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# The Rendezvous Dynamics under Linear Quadratic Optimal Control

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*Abstract*— **This paper investigates the dynamics of networks of systems achieving rendezvous under linear quadratic optimal control. While the dynamics of rendezvous were studied extensively for the symmetric case, where all systems have exactly the same dynamics (such as simple integrators), this paper investigates the rendezvous dynamics for the general case when the dynamics of the systems may be different. We show that the rendezvous is stable and that the post-rendezvous dynamics of the network of systems is entirely defined by the common eigenvalues with common eigenvectors output image. The approach is also extended to the case of constraints on systems states,inputs, and outputs.**

## I. Introduction

Controlling a network of systems is a challenging task that occurs in many applications. As discussed in [1], studies on this subject started already in the eighties for the case of teams of mobile robots, and grew considerably in the nineties because of the availability of low-cost and effective wireless communication systems. During the last decade, studies in cooperative control continued because of the increased interest in unmanned aerial vehicles (UAVs). The possibility to control a team of several agents has led to new intriguing applications such as search and rescue missions and pursuit and evasion games [2], cooperative exploration [3], distributed sensor fusion [4], and sensor networks [5].

We consider the rendezvous problem (see, e.g., [6]), where several systems need to reach and maintain the same output value. This output rendezvous value is not specified a priori and imposing a specific rendezvous value would add unnecessary constraints, and thus, in general, lower performance. The rendezvous problem is also related to the consensus problem [7], where the goal is to find an *agreement* on a common variable through an iterative, distributed scheme.

Most of the existing contributions focus on rendezvous of *symmetric* systems and adopt a distributed control approach. However, many systems in practical applications are *asymmetric*, i.e., the subsystems have (possibly) different dynamics and even different state-space dimensions. For example, the DARPA System F6 program [8], a satellite composed of a cluster of stand-alone modules that can share resources, requires rendezvous of asymmetric systems (under a central coordinator) to reconfigure the satellite capabilities while in flight. Motivated by such a class of practical applications, in this paper we aim at characterizing the properties of rendezvous for networks of possibly asymmetric linear systems. We focus on the general properties of the rendezvous dynamics under a centralized linear-quadratic (LQ) optimal control law and a fixed communication topology, obtaining basic results that might be relevant for further extensions to the case of distributed control laws and changes in network topology. We show that the optimal closed-loop dynamics depends on the eigenvalues (if any exist) that are common to all the systems and whose image satisfies a condition that depends on the system output matrix. In particular, the rendezvous may occur and be stable, but still the rendezvous value may diverge in time. We also indicate how to extend the LQ design to enforce constraints on states, inputs, and outputs of the system.

The rest of the paper is structured as follows. After defining some preliminary concepts and reviewing some existing results in Section II, we propose the optimal LQ control design and analyze the closed-loop system properties in Section III. Some simulation results are presented in Section IV. The extension to constraint enforcement in presented in Section V, and conclusions are summarized in Section VI.

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## II. Preliminaries

Due to limited space, only sketches of the main theorems are provided. First the notation is introduced and some existing results are recalled.

### A. Notation

$\mathbb{R}$, $\mathbb{R}_+$, $\mathbb{R}_{0+}$ denote the set of real, positive real, and nonnegative real numbers, respectively. $\mathbb{Z}$, $\mathbb{Z}_+$, $\mathbb{Z}_{0+}$ denote the set of integer, positive integer, and nonnegative integer numbers, respectively, and $\mathbb{Z}_{[a,b]} = \{v \in \mathbb{Z} : a \leq v \leq b\}$. We indicate the non-ordered set of $n$ given distinct elements $o_1$, $o_2$,..., $o_n$ by $\mathcal{O} = \{o_1, o_2, \ldots, o_n\}$, and a particular ordered set out of the existing different $n!$ sets by $\mathcal{O} = (o_1, o_2, \ldots, o_n)$. The cardinality of $\mathcal{O}$ is denoted by $|\mathcal{O}|$.

