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Constraint Fulfilment in Control Systems via Predictive Reference Management

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Abstract: The problem of satisfying input and state-dependent inequality constraints in feedback control systems is addressed. The proposed solution is based on predicting the evolution of the constrained vector and, accordingly, selecting on line the future reference based on both the current state and the desired set-point changes. An analysis is presented so as to estabilish stability and offset-free properties of the method when embodied in an LQ regulated system. Finally, simulations are used to evaluate the achievable performance.

Keywords: **Control with hard constraints; Predictive control; Intelligent control; Nonlinear systems; Trajectory planning.**

1. Introduction

The problem of determining a feedback control law capable of stabilizing a given plant in the presence of input and state-related inequality constraints is one of the fundamental issues in control engineering. In this context even the conceptually simple case of a linear plant with saturated inputs gives rise to challenging stability problems [6]. For the discrete-time regulation problem, [7] showed that, under feasibility conditions, zero terminal state receding horizon control [9] with input and state-related constraints yields a stable feedback system. Under quite general conditions, [7] proved in fact this to hold true even if the plant to be regulated is nonlinear and time-varying. Extensions of similar results to the continuous-time regulation problem are tackled in [11] and discussed in [12]. Specific feedback regulation systems for linear plants which avoid input saturations are treated in [8] and [4].

For 2-DOF (two degrees of freedom) control problems with hard constraints, in recent years a great deal of interest has been focussed on applying predictive control techniques [19], [17], via the on-line use of a mathematical programming solver [18], [16].

The present paper tackles the control problem with constraints along the lines of predictive control but, unlike the previous contributions, sidesteps the need of using a mathematical programming solver by adopting a suitable on-line management of the reference to be tracked. As shown in [2], this considerably lightens the computational load and at the same time gives indistinguishable performances. The solution we provide can be also used in trajectory generation problems ([5], [3], and [10]) wherein, given a control system, the command waveform has to been chosen so as to achieve the desired tasks and fulfil the prescribed constraints.

The paper is organized as follows. Sect. 2 describes the on-line *Predictive Reference Management* (PRM). Sect. 3 analyses the stability of a particular feedback control system embodying the proposed on-line PRM. Via simulation experiments Sect. 4 shows that the proposed on-line PRM fulfils input and state-dependent inequality constraints. Some conclusive remarks are finally presented in Sect. 5.

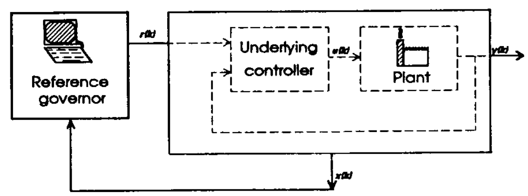

Fig. 1. Control system with PRM

2. On-line Predictive Reference Management

Consider the control system depicted in Fig. 1 where $y(t)$ and $u(t)$ are respectively the output and the input of the plant, and $x(t)$ is the state of the closed-loop system. Further, $c(t) = c(x(t), u(t))$ is the constrained vector. The underlying controller can implement any stabilizing (discrete/continuous time, linear/nonlinear, causal/anticipative) control law designed so as to make the output track the reference sequence $r(\cdot)$ in the absence of constraints.

The *reference governor* constructs $r(\cdot)$ in such a way that $c(t)$ fulfils the constraints. As explained in detail below, the selection of $r(\cdot)$ is based on the current state $x(t)$. Two possible operations are considered:

- Smoothing out (or filtering) a given set-point trajectory $w(\cdot)$ to be tracked;
- Generate $r(\cdot)$ by choosing a suitable time-parameterization of a given path in the output vector space.

The latter operation will not be treated here but will be the subject of future papers. We now focus on the former. We suppose that inside the reference governor a *set-point conditioner* is inserted in order to threshold $w(t)$ in such a way that the output in steady-state will be capable of tracking the resulting conditioned set-point trajectory while $c(\cdot)$ fulfils the constraints.

At each step t the reference governor builds a virtual reference sequence $\{r(t+i|t)\}_{i=0}^{\infty}$ which smoothly connects over the future the output from its current value $y(t)$ to the desired $w(t)$. Accordingly, the virtual reference pattern is defined as follows

$$r(t+i|t) = \lambda^{i+1}(t)y(t) + [1 - \lambda^{i+1}(t)]w(t) \qquad (1)$$

where t, $i \geq 0$ and $\lambda(t) \in [0,1)$. For the sake of simplicity the reference has been supposed to be scalar-valued.

