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Constrained Model Predictive Control Based on Reduced-Order Models

Pantelis Sopasakis⋆, Daniele Bernardini⋆ and Alberto Bemporad⋆

***Abstract*— The need for reduced-order approximations of dynamical systems emerges naturally in model-based control of very large-scale systems, such as those arising from the discretisation of partial differential equation models. The controller based on the reduced-order model, when in closed-loop with the large-scale system, ought to endow certain properties, *in primis* stability, but also satisfaction of state constraints and recursive computability of the control law in the case of constrained control.**

In this paper we introduce a new approach to the design of model predictive controllers to meet the aforementioned requirements while the on-line complexity is essentially tantamount to the one that corresponds to the low-dimensional approximate model.

I. Introduction

In a multitude of industrial control applications one must deal with dynamical models involving a large-scale number of states. Finite-state approximations of infinite-dimensional dynamical systems described by Partial Differential Equations (PDEs) are examples of such cases. These include the modelling of sloshing of liquids [1] (which is of particular interest in aerospace applications), distribution of anti-tumor drugs in the organism [2], heating and air-conditioning of buildings [3], seismic excitation of buildings [4], control of flexible structures [5], and many others. The infinite-dimensional nature of PDEs requires approximate solution methods such as the well established finite-elements method [6]. Such an approximation leads eventually to continuous or discrete-time systems with a large number of states (such as a few hundreds up to tens of thousands). However, in many cases, the dynamics of the most significant states of such systems that one needs to control can be often approximated using no more but a few state variables, at the price of introducing some (desirably small) modelling error. Model-order reduction theory provides methods that enable us to decompose a dynamical system into its dominant and neglected dynamics [7], [8]. Among many model-reduction techniques, it is worth mentioning the method of goal-oriented model-based reduction [9] where the decomposition is conditioned by the optimisation of a performance index.

Model Predictive Control (MPC) owes its popularity – *inter alia* – to its ability to take inherently into account hard state and input constraints [10]. Because of the uncertainty introduced by reducing the order of the model, it is common practice in MPC based on reduced-order models to impose only soft constraints on the state variables [11]. In other studies the dynamics of the actual high-dimensional system is completely disregarded, thus no guarantee is provided in regard to the satisfaction of constraints or the recursive feasibility of the control algorithm [12]. These shortcomings reflect on the fact that the vast majority of applications of reduced-order control does not take into account the physical borders of the underlying system [13].

Theoretically, the problem of designing an MPC controller for reduced-order systems can be cast as an output MPC problem [14]. This, however, would lead to the formulation of an optimisation problem in a high dimensional space. Despite the recent developments in the field of MPC for large-scale systems [15], optimisation in high-dimensional spaces remains cumbersome.

The MPC approach proposed in this paper does not require online operations in the original high-dimensional space while it provides stability and constraint satisfaction guarantees for the full-order system.

II. Notation

Let $\mathbb{R}_+$, $\mathbb{R}^n$ and $\mathbb{R}^{m\times n}$ denote the set of non-negative numbers, the set of column real vectors of length n, and the set of m-by-n real matrices respectively. For a real n-by-n matrix P, $P \succ 0$ ($P \succeq 0$) denotes positive (semi)definiteness. For any $s \in \mathbb{R}$, $\lfloor s \rfloor$ denotes the maximum integer that does not exceed s. For any nonnegative integers $k_1 \leq k_2$, the finite set $\{k_1, \ldots, k_2\}$ is denoted by $\mathbb{N}_{[k_1,k_2]}$. We denote by $\|\cdot\|_2$ and $\|\cdot\|_\infty$ the Euclidean and the infinity norms, and by $\mathcal{B}_2$ and $\mathcal{B}_\infty$ their unitary balls. For a $x \in \mathbb{R}^n$, we denote by $|x|$ the vector of $\mathbb{R}^n$ whose i-th element is $|x_i|$. Any set of the form $\mathcal{P} := \{x \in \mathbb{R}^n \mid \|P(x-x_0)\|_\infty \leq 1\}$, where $P \in \mathbb{R}^{n\times n}$ is an invertible matrix and $x_0 \in \mathbb{R}^n$, is called a *parallelotope*. The *volume* of $\mathcal{P}$ is $\mathrm{vol}(\mathcal{P}) := \mathrm{vol}(\mathcal{B}_\infty) \cdot |\det(P^{-1})|$, where $\mathrm{vol}(\mathcal{B}_\infty) = 2^n$. For any two convex sets $\mathcal{A}$ and $\mathcal{B}$ in $\mathbb{R}^n$ we denote by $\mathcal{A} \oplus \mathcal{B}$ their Minkowski sum, that is $\mathcal{A} \oplus \mathcal{B} := \{x \in \mathbb{R}^n \mid x = a + b,\, a \in \mathcal{A},\, b \in \mathcal{B}\}$. We denote by $\mathcal{A} \ominus \mathcal{B}$ their Pontryagin difference, that is $\mathcal{A} \ominus \mathcal{B} := \{x \in \mathbb{R}^n \mid x + b \in \mathcal{A},\, b \in \mathcal{B}\}$. For $K \in \mathbb{R}^{m\times n}$, $K\mathcal{A} = \{x \in \mathbb{R}^m \mid x = Ka,\, a \in \mathcal{A}\}$. For a set-valued function $F : \mathbb{R}^n \to \mathbb{R}^m$, we define its domain to be the set $\operatorname{dom} F = \{x \mid F(x) \neq \varnothing\}$.