For a matrix $A$, $\mathrm{rank}(A)$, $\ker(A)$, $\mathrm{nul}(A)$ denote the rank, the kernel (or nullspace) and the nullity (the dimension of the nullspace) of $A$, respectively. The Kronecker product between matrices is denoted by $\otimes$, $0$ denotes a matrix of suitable dimensions entirely composed of zeros, and $I_p$ is

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the identity matrix in $\mathbb{R}^{p\times p}$. $[A]_i^j$ denotes the $i^{th}$-row, $j^{th}$-column element of $A$, while for a vector $v$, $[v]_i$ denotes the $i^{th}$ component of $i$. For a symmetric matrix $Q \in \mathbb{R}^{n\times n}$, $Q > 0$, $(Q \geq 0)$ denotes positive (semi) definiteness, and for a vector $v$, $\|v\|_Q^2 = x'Qx$.

### B. Output-weighted Linear Quadratic Optimal Control

*Theorem 1:* Consider the linear system

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k) &\text{(1a)}\\ y(k) &= Cx(k) &\text{(1b)}\end{aligned}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $y \in \mathbb{R}^p$, and the cost function

$$\sum_{k=0}^{\infty} y(k)'Q_y y(k) + u(k)'Ru(k) \tag{2}$$

where $Q_y$, $R$ are symmetric and $Q_y > 0$, $R > 0$. Let $(A, B)$ be controllable, and $(C, A)$ be observable. The stabilizing control law that minimizes (2) is the Linear Quadratic Regulator (LQR) [9], $K = -(B'PB + R)^{-1}B'PA$, where $P$ is the solution of the Riccati equation $P = C'QC + A'PB(B'PB + R)^{-1}B'PA$. □

If system (1) is not observable, consider an observability decomposition via an appropriate change of coordinates $T$,

$$x_{obs} = Tx, \quad x_{obs} = \left[\begin{array}{cc} x_o' & x_{no}' \end{array}\right]'$$

and

$$\begin{aligned} x_{obs}(k+1) &= \left[\begin{array}{cc} A_o & 0 \\ A_{no,o} & A_{no}\end{array}\right]\left[\begin{array}{c} x_o(k) \\ x_{no}(k)\end{array}\right] + \\ &\quad + \left[\begin{array}{c} B_o \\ B_{no}\end{array}\right] u(k) &\text{(3a)}\\ y(k) &= \left[\begin{array}{cc} C_o & 0\end{array}\right]\left[\begin{array}{c} x_o(k) \\ x_{no}(k)\end{array}\right] &\text{(3b)}\end{aligned}$$

where $x_{no}$ are the coordinates of the state vector with respect to a basis of the unobservable space, and the pair $(A_o, C_o)$ is observable. For a system in the form (3) the LQ optimal control problem is

$$\begin{aligned} \min \sum_{k=0}^{\infty} \quad & y(k)'Q_y y(k) + u(k)'Ru(k) &\text{(4a)}\\ & x_o(k+1) = A_o x_o(k) + B_o u(k) &\text{(4b)}\\ & x_{no}(k+1) = A_{no,o}x_o(k) + \\ & \quad + A_{no}x_{no}(k) + B_{no}u(k) &\text{(4c)}\\ & y(k) = C_o x_o(k) &\text{(4d)}\end{aligned}$$

By (4) it is clear that the unobservable component of the state, $x_{no}$, has no impact on the optimal solution, because the unobservable states do not contribute to the output $y \in \mathbb{R}^p$ in the cost function, and because the unobservable state dynamics do not affect the observable states. Thus, the following result holds straightforwardly.

*Theorem 2:* The solution to the optimal control problem (4) is

$$\begin{aligned} u = K_o x_o &= \left[\begin{array}{cc} K_o & 0\end{array}\right] x_{obs} \\ &= \left[\begin{array}{cc} K_o & 0\end{array}\right] Tx = Kx &\text{(5)}\end{aligned}$$

where $K_o$ is the LQR controller computed on (4a), (4b), (4d). The observable subspace of the closed-loop system (1), (5)

$$\begin{aligned} x(k+1) &= (A + BK)x(k) &\text{(6a)}\\ y(k) &= Cx(k) &\text{(6b)}\end{aligned}$$

is Lyapunov stable. The unobservable eigenvalues of (1) (i.e., the eigenvalues of $A_{no}$) are not modified in (6) by (5). □