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Let $c(\cdot|t) = \{c(t+i|t)\}_{i=0}^{\infty}$ be the hypothetical evolution of the constrained vector corresponding to the use of the reference pattern (1). It will be called the *prediction* of the constrained variable for a given $r(\cdot|t)$ and $x(t)$. Then, one can compress the dynamic range of $c(\cdot|t)$ by choosing the "time constant" $\lambda(t)$ so as to possibly keep $c(\cdot|t)$ admissible. In fact the closer $\lambda(t)$ to 1, the smoother $c(\cdot|t)$ will be and the larger the resulting settling-time. The idea is to choose at each t the time constant $\lambda(t)$ which gives the shortest settling time while keeping $c(\cdot|t)$ admissible. The reference governor strategy can be described in the following algorithm. At each time step t:

1. Construct the reference pattern $r(t+i|t) \equiv w(t)$, $\forall i \geq 0$ (i.e. $\lambda(t) = 0$);
2. Make a prediction $c_M(\cdot|t) := \{c(t+i|t)\}_{i=0}^{M-1}$ over an M-step horizon by iterating the plant model and the underlying controller fed by the selected reference pattern;
3. Does $c_M(\cdot|t)$ fulfil the constraints?
   (a) Yes: Use the current reference pattern as the actual reference, compute $u(t)$ to be given to the actuator, and go to 5;
   (b) No: Go to 4
4. Can other reference patterns corresponding to a larger value of $\lambda(t) < 1$ be constructed?
   (a) Yes: Construct the reference pattern by increasing the parameter $\lambda(t)$, and go to 2;
   (b) No: Set $r(t+i|t) = r(t+i|t-1)$. Go to 5;
5. Stop

In the deterministic case, step (3a) ensures that the constraints will be fulfilled at least for the next M steps. Ideally M should be infinite. In practice the prediction horizon M has to be finite. In fact, the shorter it is, the lighter the computational burden. However, too small values of M can lead the system into a "blind alley", where no choice of future reference patterns will avoid the violation of the constraints. A rule of thumb is to set MT_s (T_s=sampling time) equal to the settling time of $c(\cdot)$ in the presence of the prescribed constraints given that, because of set-point conditioning, the constraints can be violated only during transients.

Another issue is how many sequences the governor can try before giving up and execute step (4b). Our solution is to set up a grid G made up of n_G values $0 = \lambda_0 < \lambda_1 < \ldots < \lambda_{n_G-1}$ suitably distributed on $[0,1)$. The higher is n_G the better is the performance but the heavier the computational burden.

Notice that if the underlying control law is linear, causal, and with 1-DOF (for example a lead-lag or a PID controller)

$$Ru(t) = S[r(t) - y(t)]$$

where R and S are polynomials (in the unit delay operator d or in the complex variable s), in the presence of the reference governor the actual control law becomes

$$Ru(t) = S[1-\lambda(t)][w(t) - y(t)]$$

and hence the governor introduce a sort of "valve" on the actual tracking error $w(t) - y(t)$. This form is similar to the control structure reported in [8] and [4], where the valve, in contrast with our scheme, is operated taking only into account the controller dynamics, just allowing input constraints. Moreover the plant performance resulting from the choice of $\lambda(t)$, which alters the originally designed closed loop, is disregarded.

As will be clear from simulations results, the effect of PRM is to slow down the overall closed-loop. This suggests that the underlying control law be designed in a way to ensure, without governor, a fast feedback system.

We finally point out that, even if the plant and underlying control law are linear, the resulting controller is nonlinear. Indeed $\lambda(t) = \lambda(x(t))$ since it is chosen by a prediction which depends on the current state $x(t)$ and the prescribed constraints.

3. Analysis

Although the strategy described so far can be applied to nonlinear MIMO plants and any underlying stabilizing control law, we now investigate how to use it in order to control a discrete-time SISO linear time-invariant plant

$$\begin{cases} (1-d)A(d)y(t) = B(d)\delta u(t) \\ (1-d)A(d) \ \& \ B(d) \text{ coprime} \end{cases} \tag{2}$$

with $\delta u(t) := u(t) - u(t-1)$. This can be described in the form

$$\begin{cases} x(t+1) &= \Phi x(t) + G\delta u(t) \\ y(t) &= Hx(t) \end{cases} \tag{3}$$

where $x(t)$, for example, collects input/output pairs. The constrained vector is $c(t) = Cx(t) + D\delta u(t) \in \mathbb{R}^{n_c}$. We consider the 2-DOF underlying control law

$$R(d)\delta u(t) = -S(d)y(t) + \sum_{i=0}^{\infty} h_i r(t+i) \tag{4}$$

which minimizes the quadratic performance index

$$J = \sum_{i=0}^{\infty}\{[(y(t+i) - r(t+i)]^2 + \rho[\delta u(t+i)]^2\} \tag{5}$$