III. Reduced-Order Systems

Consider a linear time-invariant system of the form

$$x_{k+1} = A_{11}x_k + A_{12}w_k + B_1u_k, \tag{1a}$$

$$w_{k+1} = A_{21}x_k + A_{22}w_k + B_2u_k, \tag{1b}$$

where $x_k \in \mathbb{R}^{n_x}$ is the measured state, $w_k \in \mathbb{R}^{n_w}$ is the unmeasured state and $u_k \in \mathbb{R}^{n_u}$ is the input. We assume that that the pair (A_{11}, B_1) is stabilizable. The vectors $x \in \mathbb{R}^{n_x}$

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⋆ IMT Institute for Advanced Studies Lucca, Piazza San Ponziano 6, Lucca, Italy. Emails: pantelis.sopasakis@imtlucca.it, daniele.bernardini@imtlucca.it, alberto.bemporad@imtlucca.it.

and $w \in \mathbb{R}^{n_w}$ stand, respectively, for the dominant and the neglected part of the overall state of the system and it is desirable that $n_x \ll n_w$. We aim at enforcing the following constraints on the dominant state and input variables

$$x_k \in \mathcal{X},\ u_k \in \mathcal{U}, \tag{2}$$

for all $k \in \mathbb{N}$, where $\mathcal{X}$ and $\mathcal{U}$ are closed polyhedra containing the origin in their interiors, and $\mathcal{U}$ is compact. We do not consider constraints on the unmeasured state w_k. Without loss of generality, we also assume that some information on the initial value of w_k is available, namely that

$$w_0 \in \mathcal{W} := \{w \in \mathbb{R}^{n_w} \mid w' W^{-1} w \leq 1\}, \tag{3}$$

i.e., w_0 lies in a non-degenerate ellipsoid centered at the origin. In general, $\mathcal{W}$ can be an ellipsoid of the form $\{w \in \mathbb{R}^{n_w} \mid (w - w_c)' W^{-1}(w - w_c) \leq 1\}$ as long as it includes the origin in its interior. The set $\mathcal{W}$ is taken to be an ellipsoid and not a polytope for reasons that will be elucidated later.

We assume that there is an $\varepsilon \in (0,1)$ so that $A_{22}\mathcal{W} \subseteq \varepsilon\mathcal{W}$, i.e., $\varepsilon W - A_{22} W A_{22}' \succeq 0$. For instance, if $\mathcal{W}$ is a Euclidean ball, this condition boils down to $\|A_{22}\|_2 < \varepsilon$.

The term $A_{12} w_k$ in (1) plays the role of an additive disturbance and enables us to make use of principles of robust model predictive control. The nominal counterpart of (1), obtained by neglecting w_k, is given by

$$z_{k+1} = A_{11} z_k + B_1 v_k, \tag{4}$$

where $z \in \mathbb{R}^{n_x}$ and $v \in \mathbb{R}^{n_u}$ are the nominal state and input vectors, respectively. Let K be a stabilizing gain for the pair (A_{11}, B_1) and define

$$A_K := A_{11} + B_1 K. \tag{5}$$

We introduce now the deviation variable $e \in \mathbb{R}^{n_x}$ such that

$$e = x - z, \tag{6a}$$
$$u = v + Ke. \tag{6b}$$

The evolution of e is described by

$$e_{k+1} = A_{11} e_k + B_1(u_k - v_k) + A_{12} w_k. \tag{7}$$

By substituting (6) in (1a), the dynamics of x is given by $x_{k+1} = A_{11}x_k + A_{12}w_k + B_1 v_k + B_1 K e_k$, and the dynamics of the deviation variable becomes:

$$e_{k+1} = A_K e_k + A_{12} w_k. \tag{8}$$

The following lemma is instrumental for the the proposed reduced-order MPC methodology.

Lemma 1: Let

$$\hat{\mathcal{S}}_K^{(\infty)} := T_1 \mathcal{W} \oplus T_2 \mathcal{X} \oplus T_3 \mathcal{U}, \tag{9}$$

where

$$T_1 := (I - A_K)^{-1} A_{12}, \tag{10a}$$
$$T_2 := (I - A_K)^{-1} A_{12} (I - A_{22})^{-1} A_{21}, \tag{10b}$$
$$T_3 := (I - A_K)^{-1} A_{12} (I - A_{22})^{-1} B_2, \tag{10c}$$

and assume that $x_k \in \mathcal{X}$ and $u_k \in \mathcal{U}$ for all $k \in \mathbb{N}$. If $e_0 \in \hat{\mathcal{S}}_K^{(\infty)}$, then $e_k \in \hat{\mathcal{S}}_K^{(\infty)}$ for all $k \in \mathbb{N}$.