*Proof (sketch):* By Theorem 1, $u = K_o x_o$ minimizes (4a), (4b), (4d). Since the unobservable component of the state $x_{no}$ does not affect the cost function neither directly nor indirectly, any additional input effort applied to used to stabilize $x_{no}$ will not decrease the regulation term of the cost function, while it will increase the input energy term, thus it is not optimal. Hence, $x_{no}$ must have no effect on the control input, which results in (5). By applying the observability decomposition (3) to the open-loop system (1), and closing the loop with (5), the observable component $x_o$ is controlled by an LQR law, hence it is asymptotically stable. The unobservable component is not modified by the input. Since the eigenvalues are not modified by the change of coordinates, the same set of eigenvalues will appear in (6) in the original state coordinates. □

*Remark 1:* Controller (5) is not LQR in the proper sense, because it does not necessarily stabilizes the whole system, but only its observable component. In particular, if the system has unstable unobservable modes (with no unobservable modes on the unit circle), the LQR [9] is different from what obtained by Theorem 2, since the LQR is stabilizing. □

### C. Graph notation and basic definitions

A graph $G(V, E)$ is defined by the set of vertices $V = \{v_1, \ldots, v_h\}$, $h \in \mathbb{Z}_+$, with $v_k \neq v_l$ for all $k \neq l$, $k, l \in \mathbb{Z}_{[1,h]}$, and the set of edges $E \subseteq V \times V = \{e_1, \ldots, e_r\}$, $r \in \mathbb{Z}_+$, $r \leq h^2$. An edge is identified by a pair of vertices $v_k, v_l, \in V$, $k \neq l$. If the order of the vertices in the pair does not matter, $e_i = \{v_k, v_l\}$, the graph is called *non-directed*; otherwise, each edge $e_i \in E$ is identified by an ordered pair, $e_i = (v_k, v_l)$, and the graph is called *directed*. Given a directed graph $G(V, E)$, for any edge index $i \in \mathbb{Z}_{[0,r]}$, where $e_i = (v_k, v_l) \in E$, we define the starting and the ending vertex indices as $\text{pre}(i) = k$, $\text{post}(i) = l$, respectively. A *path* $\Pi$ on $G(V, E)$ is a sequence of edges $\Pi = (e_{\Pi_e(1)}, \ldots, e_{\Pi_e(\ell)})$, such that $\text{pre}(\Pi_e(i+1)) = \text{post}(\Pi_e(i))$, $i \in \mathbb{Z}_{[1,\ell-1]}$. For undirected graphs, a path is a sequence of $\ell \in \mathbb{Z}_+$ edges $(e_{\Pi(1)}, \ldots, e_{\Pi(\ell)})$ where to each edge a direction can be arbitrarily assigned such that $\text{pre}(\Pi(i+1)) = \text{post}(\Pi(i))$, $i \in \mathbb{Z}_{[1,\ell-1]}$. A graph is *connected* if for any pair of nodes $v_l, v_k \in V$, there exists a path $\Pi = (e_{\Pi(1)}, \ldots, e_{\Pi(\ell)})$ such that $v_l = \text{pre}(\Pi(1))$, $v_k = \text{post}(\Pi(\ell))$.

Given a directed graph $G(V, E)$ its associated non-directed graph $\tilde{G}(V, \tilde{E})$ is such that for any $(v_h, v_l) \in E$, $\{v_h, v_l\} \in \tilde{E}$. The *incidence matrix* $\mathcal{I}$ of a directed graph $G(V, E)$, is $\mathcal{I} \in \mathbb{R}^{|E|\times|V|}$, such that $[\mathcal{I}]_i^j = -1$ if $\text{pre}(i) = j$, $[\mathcal{I}]_i^j = +1$ if $\text{post}(i) = j$, $[\mathcal{I}]_i^j = 0$, otherwise. A *coverage tree* is a set of edges $CT \subseteq E$ such that a path from any node to any other node can be constructed with edges in $CT$. A coverage tree $CT$ is *minimal* when, taken any $e_i \in CT$, $\overline{CT} = CT/\{e_i\}$ is not a coverage tree.