$\rho > 0$, assuming that $\{r(t+i)\}_{i=0}^{\infty}$ is bounded and known. The resulting control law will be referred to as *LQ control with preview*. It consists of the control law (4) where $R(d)$, $S(d)$ are polynomials satisfying the Diophantine equation

$$(1-d)A(d)R(d) + B(d)S(d) = E(d)/E(0), \tag{6}$$

with $S(d)$ of minimum degree and $E(d)$ is a strictly Hurwitz polynomial which solves the spectral factorization problem

$$E(d)E(d^{-1}) = B(d)B(d^{-1}) + \rho(1-d)A(d)A(d^{-1})(1-d^{-1}) \tag{7}$$

and

$$V(d) = \sum_{i=0}^{\infty} v_i d^i = \frac{B(d)}{E(0)E(d)}. \tag{8}$$

The solution of the LQ control problem with preview in the form (4) is given in [14] and [13]. Since the transfer function from $r(t)$ to $y(t)$ for the system (2), (4), is given by $\frac{B(d)B(d^{-1})}{E(d)E(d^{-1})}$, zero-offset results.

We shall show how PRM together with LQ guarantees constraints fulfilment, stabilizing properties, and zero-offset in steady-state. Weak assumptions are made on the initial plant state. We consider the case in which the desired trajectory is constant for $t \geq t_w$. The reference pattern considered is

$$r(t+i|t) = \begin{cases} w(t) & \lambda(t) = 0 \\ \lambda^{i+1}(t)y(t) + \\ +[1-\lambda^{i+1}(t)]w(t) & 0 < \lambda(t) \leq \lambda_{max} \\ r(t+i|t-1) & \lambda(t) = 1 \end{cases} \tag{9}$$

$$= \lambda(t-h)^{i+h+1}y(t-h) + +[1-\lambda(t-h)^{i+h+1}]w(t-h) \tag{10}$$

where $i \geq 0$, $h = h(x(t)) = t - t_h$ and $t_h = \max_{t^* \leq t}\{t^* : \lambda(t^*) < 1\}$. The PRM algorithm is slightly changed as follows:

- Let t_1 such that $\lambda(t_1) < 1$ and $t_1 \geq t_w$. Lemma 1 will show that such a t_1 exists finite. The time step t_1 is not required to be known a priori
- Define a value for the *expected settling time* N_p. For example set $N_p = 1000$.
- "If $t \geq t_1 + N_p$ and find $\lambda(t) = 1$, $\lambda(t) = 1$ until $\|x(t) - x_w\| < \frac{\delta}{2\|C_F\|\phi_1}$ (δ, x_w, C_F and ϕ_1 will be defined later). Thereafter, set $\lambda(t) = 0$". This rule will be further modified to improve the tracking performance.

In practice, choosing a high value for N_p, this modified PRM algorithm will not differ from the original. PRM embodied in LQ with preview gives the control law

$$\left\{ \begin{array}{rcl} \delta u(t) & = & Fx(t) + v(t) \\ v(t) & = & \tilde{V}(t)y(t-h) + [V(1) - \tilde{V}(t)]w(t-h) \end{array} \right. \quad (11)$$

where $\tilde{V}(t) := \lambda^{h+1}(t-h)V(\lambda(t-h))$. The closed loop can be described as

$$\left\{ \begin{array}{rcl} x(t+1) & = & \Phi_F x(t) + Gv(t) \\ c(t) & = & C_F x(t) + Dv(t) \end{array} \right. \quad (12)$$

where $\Phi_F := \Phi + GF$ and $C_F := C + DF$. Define $\mathcal{X}(r(\cdot|t)) := \{x \in \mathbb{R}^n : c(k+t|t) \in \mathcal{C}\ \forall k \geq 0\}$ where $\{c(k+t|t)\}$ is the predicted evolution of the constrained vector obtained by (11)-(12) initialized at $x(t) = x$. In the following, given $w \in \mathbb{R}$, we denote $x_w := (I - \Phi_F)^{-1}GV(1)w$, $c_w := C_F x_w + Dv_w$, and $v_w := V(1)w$ which are respectively the values for $x(t)$, $c(t)$ and $v(t)$ in steady-state when $r(\cdot|t) \equiv w\ \forall t \geq 0$. Then, chosen an arbitrary small "tolerance" $\delta > 0$, define the *admissible set-point set*

$$W_\delta := \{w \in \mathbb{R} : B(c_w, \delta) \subset \mathcal{C}\}$$

Assume that $x(0)$ is a vector such that, in the absence of constraints, is reachable from a generic steady-state state vector, and such that

$$\begin{array}{l} \exists \bar{y} \in \mathbb{R}, \bar{h} \geq 0, \bar{\lambda} \in [0,1), \bar{w} \in W_\delta \text{ s.t.} \\ x(0) \in \mathcal{X}(r(\cdot|-1)), \\ r(i|-1) := \bar{\lambda}^{i+\bar{h}+1}\bar{y} + (1 - \bar{\lambda}^{i+\bar{h}+1})\bar{w}\} \end{array} \quad (13)$$

For example, (14) is satisfied for $x(0) = x_w$ ($\bar{h} = 0$, $\bar{y} = Hx_{\bar{w}}$, $\bar{\lambda} = 0$, and $\bar{w} = w$).