Proof: Because of (1b) we have that

$$w_k = A_{22}^k w_0 + \sum_{j=0}^{k-1} A_{22}^j (A_{21} x_j + B_2 u_j). \tag{11}$$

Hence for all $x_k \in \mathcal{X}$ and $u_k \in \mathcal{U}$, it is $w_k \in \mathcal{W}_k$, where $\mathcal{W}_0 = \mathcal{W}$ and

$$\mathcal{W}_k = A_{22}^k \mathcal{W} \oplus \sum_{j=0}^{k-1} A_{22}^j (A_{21}\mathcal{X} \oplus B_2 \mathcal{U}), \tag{12}$$

for all $k \in \mathbb{N}$. It holds $\mathcal{W}_k \subseteq \hat{\mathcal{W}}$ for all $k \in \mathbb{N}$, where

$$\hat{\mathcal{W}} := \mathcal{W} \oplus (I - A_{22})^{-1}(A_{21}\mathcal{X} \oplus B_2\mathcal{U}). \tag{13}$$

Then, $e_k \in \mathcal{S}_K^{(k)}$ with

$$\mathcal{S}_K^{(k)} := \bigoplus_{j=0}^{k-1} A_K^j A_{12} \mathcal{W}_j, \tag{14}$$

for all $k \in \mathbb{N}$. Let $\mathcal{S}_K^{(\infty)}$ denote the Painlevé-Kuratowski limit of $\mathcal{S}_K^{(k)}$ as $k \to \infty$ [16]. An over-approximation of $\mathcal{S}_K^{(\infty)}$, namely $\hat{\mathcal{S}}_K^{(\infty)} \supseteq \mathcal{S}_K^{(\infty)}$, can be easily computed as

$$\hat{\mathcal{S}}_K^{(\infty)} = (I - A_K)^{-1} A_{12} \hat{\mathcal{W}}, \tag{15}$$

using the fact that $\hat{\mathcal{W}} \supseteq \mathcal{W}_k$ for all $k \in \mathbb{N}$. By substituting (13), Equation (15) can be rewritten as in (9). ■

Throughout the rest of the paper we assume that

$$\hat{\mathcal{S}}_K^{(\infty)} \subseteq \mathcal{X} \tag{16a}$$
$$K\hat{\mathcal{S}}_K^{(\infty)} \subseteq \mathcal{U}. \tag{16b}$$

For computational convenience, we shall overapproximate the ellipsoid $T_1\mathcal{W}$ in (9) by a polytope. Various bounding-polytope methods have been proposed in the literature such as minimum-volume parallelotopes [17], zonotopes [18] or rectangular polytopes [19]. In this paper we employ the following result to calculate minimum-volume outbounding parallelotopes.

Proposition 1: Let $\mathcal{E} = \{x \in \mathbb{R}^n \mid x' W^{-1} x \leq 1\}$ be an ellipsoid and $\mathcal{P}$ a minimum-volume outbounding parallelotope for it. Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a linear invertible transformation. Then $F(\mathcal{P})$ is a minimum volume outbounding parallelotope for the ellipsoid $F(\mathcal{E})$.

Proof: The proof follows from the fact that $\mathrm{vol}(F(\mathcal{P})) = |\det(F)|\,\mathrm{vol}(\mathcal{P})$ [20]. Let $\hat{\mathcal{P}}$ be an outbounding parallelotope for $F(\mathcal{E})$; then $F^{-1}(\hat{\mathcal{P}})$ is an outbounding parallelotope for $\mathcal{E}$, and $\mathrm{vol}(F^{-1}(\hat{\mathcal{P}})) \geq \mathrm{vol}(\mathcal{P})$, so multiplying by $|\det F|$ we get $\mathrm{vol}(\hat{\mathcal{P}}) \geq \mathrm{vol}(F(\mathcal{P}))$, which completes the proof. ■

Given an ellipsoid $\mathcal{E} = \{x \mid x' W^{-1} x \leq 1\}$, define $F := W^{\frac{1}{2}}$. Since $\mathcal{B}_\infty := \{x \mid -1 \leq x \leq 1\}$ is a minimum-volume outbounding parallelotope for $\mathcal{B}_2 := \{x \mid x'x \leq 1\}$ and $F\mathcal{B}_2 = \mathcal{E}$, then $F\mathcal{B}_\infty$ is a minimum-volume outbounding parallelotope for $\mathcal{E}$. Hence, a minimum volume outbounding parallelotope for

$$T_1\mathcal{W} = \{x \mid x'(T_1 W T_1')^{-1} x \leq 1\} \tag{17}$$

is given by $\Gamma\mathcal{B}_\infty$, with $\Gamma := (T_1 W T_1')^{\frac{1}{2}}$.

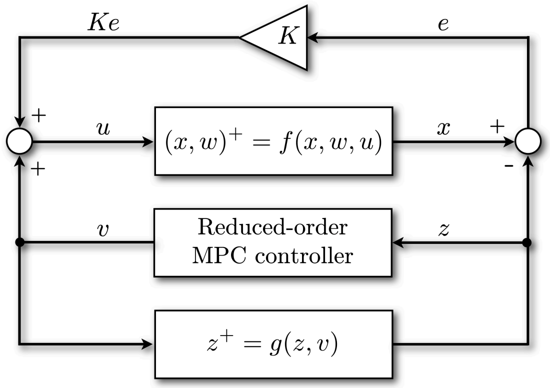

Fig. 1. Feedback control using a reduced-order model.