## III. LQ-optimal Rendezvous Control and its Closed-Loop Dynamical Properties

Consider a set of $N$ linear time-invariant dynamical systems $\{\Sigma_i\}_{h=1}^N$, where $\Sigma_h$, $h \in \mathbb{Z}_{[1,N]}$, is defined by

$$\bar{x}_h(k+1) = \bar{A}_h\bar{x}_h(k) + \bar{B}_h\bar{u}_h(k) \quad (7a)$$
$$y_h(k) = \bar{C}_h\bar{x}_h(k) \quad (7b)$$

and where $\bar{x}_h \in \mathbb{R}^{\bar{n}_h}$, $\bar{u}_h \in \mathbb{R}^{\bar{m}_h}$, $y_h \in \mathbb{R}^p$, for all $h \in \mathbb{Z}_{[1,N]}$. Define a connected directed graph $G(V,E)$, where $V = \{v_h\}_{h=1}^N$, so that $v_h$ is associated to $\Sigma_h$. The edges of the graph will be required to satisfy a few properties, defined in the sequel of the paper.

*Assumption 1:* For all $h \in \mathbb{Z}_{[1,N]}$, system (1) is observable and reachable, $\text{rank}(B_h) = \bar{m}_h$, and $\text{rank}(C_h(I - A_h)^{-1}B_h) = \bar{m}_h$.

*Assumption 2:* For all $h \in \mathbb{Z}_{[1,N]}$, $\bar{m}_h = \bar{m} = \bar{p}$ and $A_h$ is diagonalizable.

Assumption 2, is made principally for easing notation. It is worth to point out that the results developed in the next sections extend with minor modifications to the cases when Assumption 2 does not hold.

*Definition 1 (Rendezvous):* Given the set $\{\Sigma_i\}_{h=1}^N$ of dynamical systems (7), an (asymptotic) rendezvous occurs if for any $i, j \in \mathbb{Z}_{[1,N]}$, $\lim_{k\to\infty} \|y_i(k) - y_j(k)\| = 0$.
Note that the rendezvous condition does not necessarily imply convergence of all outputs to a steady value, that is $\lim_{k\to\infty} y_i(k)$ may not exists, $\forall i \in \mathbb{Z}_{[1,N]}$.

### A. LQ-optimal control for rendezvous

In order to achieve rendezvous by means of LQ-optimal control, introduce the following rendezvous stage cost

$$J_{\text{rdv}}(y_1,\ldots,y_N) = \sum_{e\in E, e=(v_i,v_j)} \|y_i - y_j\|^2_{Q_{ij}} \quad (8)$$

where $Q_{ij} \in \mathbb{R}_+$, $i, j \in \mathbb{Z}_{[1,N]}$ are (edge) weights.

*Proposition 1:* Let $\tilde{G}(V,E)$ be the non-directed graph associated with $G(V,E)$, and let $\tilde{G}(V,E)$ be connected. Then, $\lim_{k\to\infty} J_{\text{rdv}}(y_1(k), \ldots, y_N(k)) = 0$ if and only if the rendezvous occurs. □

In order to setup the control problem as in (4), consider the incremental form

$$\bar{u}_h(k) = \bar{u}_h(k-1) + u_h(k), \quad (9)$$

where $u_h(k) \in \mathbb{R}^{m_h}$, $u_h(k) = \Delta\bar{u}_h(k) = \bar{u}_h(k) - \bar{u}_h(k-1)$ is the input increment at time $k$, so that for all $h \in \mathbb{Z}_{[1,N]}$ the augmented system becomes

$$x_h(k+1) = A_hx_h(k) + B_hu_h(k) \quad (10a)$$
$$y_h(k) = Cx_h(k)\ , \quad (10b)$$
$$A_h = \begin{bmatrix} \bar{A}_h & \bar{B}_h \\ 0 & I \end{bmatrix}, B_h = \begin{bmatrix} \bar{B}_h \\ I \end{bmatrix}, C = \begin{bmatrix} \bar{C} & 0 \end{bmatrix},$$

where $x_h \in \mathbb{R}^{n_h}$, $x_h(k) = \left[\bar{x}_h(k)' \ \bar{u}_h(k-1)'\right]'$, $u_h \in \mathbb{R}^{m_h}$, $y_h \in \mathbb{R}^p$. For the augmented systems (10), we can have $\lim_{k\to\infty} x_h(k) = x_h^{\text{eq}}$, $x_h^{\text{eq}} \neq 0$, and at the same time $\lim_{k\to\infty} u(k) = 0$.