In the sequel we shall assume that W_δ is bounded. It can be shown that if $\mathcal{C}$ is bounded and the dc-gain from w to c is nonzero this is always verified. However, this assumption entails no loss of generality in that all sequences $w(\cdot)$ to be tracked are uniformly bounded.

Lemma 1 *Consider plant (3) controlled by (11) with $x(0)$ as in (14). Suppose $w(t) \equiv w\ \forall t \geq t_w$, and $\mathcal{C}$ convex and bounded. Then $\exists t_1 \geq t_w$ such that $\lambda(t_1) < 1$.*

Proof. Define $t_0 := \max[\{-1\} \bigcup \{t : \lambda(t) < 1,\ 0 \leq t \leq t_w\}]$. If $t_0 = t_w$ the proof is immediate. Suppose $t_0 < t_w$. If $t_0 \geq 0$ define $w_0 := w(t_0)$, $\lambda_0 := \lambda(t_0)$, $y_0 := y(t_0)$, else, recalling (14), define $w_0, \lambda_0, y_0 := \bar{w}, \bar{\lambda}, \bar{y}$ respectively. Assume now that the reference governor keeps $\lambda(t) \equiv 1$ $\forall t > t_0$. Without loss of generality consider $t_0 = 0$.

Because of the closed-loop asymptotic stability,

$$\lim_{t \to \infty} x(t) = (I - \Phi_F)^{-1}GV(1)w_0 = x_{w_0}$$

and hence $\forall \epsilon > 0\ \exists t_\epsilon \geq t_w$ such that $\forall t \geq t_\epsilon\ \|x(t) - x_{w_0}\| \leq \epsilon$. Define $\delta x(t) := x(t) - x_{w_0}$ and $\delta y(t) := H\delta x(t)$. Being $t_\epsilon \geq t_w > t_0$, $\lambda(t_\epsilon) = 1$. This implies that $\forall \lambda \in [0,1)$ $x(t) \notin \mathcal{X}(r_\lambda(\cdot|t_\epsilon))$, i.e. $c(k + t_\epsilon|t_\epsilon) \notin \mathcal{C}$ for some $k \geq 0$. We show that this can be contradicted for some $\lambda > \lambda_F$, where $\lambda_F := \max\{\text{eigenvalues of } \Phi_F\}$. Temporarily assume that the reference governor can choose λ arbitrarily close to 1. Because $r_\lambda(k + t_\epsilon|t_\epsilon) = \lambda^{k+1}[w_0 + \delta y(t_\epsilon)] + (1 - \lambda^{k+1})w$ then $v(k + t_\epsilon|t_\epsilon) = V(1)w_0 + [V(1) - \lambda^{k+1}V(\lambda)](w - w_0) + \lambda^{k+1}V(\lambda)\delta y(t_\epsilon)$. Denoting with $\varphi(k, x(0), 0, u(k))$ the value of $x(k)$ resulting at time k by applying the input sequence $\{u(j)\}$, $j = 0, 1, \ldots, k$, we have

$$\begin{array}{rcl} x(k + t_\epsilon|t_\epsilon) & = & \varphi(k, x(t), 0, 0) + \varphi(k, 0, 0, V(1)w_0) + \\ & & +\varphi(k, 0, 0, [V(1) - \lambda^{k+1}V(\lambda)](w - w_0)) + \\ & & +\varphi(k, 0, 0, \lambda^{k+1}V(\lambda)\delta y(t_\epsilon)) \\ & =: & \varphi_1(k) + \varphi_2(k) + \varphi_3(k) + \varphi_4(k) \end{array}$$

where

$$\begin{array}{rcl} \varphi_1(k) & = & \Phi_F^k x(t) \\ \varphi_2(k) & = & (I - \Phi_F^k)x_{w_0} \\ \varphi_3(k) & = & [I - L(\lambda, k)](x_w - x_{w_0}) \\ \varphi_4(k) & = & \lambda(\lambda^k I - \Phi_F^k)(\lambda I - \Phi_F)^{-1}GV(\lambda)\delta y(t_\epsilon) \end{array}$$

with

$$\begin{array}{rcl} L(\lambda, k) & := & \Phi_F^k + \lambda(\lambda^k I - \Phi_F^k)[I + (1 - \lambda)(\lambda I - \Phi_F)^{-1}]\bar{v}(\lambda) \\ & = & \lambda^{k+1}\bar{v}(\lambda)I + P(\lambda, k) \\ P(\lambda, k) & := & \lambda^{k+1}(1 - \lambda)(\lambda I - \Phi_F)^{-1}\bar{v}(\lambda) + \\ & & +[1 - \lambda\bar{v}(\lambda)]\Phi_F^k - \lambda(1 - \lambda)(\lambda I - \Phi_F)^{-1}\bar{v}(\lambda)\Phi_F^k \\ \bar{v}(\lambda) & := & \dfrac{V(\lambda)}{V(1)} \end{array}$$