IV. Reduced-Order Model Predictive Control

A. Basic Setup

In this section we describe a methodology for the design of an MPC controller for the nominal, low-dimensional system (4), in such a way that the measured state x of (1) is stabilized and the constraints (2) are fulfilled. The computation of $\hat{\mathcal{S}}_K^{(\infty)}$ is performed offline and no high-complexity operations are performed online.

The controlled system is given by (1a) where the term $A_{12}w_k \in A_{12}\hat{\mathcal{W}}$ plays the role of a bounded additive uncertainty and promotes the use of principles of tubed-based robust MPC.

Let us introduce the sets

$$\mathcal{Z} := \mathcal{X} \ominus \hat{\mathcal{S}}_K^{(\infty)} \tag{18a}$$

$$\mathcal{V} := \mathcal{U} \ominus K\hat{\mathcal{S}}_K^{(\infty)} \tag{18b}$$

which are tightened variants of $\mathcal{X}$ and $\mathcal{U}$, respectively and $\mathcal{Z}$ and $\mathcal{V}$ are nonempty because of (16). Let $N \in \mathbb{N}$ be the prediction horizon. We impose the following constraints on the state and the input of the nominal system:

$$z_k \in \mathcal{Z},\ \forall k \in \mathbb{N}_{[1,N-1]}, \tag{19a}$$

$$v_k \in \mathcal{V},\ \forall k \in \mathbb{N}_{[0,N-1]}, \tag{19b}$$

and the terminal constraint

$$z_N \in \mathcal{Z}_f, \tag{20}$$

where $\mathcal{Z}_f \subseteq \mathcal{Z}$. The MPC optimisation problem amounts to minimizing the following cost function:

$$V_N(z, \mathbf{v}) := \sum_{k\in\mathbb{N}_{[0,N-1]}} \ell(z_k, v_k) + V_f(z_N) \tag{21}$$

subject to the dynamics of z, i.e., $z_{k+1} = A_{11}z_k + B_1 v_k$ and the aforementioned constraints. We assume that $\ell(z, v) = z'Qz + v'Rv$ where $Q = Q' \succeq 0$, $R = R' \succ 0$, and that $V_f(x) = x'Px$ where $P = P' \succeq 0$. We formulate the following optimisation problem:

$$\mathbb{P}_N(z) : V_N^\star(z) = \min_{\mathbf{v}\in\mathcal{V}(z)} V_N(z, \mathbf{v}) \tag{22}$$

where $\mathcal{V}(z)$ is the following multivalued mapping:

$$\mathcal{V}(z) := \left\{ \mathbf{v} \,\middle|\, \begin{array}{l} z_0 = z, \\ z_{k+1} = A_{11}z_k + B_1 v_k,\ \forall k \in \mathbb{N}_{[1,N-1]}, \\ z_k \in \mathcal{Z},\ \forall k \in \mathbb{N}_{[1,N-1]}, \\ z_N \in \mathcal{Z}_f, \\ v_k \in \mathcal{V},\ \forall k \in \mathbb{N}_{[0,N-1]} \end{array} \right\}. \tag{23}$$

The solution of the optimisation problem (22) yields the optimal sequence $\mathbf{v}^\star(z) = \{v_k^\star(z)\}_{k\in\mathbb{N}_{[0,N-1]}}$ and leads to the MPC control law:

$$\kappa_N(z) = v_0^\star(z), \tag{24}$$

where z is the current state. The terminal cost matrix P and the set $\mathcal{Z}_f$ are chosen so that the MPC control law $v_k = \kappa_N(z_k)$ stabilizes $z_{k+1} = A_{11}z_k + B_1 v_k$ over the domain $\mathcal{Z}_N := \operatorname{dom} \mathcal{V}$ by the standard stabilizing conditions [10, Theorem 2.24].

The actual control action applied to the system will then be

$$\tilde{\kappa}_N(z, x) = K(x - z) + \kappa_N(z) \tag{25}$$

As a result, the closed-loop state trajectory in presence of κ_N will satisfy:

$$x_{k+1} = A_{11}x_k + B_1(\tilde{\kappa}_N(z_k, x_k)) + A_{12}w_k \tag{26}$$

and the nominal system's trajectory will be such that

$$z_{k+1} = A_{11}z_k + B_1\kappa_N(z_k) \tag{27}$$

with $z_0 = x_0$. If $e_0 \in \hat{\mathcal{S}}_K^{(\infty)}$, then $e_k \in \hat{\mathcal{S}}_K^{(\infty)}$ for all $k \in \mathbb{N}$, thus $x_k \in \{z_k\} \oplus \hat{\mathcal{S}}_K^{(\infty)}$.

Theorem 1: The set $\hat{\mathcal{S}}_K^{(\infty)} \times \{0\}$ is exponentially stable for the system (26) and (27) (with state variable $\left[\begin{smallmatrix} x \\ z \end{smallmatrix}\right]$) over the domain of attraction $\hat{\mathcal{S}}_K^{(\infty)} \times \mathcal{Z}_N$.

