n_o-1}(w)$ is uniformly bounded with respect to $w \in W_\delta$. By definition (20), $\mathcal{X}_k(w) \subseteq \mathcal{X}_{n_o-1}(w)$, $\forall k \ge n_o - 1$, $\forall w \in W_\delta$. □

Lemma 3 *$\forall \delta > 0 \; \exists \hat{k} = \hat{k}(\gamma,\delta) \ge 0$ such that $\mathcal{X}_{\hat{k}+1}(w) = \mathcal{X}_{\hat{k}}(w) = \mathcal{X}_\infty(w)$, $\forall w \in W_\delta$.*

Proof. Let $k \ge n_o - 1$ and $\xi_o \in \mathcal{X}_k(w)$. Then, being $c(k,\xi_o,w) - c_w = C_oA_o^k\xi_o$, for some $\alpha_1, \lambda_1 \in \mathbb{R}$, $\alpha_1 > 0$, $\max(|\lambda_M|, \gamma) < \lambda_1 < 1$

$$\|c(k,\xi_o,w) - c_w\| \le \alpha_1\lambda_1^k\|\xi_o\|.$$

By Lemma 2, $\|\xi_o\|$ is bounded. Then $\exists\hat{k} \ge n_o - 1$ such that $\|c(k,\xi_o,w) - c_w\| \le \alpha_2\lambda_1^k < \delta$, $\forall k \ge \hat{k}$. Thus $\forall w \in W_\delta$, $c(k,\xi_o,w) \in \mathcal{C}$, $\forall k \ge \hat{k}$, and hence $\mathcal{X}_{\hat{k}}(w) \subseteq \mathcal{X}_\infty(w)$. Being $\mathcal{X}_\infty(w) \subseteq \mathcal{X}_{\hat{k}}(w)$, $\mathcal{X}_\infty(w) = \mathcal{X}_{\hat{k}}$. In particular $c(\hat{k}+1,\xi_o,w) \in \mathcal{C} \; \forall\xi_o \in \mathcal{X}_{\hat{k}}(w)$, or $\xi_o \in \mathcal{X}_{\hat{k}+1}(w)$ which gives $\mathcal{X}_{\hat{k}+1}(w) = \mathcal{X}_{\hat{k}}(w)$ □

Theorem 2 *For all $x(t) \in \mathbb{R}^n$ and for all $w(t) \in W_\delta$, $\mathcal{M}(t)$ can be determined by a finite number $\bar{k}$ of constraints.*

Proof. Lemma 3 shows the existence of a number $\bar{k} \le \hat{k}$ such that $\mathcal{X}_\infty(w(t)) = \mathcal{X}_{\bar{k}}(w(t))$. Because $\mu \in \mathcal{M}(t) \Leftrightarrow S[\mu' \; x'(t) - x'_{w(t)}]' \in \mathcal{X}_\infty(w(t))$, then $\mu \in \mathcal{M}(t) \Leftrightarrow S[\mu' \; x'(t) - x'_{w(t)}]' \in \mathcal{X}_{\bar{k}}(w(t))$, or equivalently, $c(k, x(t), \mu, w(t)) \in \mathcal{C}$, $\forall k \le \bar{k}$. □

We are interested in determining the minimum k such that $\mathcal{X}_\infty(w) = \mathcal{X}_k(w)$. To this end, we introduce the following lemma.

Lemma 4 *For all $w \in W_\delta$, if $\mathcal{X}_{\bar{k}}(w) = \mathcal{X}_{\bar{k}+1}(w)$ then $\mathcal{X}_\infty(w) = \mathcal{X}_{\bar{k}}(w)$*

Proof. We shall prove by induction that $\mathcal{X}_k(w) = \mathcal{X}_{k+1}(w)$, $\forall k \ge \bar{k}$. For $k = \bar{k}$ this is true by assumption. Assume that $\mathcal{X}_{\bar{k}+h-1}(w) = \mathcal{X}_{\bar{k}+h}(w)$ and let $\xi_o \in \mathcal{X}_{\bar{k}+h}(w)$. For some $\mu \in \mathbb{R}^p$, $x \in \mathbb{R}^n$, we have $\xi_o = S[\mu' \; x' - x'_w]'$. Being

$$c(\bar{k}+h-1, S\begin{bmatrix} \gamma\mu \\ \Phi x + G(\mu+w) - x_w \end{bmatrix}, w) = c(\bar{k}+h, \xi_o, w) \tag{22}$$

then $S\left[\begin{array}{c}\gamma\mu \\ \Phi x + G(\mu + w) - x_w\end{array}\right] \in \mathcal{X}_{\bar{k}+h-1}(w) \subseteq \mathcal{X}_{\bar{k}+h}(w)$. This and (22) imply that $c(\bar{k}+h+1, \xi_o, w) \in \mathcal{C}$, and hence $\mathcal{X}_{\bar{k}+h}(w) \subseteq \mathcal{X}_{\bar{k}+h+1}(w)$. Then, $\mathcal{X}_{\bar{k}+h+1}(w) = \mathcal{X}_{\bar{k}+h}(w)$. Finally $\mathcal{X}_\infty(w) = \bigcap_{k=0}^{\bar{k}} \mathcal{X}_k(w) = \mathcal{X}_{\bar{k}}(w)$ □