then it is easy to verify that

$$c(k + t_\epsilon|t_\epsilon) = C(\lambda, k) + R(\lambda, k)\delta x(t_\epsilon) + CP(\lambda, k)(x_{w_0} - x_w)$$

where

$$\begin{array}{rcl} C(\lambda, k) & := & \lambda^{k+1}\bar{v}(\lambda)c_{w_0} + [1 - \lambda^{k+1}\bar{v}(\lambda)]c_w \\ R(\lambda, k) & := & C\Phi_F^k + V(\lambda)[\lambda C(\lambda^k I - \\ & & \Phi_F^k)(\lambda I - \Phi_F)^{-1}G + D]H \end{array}$$

Notice that $C(\lambda, k)$ is a convex average of elements in $\mathcal{C}$, and hence $C(\lambda, k) \in \mathcal{C}$. Moreover $\lim_{k\to\infty} \Phi_F^k(x_{w_0} - x_w) = 0$, which implies that exists a constant ϕ_1 such that

$$\|\Phi_F^k(x_{w_0} - x_w)\| \leq \phi_1\|x_{w_0} - x_w\|\ \forall k \geq 0 \quad (14)$$

Then

$$\begin{array}{rcl} \|P(\lambda, k)(x_{w_0} - x_w)\| & \leq & [1 - \lambda\bar{v}(\lambda)]\phi_1\|x_{w_0} - x_w\| + \\ & & +\lambda(1 - \lambda)\bar{v}(\lambda)\|(\lambda^k I - \Phi_F^k) \\ & & (\lambda I - \Phi_F)^{-1}(x_{w_0} - x_w)\| \\ & \leq & \ell(\lambda)\|x_{w_0} - x_w\| \end{array}$$

where

$$\begin{array}{rcl} \ell(\lambda) & := & [1 - \lambda\bar{v}(\lambda)]\phi_1 + \lambda(1 - \lambda)\bar{v}(\lambda)(1 + \phi_1)\phi_2 \\ \phi_2 & := & \sup_{\lambda \in [0,1)} \bar{\sigma}[(\lambda I - \Phi_F)^{-1}] \end{array}$$

Because $\lim_{\lambda \to 1^-} \ell(\lambda) = 0$, then $\exists \lambda_{max} < 1$ such that

$$\ell(\lambda_{max}) < \frac{\delta}{2\|C\|X_\delta} \quad (15)$$

where

$$X_\delta := \sup_{w_0, w \in W_\delta} \|x_{w_0} - x_w\|$$

($X_\delta < \infty$ by the assumption that W_δ is bounded). Chosen

$$\epsilon := \frac{\delta}{2[\|C\|\phi_1 + (\|C\|(1+\phi_1)\phi_2\|G\| + \|D\|)\|H\|]}$$

for $\lambda = \lambda_{max}$ then $c(k+t_\epsilon|t_\epsilon) \in \mathcal{C}\ \forall k \geq 0$. We can now assume that the reference governor selects $\lambda \in [0, \lambda_{max}] \bigcup \{1\}$. □

Lemma 2 *$\forall \lambda_{max} \in [0,1)\ \exists \mu_{max} < 1$ such that $\left|\frac{B(\lambda)}{E(\lambda)}\right| \leq \mu_{max}\ \forall \lambda \in [0, \lambda_{max}]$.*

Proof. See [1], lemma 5.1, p. 151

Theorem 1 *Consider the plant (3) controlled by (11) with λ_{max} as in (15). Assume that $\mathcal{C}$ is convex and bounded, $x(0)$ and $r(\cdot|-1)$ are as in (14), and the desired trajectory $\{w(t)\}_{t=0}^{\infty} \subset W_\delta$ is such that $w(t) \equiv w\ \forall t \geq t_w$. Then*

i. $\lim_{t\to\infty} x(t) = x_w$

ii. $\exists T \geq 0$ *such that* $\lambda(t) = 0\ \forall t \geq T$

iii. $c(t) \in \mathcal{C}\ \forall t \geq 0$

Proof. By Lemma 1, $\exists t_1 \geq t_w$ such that $\lambda(t_1) < 1$. This implies $w(t - h(t)) \equiv w\ \forall t \geq t_1$. Define $t_2 := t_1 + N_p$. Distinguish between two situations:
CASE 1: $\lambda(t) < 1$ (and hence $h(t) \equiv 0$) $\forall t \geq t_2$.
CASE 2: $\exists t_3 \geq t_2$ such that $\lambda(t_3) = 1$.