Proof: The proof goes along the lines of [10, Proposition 3.15]. ■

B. Set-membership estimation of the neglected variables

In the previous section we assumed that $w_k \in \hat{\mathcal{W}}$ for all $k \in \mathbb{N}$ provided that $x_k \in \mathcal{X}$ and $u_k \in \mathcal{U}$. In this section we employ a set-valued observer for $\mathcal{W}_{k|k}$ so that $w_k \in \mathcal{W}_{k|k}$ using online measurements of x_k and u_k. Additionally, we predict a series of sets $\mathcal{W}_{k+j|k}$ for $j \in \mathbb{N}_{[1,N]}$ so that $w_{k+j} \in \mathcal{W}_{k+j|k}$. These sets will be used to enable time-varying constraint tightening, so as to reduce the conservativeness arising when the time-invariant constraints (19) are enforced throughout the prediction horizon.

At every $k \in \mathbb{N}$ we measure the state x_k and we compute $\mathcal{W}_{k|k}$ so that $\mathcal{W}_{0|0} = \bar{\mathcal{W}}$, where $\bar{\mathcal{W}}$ is a polytopic overapproximation of $\mathcal{W}$, and for $k \in \mathbb{N}$ we apply the following correction step:

$$\mathcal{W}_{k-1|k} = \left\{ w \in \mathcal{W}_{k-1|k-1} \middle| A_{12}w = x_k - A_{11}x_{k-1} - B_1 u_{k-1} \right\} \tag{28}$$

and the prediction step:

$$\mathcal{W}_{k|k} = A_{21}x_{k-1} + B_2 u_{k-1} \oplus A_{22}\mathcal{W}_{k-1|k}.$$

Subsequently, we may compute a series of sets $\mathcal{W}_{k+j|k}$ for $j \in \mathbb{N}_{[1,N-1]}$ as

$$\mathcal{W}_{k+j|k} = A_{21}\mathcal{X}_{k+j-1|k} \oplus B_2\mathcal{U} \oplus A_{22}\mathcal{W}_{k+j-1|k}, \quad (29)$$

where $\mathcal{X}_{k+j|k}$ is a sequence of sets such that $\mathcal{X}_{k|k} = \{x_k\}$ and for $j \in \mathbb{N}_{[1,N-1]}$

$$\mathcal{X}_{k+j|k} = \mathcal{X} \cap (A_{11}\mathcal{X}_{k+j-1|k} \oplus B_1\mathcal{U} \oplus A_{12}\mathcal{W}_{k+j-1|k}). \quad (30)$$

For the deviation variables we have that $e_{k+j} \in S_{k+j|k}$ for $j \in \mathbb{N}_{[0,N]}$, where

$$S_{k+j|k} = \bigoplus_{i=0}^{j} A_K^i A_{12}\mathcal{W}_{k+i|k}. \quad (31)$$

We shall use $\mathcal{S}_{k+j|k}$ as a time-varying set-membership estimation for the deviation variable e_{k+j}. However, the computations involved in the aforementioned procedure bring about high complexity. In order to avoid online computations of high complexity we introduce the following estimator:

$$\mathcal{H}_{k|k} = A_{12}A_{22}^k\mathcal{W} \oplus A_{12}\sum_{j=0}^{k-1} A_{22}^j(A_{21}x_j + B_2u_j). \quad (32)$$

Since $A_{22}\mathcal{W} \subseteq \varepsilon\mathcal{W}$, the expression for $\mathcal{H}_{k|k}$ can be overapproximated by:

$$\mathcal{H}_{k|k} = \varepsilon^k A_{12}\bar{\mathcal{W}} \oplus A_{12}\sum_{j=0}^{k-1} T(x_j, u_j, \varepsilon^j)\mathcal{B}_\infty, \quad (33)$$

where

$$T(x_j, u_j, \varepsilon^j) = \varepsilon^j \operatorname{diag}(|A_{21}x_j + B_2u_j|). \quad (34)$$

However, this estimator does not consider the current measurement x_k. For all subsequent time instants $j \in \mathbb{N}_{[1,N-1]}$ we have

$$\mathcal{H}_{k+j|k} = A_{12}A_{21}\mathcal{X}_{k+j-1|k} \oplus A_{12}B_2\mathcal{U} \oplus \varepsilon\mathcal{H}_{k+j-1|k}, \quad (35a)$$

$$\mathcal{X}_{k+j|k} = \mathcal{X} \cap (A_{11}\mathcal{X}_{k+j-1|k} \oplus \mathcal{H}_{k+j-1|k} \oplus B_1\mathcal{U}). \quad (35b)$$

The number of inequalities needed to represent $\mathcal{H}_{k+j|k}$ is bounded and does not increase with time.