*Proposition 2:* Under Assumption 1, the augmented system (10) is observable. □

The assumption $\text{rank}(\bar{B}_h) = \bar{m}_h$ is needed to maintain complete observability of the augmented system. In reality, an observability loss in the augmented model will be only fictitious, because the full command input is actually known, and it does not contribute to the output vector only because in this problem formulation we use the output as rendezvous variable. Indeed, if the assumption does not hold, which is the case for over-actuated systems, multiple steady state inputs can correspond to a single steady-state output. This will not be a problem for the results developed next, although it would complicate the notation.

We formulate the rendezvous on (10) as

$$\min_{\{u_h(\cdot)\}_{h=1}^N} \sum_{k=0}^{\infty} \sum_{(v_i,v_j)\in E} \|y_i(k) - y_j(k)\|^2_{Q_{ij}} + \sum_{h=1}^N \|u_h(k)\|^2_{R_h}$$
$$x_h(k+1) = A_hx_h(k) + B_hu_h(k)$$
$$y(k) = C_hx_h(k), \qquad h \in \mathbb{Z}_{[1,N]} \quad (11)$$

where $Q_{ij} > 0$, $R_h > 0$ for all $i, j, h \in \mathbb{Z}_{[1,N]}$. Problem (11) is converted in the form (4) by the observability decomposition (3) of the dynamics of the network of systems

$$x(k+1) = Ax(k) + Bu(k) \quad (12a)$$
$$y(k) = Cx(k) \quad (12b)$$

where

$$A = \begin{bmatrix} A_1 & \ldots & 0 \\ 0 & \ddots & 0 \\ 0 & \ldots & A_h \end{bmatrix}, B = \begin{bmatrix} B_1 & \ldots & 0 \\ 0 & \ddots & 0 \\ 0 & \ldots & B_h \end{bmatrix} \quad (13a)$$
$$C = (\mathcal{I} \otimes I_p) \cdot \begin{bmatrix} C_1 & \ldots & 0 \\ 0 & \ddots & 0 \\ 0 & \ldots & C_h \end{bmatrix}, \quad (13b)$$

and $\mathcal{I}$ is the incidence matrix of $G(V,E)$. For dynamics (13), $x \in \mathbb{R}^{\mathcal{N}}$, $\mathcal{N} = \sum_{h=1}^N n_h$, $x = \left[x_1' \cdots x_N'\right]'$, $u \in \mathbb{R}^{\mathcal{M}}$, $\mathcal{M} = \sum_{h=1}^N m_h$, $u = \left[u_1' \cdots u_N'\right]'$, $y \in \mathbb{R}^{\mathcal{P}}$, $\mathcal{P} = Np$, $y = \left[y_1' \cdots y_N'\right]'$. Controller (5) guarantees the asymptotic stability of the observable component of (13), which then guarantees asymptotic stability of the rendezvous by (3). On the other hand, the dynamics of the network of systems depends also on the unobservable component of (13), which motivates the next study.

### B. LQ-optimal rendezvous for asymptotically stable systems

The dynamical properties of the overall closed-loop system derive from Theorem 2 when applied to the network of systems defined by (13). The observability matrix of (13), $\Theta = \left[C' \ (CA)' \ \ldots \ (CA^{\mathcal{N}})'\right]'$ can be rearranged as

$$\Theta = \begin{bmatrix} \Theta^{(\mathcal{N})}_{\text{pre}(1)} & -\Theta^{(\mathcal{N})}_{\text{post}(1)} & 0 & \cdots & 0 \\ & & \vdots & & \\ \cdots & \Theta^{(\mathcal{N})}_{\text{pre}(i)} & \cdots & -\Theta^{(\mathcal{N})}_{\text{post}(i)} & 0 \\ & & \vdots & & \end{bmatrix} \quad (14)$$

where $\Theta_h^{(\mathcal{N})}$ is the $\mathcal{N}$-steps observability matrix of the $h$-th system (10), thus resulting in a matrix of $\mathcal{E} = |E|$ block rows (one per edge in the graph), and where each block row belongs to $\mathbb{R}^{\mathcal{N}p\times\mathcal{N}}$. Due to the dimension of the full system, each of the observability matrices $\Theta_h^{(\mathcal{N})}$ is expanded beyond the observability index of a single subsystem.