CASE 1. Imagine that the plant (3), has been driven to $x(t_2)$ by the control law

$$\begin{cases} \delta u(t) &= Fx(t) + v(t) \\ v(t) &= V(1)\bar{w}(t) \end{cases} \quad (16)$$

starting at $x(-\infty) = x_w$ and applying a suitable sequence

$$\bar{w}(t) = \begin{cases} w & ,t < t_2 - n, \quad n = \dim x \\ w_t & ,t_2 - n < t < t_2 - 1 \end{cases}$$

For $t < t_2$ no reference governor is supposed to be active, allowing $c(t)$ to be free. Clearly $x(t) = x_w\ \forall t \leq t_2 - n$. Let $\bar{w}(t) = w(t) = w\ \forall t \geq t_2$. Moreover, consider the plant

$$\begin{cases} \hat{x}(t+1) &= \Phi_F \hat{x}(t) + G\hat{v}(t) \\ \hat{v}(t) &= V(1)w \\ \hat{x}(-\infty) &= x_w \end{cases} \quad (17)$$

Clearly $\hat{x}(t) \equiv x_w\ \forall t \in \mathbb{Z}$. Define $\tilde{x}(t) := x(t) - \hat{x}(t)$, $\tilde{w}(t) := \bar{w}(t) - w$, $t \in \mathbb{Z}$. Vector $\tilde{x}(t)$ can be seen as the state of the variable-structure plant

$$\begin{cases} \tilde{x}(t+1) &= \Phi_F \tilde{x}(t) + G\tilde{v}(t) \\ \tilde{v}(t) &= \begin{cases} V(1)\tilde{w}(t) & ,t < t_2 \\ \tilde{V}(t)H\tilde{x}(t) + \tilde{w}_1(t) & ,t \geq t_2 \end{cases} \end{cases} \quad (18)$$

where, being $h(t) \equiv 0$, $\tilde{w}_1(t) := [V(1) - \tilde{V}(t)]\tilde{w}(t)$ $\tilde{V}(t) = \lambda(t)V(\lambda(t))$. Notice that $\tilde{w}_1(t) \equiv 0$ for $t < t_2 - n$, $t \geq t_2$, and so $\tilde{w}_1(\cdot) \in \ell_2$. Equations (18) can be rewritten as

$$\begin{cases} \tilde{x} &= (I - d\Phi_F)^{-1} dG\tilde{v} =: G_1\tilde{v} \\ \tilde{v} &= G_2\tilde{x} + \tilde{w}_1 \end{cases}$$

where $\tilde{x}, \tilde{v}$ denote their respective sequences, and G_2 is an operator defined as follows:

$$[G_2\tilde{x}](t) = \begin{cases} 0 & ,t < t_2 \\ \tilde{V}(\tilde{x}(t))H\tilde{x}(t) & ,t \geq t_2 \end{cases}$$

Notice that G_1 and G_2 are both stable. Moreover

$$\begin{aligned} \|G_2G_1\| &= \sup_{\tilde{v}\neq 0} \frac{\|G_2G_1\tilde{v}\|}{\|\tilde{v}\|} = \sup_{\tilde{v}\neq 0} \sqrt{\frac{\sum_{t=-\infty}^{\infty}[G_2G_1\tilde{v}]^2(t)}{\|\tilde{v}\|^2}} \\ &= \sup_{\tilde{v}\neq 0} \sqrt{\frac{\sum_{t=t_2}^{\infty}[G_2G_1\tilde{v}]^2(t)}{\|\tilde{v}\|^2}} \\ &\leq \sup_{\tilde{v}\neq 0} \sqrt{\frac{\sum_{t=-\infty}^{t_2-1}[\mu_{max}\frac{B(d)}{E(d)}\tilde{v}]^2(t)}{\|\tilde{v}\|^2}} + \\ &\quad + \sup_{\tilde{v}\neq 0} \sqrt{\frac{\sum_{t=t_2}^{\infty}\tilde{V}^2(t)[H(I - d\Phi_F)^{-1}dG\tilde{v}]^2(t)}{\|\tilde{v}\|^2}} \\ &\leq \mu_{max} \sup_{\tilde{v}\neq 0} \frac{\|\frac{B(d)}{E(d)}\tilde{v}\|}{\|\tilde{v}\|} \\ &\leq \mu_{max} \max_{[0,2\pi]} \left|\frac{B(e^{jw})}{E(e^{jw})}\right| \leq \mu_{max} < 1 \end{aligned}$$

being $|\tilde{V}(t)| \leq \mu_{max}$ by Lemma 2, and $\left|\frac{B(e^{jw})}{E(e^{jw})}\right| \leq 1$ by the spectral factorization equation. By the *small gain* theorem, $\tilde{x} \in \ell_2$. Hence $x(t)$ is bounded and $\lim_{t\to\infty} x(t) = x_w$.
(ii). Consider now a prediction made with $r(\cdot|t) \equiv w$. Then $c(k+t|t) = C_F\Phi_F^k x(t) + C_F \sum_{i=0}^{k-1} \Phi_F^i GV(1)w + DV(1)w = C_F\Phi_F^k[x(t) - x_w] + c_w$. By (i), $\exists T > 0$ such that $\forall t \geq T$