Now $S_{k+j|k}$ can be expressed in terms of the lower dimensional sets $\mathcal{H}_{k+j|k}$ as follows:

$$S_{k+j|k} = \bigoplus_{i=0}^{j} A_K^i \mathcal{H}_{k+i|k} \quad (36)$$

$$\mathcal{Z}_{k+j|k} = \mathcal{X} \ominus \mathcal{S}_{k+j|k}, \quad (37)$$

$$\mathcal{V}_{k+j|k} = \mathcal{U} \ominus K\mathcal{S}_{k+j|k}. \quad (38)$$

This allows us to reformulate the MPC problem making use of the online estimation of $\mathcal{W}_{k|k}$. The new MPC problem has the form:

$$\bar{\mathbb{P}}_N(z_k, \mathcal{H}_{k|k}) : V_N^\star(z_k, \mathcal{H}_{k|k}) = \min_{\mathbf{v} \in \bar{\mathcal{V}}(z_k, \mathcal{H}_{k|k})} V_N(z_k, \mathbf{v}), \quad (39)$$

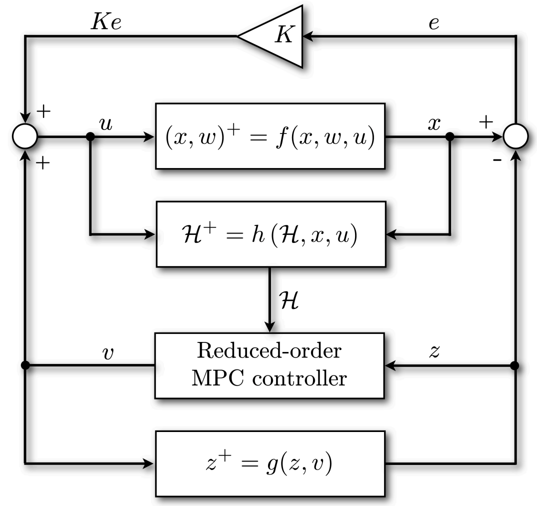

Fig. 2. Feedback control using a reduced-order model and on-line estimation of the uncertainty on w.

where

$$\bar{\mathcal{V}}(z_k, \mathcal{H}_{k|k}) := \left\{ \mathbf{v} \;\middle|\; \begin{array}{l} z_{k+j+1} = A_{11}z_{k+j} + B_1v_{k+j}, \\ \forall j \in \mathbb{N}_{[0,N-1]}, \\ z_{k+j} \in \mathcal{Z}_{k+j|k},\ \forall j \in \mathbb{N}_{[1,N-1]}, \\ z_{k+N} \in \mathcal{Z}_f, \\ v_{k+j} \in \mathcal{V}_{k+j|k},\ \forall j \in \mathbb{N}_{[0,N-1]} \end{array} \right\}. \quad (40)$$

The solution of the optimisation problem (22) yields the optimal sequence $\mathbf{v}^\star(z, \mathcal{H}_{k|k}) = \{v_k^\star(z, \mathcal{H}_{k|k})\}_{k \in \mathbb{N}_{[0,N-1]}}$ and leads to the MPC control law $\rho_N(z_k, \mathcal{H}_{k|k}) = v_0^\star(z_k, \mathcal{H}_{k|k})$. Similar stability results as the ones given in Theorem 1 hold for this case as well.

Corollary 1: The set $\hat{\mathcal{S}}_K^{(\infty)} \times \{0\}$ is exponentially stable for the system of (26) and (27) (with state variable $\left[\begin{smallmatrix} x \\ z \end{smallmatrix}\right]$) over the domain of attraction $\hat{\mathcal{S}}_K^{(\infty)} \times \mathcal{Z}_N$. Additionally, if $\mathcal{H}_{k|k} \to \mathcal{H}^\star$ and $\mathcal{S}^\star := (I - A_K)^{-1}\mathcal{H}^\star$, then $\mathcal{S}^\star \times \{0\}$ is exponentially stable for (26) and (27).

C. Computational Complexity

The minimum-volume outbounding parallelotope of $T_1\mathcal{W}$ is represented by $2n_x$ inequalities. Let p_x and p_u be the number of inequalities in $\mathcal{X}$ and $\mathcal{U}$ respectively. Let $p := \max(p_x, p_u, 2n_x)$ and, according to McMullen's formula [21],

$$q := \binom{p - \lfloor \frac{n_x+1}{2} \rfloor}{p - n_x} + \binom{p - \lfloor \frac{n_x+2}{2} \rfloor}{p - n_x}$$

is an upper-bound for the number of inequalities that represent a polytope with p vertices in $\mathbb{R}^{n_x}$. Then $\hat{\mathcal{S}}_K^{(\infty)}$ is represented by $\mathcal{O}(q^{n_x-1})$ inequalities [22] which corresponds to a reasonable complexity since the respective computations are carried out offline.

If $\mathcal{W}$ was chosen to be a polytope in $\mathbb{R}^{n_w}$ the operation $T_1\mathcal{W}$ in (9) would require iteration over its vertices, which would be at least 2^{n_w}. Such a complexity would be prohibitive for large values of n_w, even for offline computations.