For a state $x \in \mathbb{R}^{\mathcal{N}}$ to belong to the unobservable space of the network of systems (13), it must holds from (14) that

$$\Theta_{\mathrm{pre}(i)}^{(\mathcal{N})} x_{\mathrm{pre}(i)} = \Theta_{\mathrm{post}(i)}^{(\mathcal{N})} x_{\mathrm{post}(i)}, \ \forall e_i \in E \ . \qquad (15)$$

We show next that some of the equations in (15) may be redundant.

*Proposition 3:* The linearly independent equations in (15) are generated by a minimal coverage tree $CT$ of the network of systems graph $G(V, E)$. Such coverage has $N-1$ edges, so that, in general, the observability matrix (14) has at most $\mathcal{N}(N-1)p$ independent rows. □

*Theorem 3:* Let systems (7) be open-loop asymptotically stable, and apply controller (5) based on (11). Then, the network of systems (13) converges asymptotically to a steady rendezvous condition, and all states remain bounded. □

*Proof (sketch):* By (5), $\lim_{k\to\infty} \|\Delta\bar{u}_h(k)\| = 0$, for all $h \in \mathbb{Z}_{[1,N]}$. Because of the assumed asymptotic stability of (7), (10) converge to the equilibria $x_h^{(eq)}$, $h \in \mathbb{Z}_{[1,N]}$. Equations (15) reduces to $\left[y_i^{(eq)\prime} \ldots y_i^{(eq)\prime}\right]' = \left[y_j^{(eq)\prime} \ldots y_j^{(eq)\prime}\right]'$, hence $(N-1)p$ equations in $Np$ variables, resulting in $\mathrm{null}(\Theta) = p$. The solutions are obtained as any $y_1^{eq} \in \mathbb{R}^p$, and $y_i^{eq} = y_j^{eq} \in \mathbb{R}^p$, for all $(v_i, v_j) \in CT$, hence all possible rendezvous.

### C. Generalization to non-asymptotically stable systems

Next we remove the stability assumption on (1). By Theorem 2 we know that the dynamics of the network of systems is stabilized by (5) except for the unobservable subspace. Let the eigenvalues of (10) be ordered so that the ones common to all systems have the same index, and let $\mathcal{L}$ be the set of such indices.

*Theorem 4:* The rendezvous space is the subspace spanned by the eigenvectors of (10), $\phi_\ell^h$, $h \in \mathbb{Z}_{[1,N]}$, whose corresponding eigenvalues are equal, $\lambda_\ell^h = \lambda_\ell$, for all $h \in \mathbb{Z}_{[1,N]}$, $\ell \in \mathcal{L}$, and such that there exists $\gamma_\ell \in \mathbb{R}^p$, $\mu_\ell^h \in \mathbb{R}$, $h \in \mathbb{Z}_{[1,N]}$, $\ell \in \mathcal{L}$, for which $\mu_\ell^h C\phi_\ell^h = \gamma_\ell$. □