$$\|x(t) - x_w\| < \frac{\delta}{2\|C_F\|\phi_1} \quad (19)$$

Hence $\|c(k+t|t) - c_w\| < \delta$ and $c(k+t|t) \in \mathcal{C}\ \forall k \geq 0$. Then, $r(\cdot|t) \equiv w$ is admissible $\forall t \geq T$, i.e. $\lambda(t) = 0\ \forall t \geq T$.
CASE 2. Taking into account the modified PRM algorithm described after (10), $\lambda(t)$ is fixed to 1 for $t \geq t_3$ until a time step T which will be defined later. Substitute t_0 with $t_3 - 1$ in the proof of Lemma 1 and repeat the same reasoning, showing by this way that $\|x(t) - x_w\| \to 0$. Hence, exists a time $T \geq t_3$ such that (19) holds. The set $\mathcal{X}_w := \mathcal{X}(r_0(\cdot|t) \equiv w)$ is invariant, i.e. $x(t) \in \mathcal{X}_w \Rightarrow x(t+1) \in \mathcal{X}_w$. In fact, when $r(t+i|t) \equiv w$, $x(t+i|t)$ and $x(t+i)$ are identical. Then, for $t \geq T$ $\lambda(t)$ can be kept to 0 without violating the constraints.
(iii). Immediate, by PRM's definition. □

Remark 1. To be able to estabilish time t_1 and t_2, we have assumed t_w to be known by the reference governor. Actually one can ignore t_w a priori and estimate it on-line. For example one can define $t_w := t - 10$ when $w(t) = w(t-1) = \ldots = w(t-10)$. No problem occours if after this $w(t)$ changes. It is sufficient to wait again for the ultimate t_w to come.

Remark 2. A more effective rule for Case 2 can be constructed. Keeping $\lambda(t) = 1$ for $t > t_3$ means to use $r(\cdot|t_3 - 1)$ as reference. It is very probable that $\lambda(t_3) \simeq \lambda_{max}$, which makes the converge of $x(t)$ very slow. Then it is more convenient to wait for a time t_4 such that $\lambda(t_4 + 1)$ would be greater that $\lambda(t_4)$ and keep $\lambda(t) = 1$ for $t > t_4$.

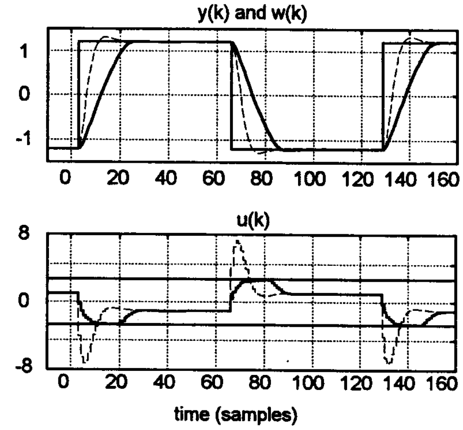

Fig. 2. LQ with preview when PRM is active (solid line) and in absence of constraints (dashed line).

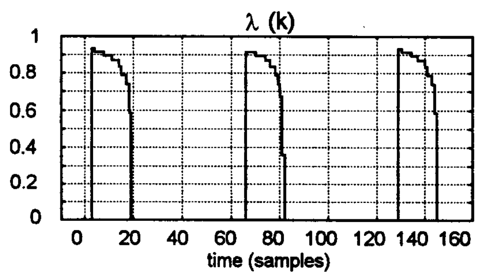

Fig. 3. LQ with preview + PRM in the presence of constraints.

4. Simulation results

Example 1. Consider the linear discrete-time plant

$$(1 - 1.9517d + 0.9517d^2)y(t) = (-0.0488d + 0.0488d^2)u(t) \tag{20}$$

obtained by sampling every $T_s = 0.005$s and zero-order holding the input of the continuous-time unstable plant

$$y(\tau) = \frac{1 + 10s}{(1 + 0.1s)(1 - 10s)}u(\tau) \tag{21}$$

for which a square-wave is chosen as desired trajectory $w(\cdot)$. Fig. 2 shows the behaviour of the LQ-with-preview regulated system without constraints ($r(\cdot) \equiv w(\cdot)$) and with PRM activated in order to fulfil the constraint

$$|u(t)| \leq 3 \tag{22}$$

Because of open-loop instability the transfer function from $w(t)$ to $u(t)$ is non-minimum phase. This explains the large unpleasant overshoots of the input.