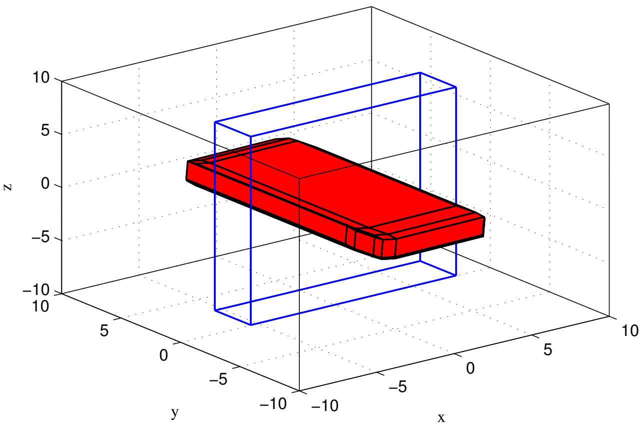

Fig. 3. Sets $\mathcal{X}$ (black bounding box), $\mathcal{Z}$ (blue), and $\hat{\mathcal{S}}_K^{(\infty)}$ (red).

Both optimisation problems $\mathbb{P}_N(z_k)$ and $\bar{\mathbb{P}}_N(z_k, \mathcal{H}_{k|k})$ involve $(n_u + n_x)N$ decision variables and $n_x N$ equality constraints. If n_f is the number of inequalities for $\mathcal{Z}_f$, $\mathbb{P}_N(z_k)$ counts $n_f + n_c(N-1)$ inequalities, hence, $\mathbb{P}_N(z_k)$ is no more complex than an MPC problem for a system of state dimension n_x, input dimension n_u and constraints as in (2).

Finally, in (40) one may impose the constraints $z_{k+j} \in \mathcal{Z}_{k+j|k}$ only for the first r predicted values, with $r \in \mathbb{N}_{[1,N-2]}$, and use simply $z_{k+j} \in \mathcal{Z}$ for $j \in \mathbb{N}_{[r+1,N-1]}$. This is particularly useful for large prediction horizons.

V. Example

In this section we test the performance of the proposed approach on a numerical example. We consider a system of the form (1) with $n_u = 2$, $n_x = 3$ and $n_w = 500$. The matrices in (1) are randomly generated such that (A_{11}, B_1) is stabilizable. The system is slightly open-loop unstable, as shown in Figure **??**. The constraints (2) are chosen to be $\mathcal{X} = \{x \in \mathbb{R}^{n_x} \mid \|x\|_\infty \leq 10\}$, $\mathcal{U} = \{u \in \mathbb{R}^{n_u} \mid \|u\|_\infty \leq 1\}$. The initial conditions on $w(k)$ are set as in (3), where W has random eigenvalues in the range between 500 and 1000. The condition $A_{22}\mathcal{W} \subseteq \varepsilon\mathcal{W}$ is satisfied with $\varepsilon = 0.012$.

For the MPC cost function (21) we consider the weights $Q = I_{n_x}$, and $R = 5I_{n_u}$. The terminal weight P and the gain K are obtained by solving a Riccati equation. We compute the set $\hat{\mathcal{S}}_K^{(\infty)}$ by (15), and the tightened constraints $\mathcal{Z} = \mathcal{X} \ominus \hat{\mathcal{S}}_K^{(\infty)}$ and $\mathcal{V} = \mathcal{U} \ominus K\hat{\mathcal{S}}_K^{(\infty)}$, which result in nonempty box sets as shown in Figures 3–4. The terminal set $\mathcal{Z}_f$ is computed as the maximal positively invariant set for the system $z_{k+1} = A_K z_k$, with $z_k \in \mathcal{Z}$.

We compare the performance of the proposed Reduced-Order MPC (ROMPC) with a Full-Order MPC (FOMPC), that exploits measurements of the full state vector $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ of system (1) to minimize the cost function (21) with $z = x$ and $v = u$, subject to constraints (2). Using a prediction horizon $N = 10$, both control schemes succeed in stabilizing the closed-loop system, as shown in Figures 5 and 6.

We observe that the full state $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ converges to the origin, both with ROMPC and FOMPC, and that constraints on x

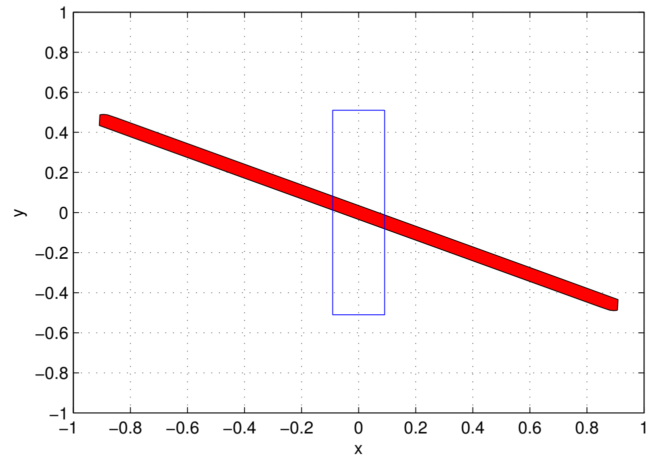

Fig. 4. Sets $\mathcal{U}$ (black bounding box), $\mathcal{V}$ (blue), and $K\hat{\mathcal{S}}_K^{(\infty)}$ (red).