From now on we shall assume that $\mathcal{C}$ is a set of the form

$$\mathcal{C} = \{c \in \mathbb{R}^q : g_i(c) \leq 0, \ \forall i = 0, 1, \ldots, m\} \qquad (23)$$

with

(i) $\mathcal{C}$ bounded and convex
(ii) $g_i : \mathbb{R}^q \to \mathbb{R}$ continuous, $\forall i = 0, \ldots, m.$ (24)

Lemma 5 *Suppose $\mathcal{C}$ is defined as in (23)-(24). Then*

(i) $\mathcal{C}$ is compact
(ii) $\mathcal{N}_k$ is compact

where

$$\mathcal{N}_k := \left\{\left[\begin{array}{c} w \\ \xi_o \end{array}\right] \in \mathbb{R}^{n_o+p} : \ w \in W_\delta, \ \xi_o \in \mathcal{X}_k(w)\right\}. \qquad (25)$$

Proof. (i) follows at once because $\mathcal{C}$ is bounded and closed. Because $F_{cv}(1)w$ is a linear function of w, W_δ is closed; moreover, being W bounded, W_δ is compact, $\forall \delta > 0$. Let

$$I_k(w) := \{\xi_o \in \mathbb{R}^{n_o} : c(k, \xi_o, w) \in \mathcal{C}\} \qquad (26)$$

Being the function $c(k, \cdot, w)$ linear in its argument ξ_o and $\mathcal{C}$ closed, then $I_k(w)$ is closed. Since $\mathcal{X}_k(w) = \bigcap_{i=0}^{k} I_i(w)$ it follows that $\mathcal{X}_k(w)$, $\forall k \geq 0$, and $\mathcal{X}_\infty(w)$ are closed. By Lemma 2 being $\mathcal{X}_\infty(w)$ bounded, $\mathcal{X}_\infty(w)$ is compact. Hence $\mathcal{N}_k$ is compact. □

The following algorithm [8] can be used to find the index

$$k^o := \min_{k \geq 0}\{k | \ \mathcal{X}_k(w) = \mathcal{X}_\infty(w)\}, \ \forall w \in W_\delta \qquad (27)$$

Algorithm 1:

1. $k \leftarrow 0$
2. Let
$$\mathcal{G}_j := \max_{[\ w' \quad \xi_o'\ ]'}\{g_j(c(k+1, \xi, w))\}$$
$$\text{subject to} \left\{\begin{array}{l} w \in W_\delta \\ g_i(c(h, \xi_o, w)) \leq 0, \\ \forall h = 0, \ldots, k, \ \forall i = 0, \ldots, M \end{array}\right.$$
3. If $\mathcal{G}_j \leq 0$, $\forall j = 0, \ldots, M$, then let $k^o \leftarrow k$ and stop
4. $k \leftarrow k + 1$
5. Go to step 2

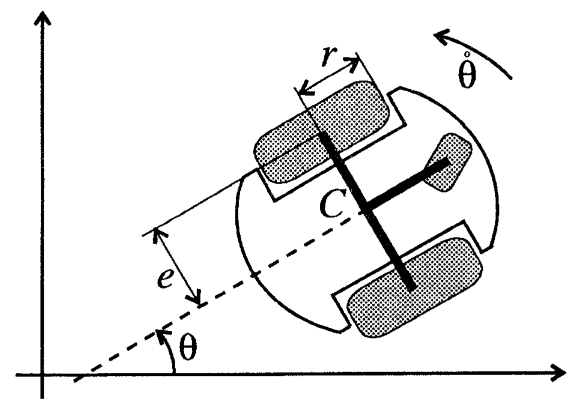

Fig. 1. Robot model.

This algorithm stops when the minimum k such that $\mathcal{X}_k(w) \subset \mathcal{X}_{k+1}(w)$, $\forall w \in W_\delta$, is found.

Theorem 3 *Let $w(t) \in W_\delta$ and $x(t) \in \mathbb{R}^n$ be respectively the desired set-point and the system state at time t. Let $\mathcal{C}$ be given as in (23)-(24). Let k^o as in (27) be determined by Algorithm 1. If the vector μ fulfils the $k^o + 1$ constraints*

$$c(k, x(t), \mu, w(t)) \in \mathcal{C}, \ \forall k = 0, \ldots, k_o \qquad (28)$$

then the virtual command $v(t + k|t, \mu) = \gamma^k \mu + w(t)$ yields a c-evolution $c(\cdot|t, \mu) \subset \mathcal{C}$.