The reference governor chooses a non-zero $\lambda(t)$, i.e. transforms $w(\cdot)$ into $r(\cdot|t)$, only during transients, when constraints would be violated. Moreover, in order to yield the shortest settling-time, it always selects the smallest value for $\lambda(t)$ compatible with the prescribed constraints.

Example 2. Consider the two-axis DC motor-driven computer numerical controlled (CNC) machine described in [5], where each axis is controlled by a conventional (1-DOF) lead-lag controller. This receives a zero-order held reference with a sampling period $T_s = 0.01s$ by the governor. The task is the positioning of the CNC tool in specified points of the x-y plane. The sequence of these points is not known a priori. Each axis is subject to the motor velocity constraint

$$|\omega_m(t)| \leq 12.57 \text{ rad/s}$$

Fig. 4 shows the actual path described by the CNC tool on the x-y plane. Fig. 5 and Fig. 6 depict respectively the motor velocities and the values selected for vector λ. Finally Fig. 7 shows how the desired x-axis trajectory $w_x(t)$ is filtered by the governor in the actual reference $r_x(t)$. In this example a small prediction of $M = 2$ steps suffices. The reference governor makes its choices with a discretized model of the closed-loop transfer functions.

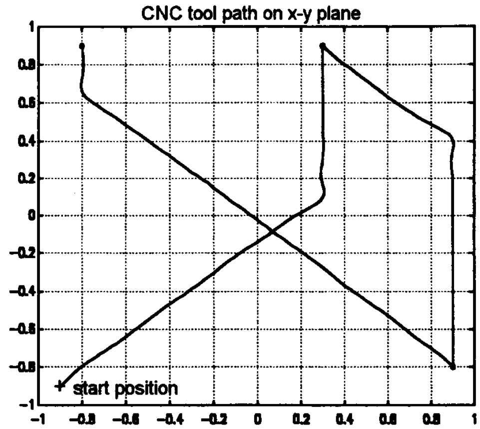

Fig. 4. CNC tool path on x-y plane.

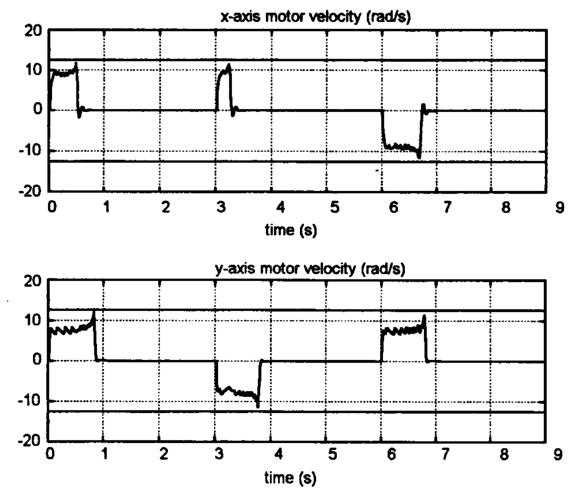

Fig. 5. Motor velocities (rad/s).

5. Conclusions.

On-line predictive reference managing schemes can be effective tools for solving feedback control problems in the presence of input and state-related constraints. These schemes can be embodied in any feedback control system, provided that model-based predictions of the constrained vector can be carried out within two subsequent sampling times. The

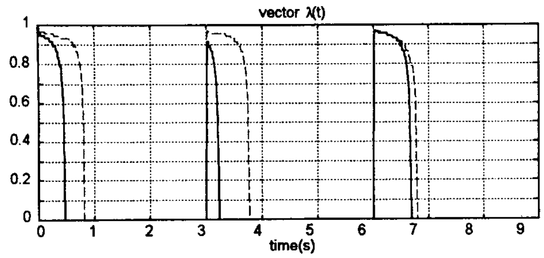

Fig. 6. Vector λ: x-component (solid-line) and y-component (dashed line).

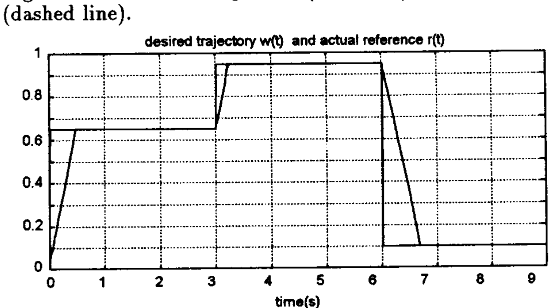

Fig. 7. Desired trajectory and actual filtered reference

specific underlying control law analyzed in this paper consists of an optimal LQ controller with preview where constraints can be input saturations, input increment saturations, output over/undershoot limitations, etc..

An open problem is how to robustify control systems with PRM when only a coarse plant model is available. Robustness could be increased by employing more stringent constraints, with margins of error dependent on the uncertainties. The tradeoff between robustness and performance deserves appropriate investigation.

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