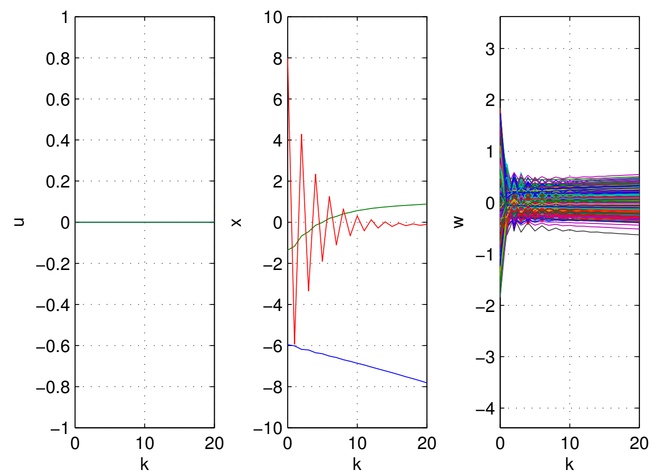

Fig. 5. Trajectories x_k and w_k with $u_k = 0$.

and u are always satisfied. Moreover, we see how tightening the constraints in x and u has the consequence of attenuating oscillations in the evolution of w. CPU times[1] required for setting up and solving the MPC problem for both ROMPC and FORMPC are reported in Table I.

The high computational complexity that accompanies the implementation of a full-order MPC renders it unsuitable for practical applications, especially for high values of n_w. The requirement of full state feedback is also hard to meet. With an average of around 14 seconds required to solve an instance of the problem, FOMPC would only be viable to control plants with very slow dynamics. On the contrary, the complexity of ROMPC depends mainly on the size of the reduced-order system, deeming it a suitable candidate for real-time control of this class of systems.

VI. Conclusions

The proposed methodology addresses the problem of designing a model predictive controller of large-scale systems

[1]Reported data have been obtained on a 2.7GHz processor, using CDD for polyhedral computations and CPLEX v12 as the QP solver.

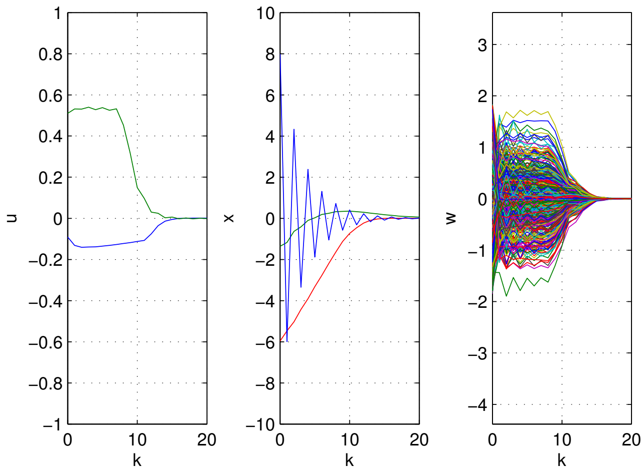

Fig. 6. Trajectories of u_k, x_k and w_k with Reduced-Order MPC.

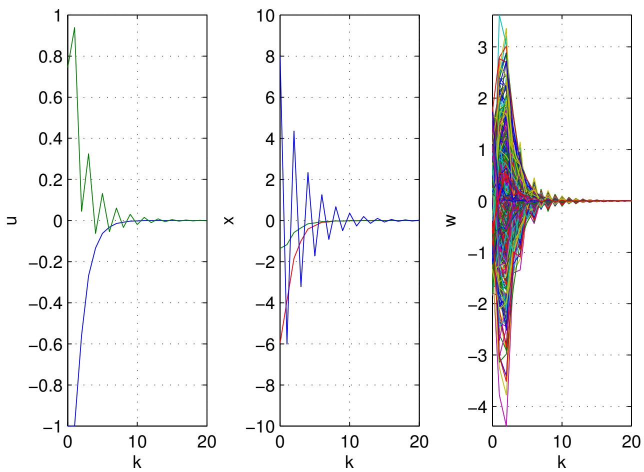

Fig. 7. Trajectories of u_k, x_k and w_k with Full-Order MPC.

that admit a decomposition into dominant and neglected dynamics. The proposed MPC schemes do not involve any high-order or complex operation to be performed online and only require feedback of the dominant states. Thus, they can be used in applications where high sampling-rate is necessary and/or reliable full state measurements are not possible to obtain. At the same time the neglected dynamics are taken implicitly into account so as to guarantee closed-loop asymptotic stability and satisfaction of constraints on the dominant state variables. At this stage, the problem of enforcing constraints on the neglected states has not been faced, as this would lead either to over-conservativeness of the approach or to complex numerical operations in the full state space.

TABLE I

COMPUTATIONAL TIMES

	Reduced-Order MPC	Full-Order MPC
Computation of P, K, $\mathcal{Z}$, $\mathcal{V}$, $\mathcal{Z}_f$	1.3s	14.4s
Solution of the MPC problem (avg.)	8.4ms	14297ms
Solution of the MPC problem (st. dev)	0.42ms	859ms

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