We have reduced a quadratic programming problem with an infinite number of constraints in one with a finite number of constraints. Notice that when $\mathcal{C}$ is a polytope, the constraints become linear and k^o can be easily computed by standard optimization routines. When the plant is SISO, the constraints are due to saturating actuators ($c = u$ is the input of the plant), and $\mathcal{C} = [C^-, C^+]$), the minimization procedure becomes trivial, as shown in [6].

6. An example

The method developed in the previous sections is used to control the differentially driven mobile robot depicted in Fig. 1 (mass M, inertial momentum J w.r.t. C, track $2e$). Two DC gearhead motors (gear ratio ρ, motor constant k_T, leak resistance R) drive independently the active wheels (radius r, inertial momentum J_ϕ), and receive a voltage (V_1 and V_2 respectively) which cannot exceed the battery level. Call v the linear velocity of the mobile robot, θ it heading in the planar work-space, $V_+ = V_2 + V_1$, $V_- = V_2 - V_1$. Newton's and Kirchhoff's laws give

$$v = \frac{\rho k_T r}{R[(Mr^2 + 2J_\varphi)s + Br^2 + 2\beta] + 2\rho^2 k_T^2} V_+$$

$$\theta = \frac{\rho k_T e r}{sR[(Jr^2 + 2J_\varphi e^2)s + Br^2 + 2\beta e^2] + 2\rho^2 k_T^2 e^2 s} V_-$$

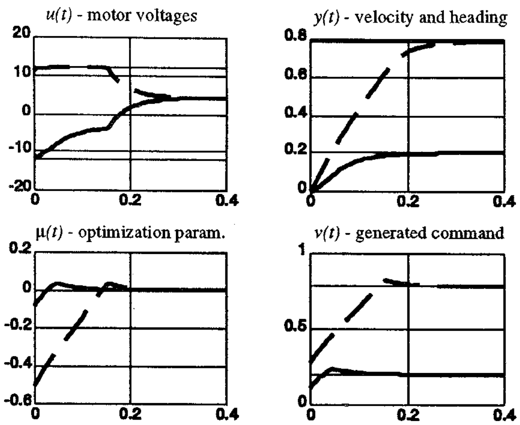

Fig. 2. Response with the command governor.

where β, B, $\mathcal{B}$ are friction coefficients. Simulations have been carried on the following model

$$v = \frac{0.0655}{0.0124s + 2.8355} V_+ \quad (29)$$

$$\theta = \frac{0.00655}{s(0.0000635s + 0.02835)} V_- \quad (30)$$

Decoupled feedback loops have been designed in order to track with zero offset the desired set-point trajectory v_d and θ_d

$$V_+ = -k_v \frac{v - v_d}{s} \quad (31)$$

$$V_- = -k_\theta(\theta - \theta_d) \quad (32)$$

Eqs. (29)-(30) are coupled by the physical constraint

$$|V_i| \leq 12V, \; i = 1, 2. \quad (33)$$

With the prescribed set-points $v_d = 0.2ms^{-1}$, $\theta_d = \pi/4$ (33) would be violated, and hence a command governor has been implemented ($y = [v\ \theta]'$, $u = [V_1\ V_2]'$, $w = [v_d\ \theta_d]'$). The continuous-time overall system (29)-(32) is sampled every $T_s = 5ms$ and a zero-order hold is used. Fig. 2 depicts the trajectories that result when the CG ($\psi_v = 0.01I_2$, $\psi_y = I_2$) is activated. The system behaviour has been simulated in $26s$ with Simulink on a 486DX2/66 computer, using Matlab 4.0 standard QP routines. We chose $\gamma = 0.95$ and $\delta = 2.5$. Algorithm 1 found $k^o = 37$. Even if the tracking performance could deteriorate, simulation times could be drastically reduced by performing scalar on-line optimization, by setting $\mu = \nu i_\nu$, where ν is a scalar minimization parameter, and i_ν is a fixed vector (e.g. $i_\nu = w$). Notice that the governor proposed by [9] is based on a scalarly parameterized command.

7. Conclusions

The command governor problem, viz. the one of on-line designing, given the reference to be tracked, a command sequence in such a way that a compensated control system can operate in a stable way with satisfactory tracking performance and no constraint violation, has been addressed by exploiting some ideas originating from predictive control.

Though some related encouraging indications have been provided by simulations [6], an important future research topic is stability and performance robustness of the command governor against exogenous disturbances and modeling errors.

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