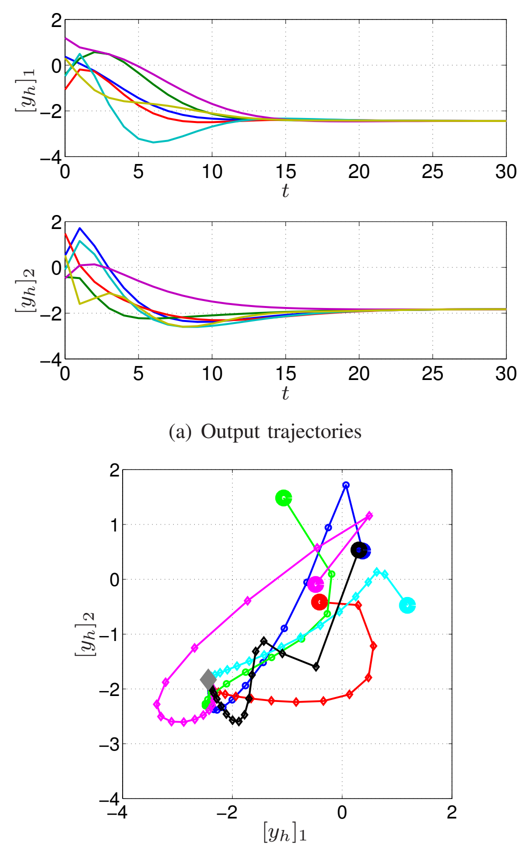
*Proof (sketch):* The rendezvous space is the unobservable subspace of the network of systems, i.e., $x \in \mathbb{R}^{\mathcal{N}}$, such that $\Theta x = 0$. To characterize such solutions one can consider the structure of $\Theta$ induced by the coverage tree, and operate the eigenvector decomposition on the dynamics of each subsystem. This results in the equations $\sum_{\zeta_h=1}^{n_i} C_i\phi_{\zeta_i}^i \mu_{\zeta_i}^i (\lambda_{\zeta_i}^i)^k = \sum_{\zeta_h=1}^{n_j} C_j\phi_{\zeta_j}^j \mu_{\zeta_j}^j (\lambda_{\zeta_j}^j)^k$, $k \in \mathbb{Z}_{[0,\mathcal{N}-1]}$, for all $(v_i, v_j) \in CT$, where $\phi_{\zeta_h}^h \in \mathbb{R}^{n_h}$, $\mu_{\zeta_h}^h \in \mathbb{R}$, for $\zeta_h \in \mathbb{Z}_{[1,n_h]}$, are the eigenvectors and the corresponding coordinates. The proof is concluded by showing that the equations admit only a trivial solution $x = 0$, unless some common eigenvalues between the different subsystems exist, satisfying the theorem's conditions. In these cases, the stacked eigenvectors with appropriate scalar weights belong to $\ker(\Theta)$.

*Remark 2:* For a non-trivial rendezvous to occur, Theorem 4 requires the systems to have an equal eigenvalue $\lambda_\ell$, $\ell \in \mathcal{L}$, and the corresponding eigenvectors to have equal image through the output matrix. The systems may have different state dimensions as long as the output dimensions are the same. The results of Theorem 3 are also in accordance to these, since the systems share the eigenvalues of the input integrators (9) for which the condition on the eigenvalues holds. □

*Remark 3:* The diagonalizability condition required by Assumption 2 is included here mostly for the ease of notation. The result of Theorem 4 can be extended to the case in which Jordan blocks are present. □

*Remark 4:* If $m_h < p$ for some $h \in \mathbb{Z}_{[1,N]}$, not all the rendezvous are achievable, see (3). Only a subset of rendezvous can occur, or none, if there is an empty intersection of achievable rendezvous between the systems. □

*Corollary 1:* The dynamics of the network of systems (13) after the rendezvous occurs are defined by the common eigenvalues $\lambda_\ell^h = \lambda_\ell$ for all $h \in \mathbb{Z}_{[1,N]}$, $\ell \in \mathcal{L}$, such that for the corresponding eigenvectors $\phi_\ell^h$, $h \in \mathbb{Z}_{[1,N]}$, there exists $\gamma_\ell \in \mathbb{R}^p$, $\mu_\ell^h \in \mathbb{R}$, $h \in \mathbb{Z}_{[1,N]}$, for which $\mu_\ell^h C\phi_\ell^h = \gamma_\ell$. The network of systems (13) is unstable only if there exists $\ell \in \mathcal{L}$, such that $\lambda_\ell > 1$ and satisfies the above conditions. □




(a) Output trajectories

(b) Output plane

Fig. 1. Rendezvous simulations by LQ control for 6 systems that have no common eigenvalues before augmentation to incremental form.

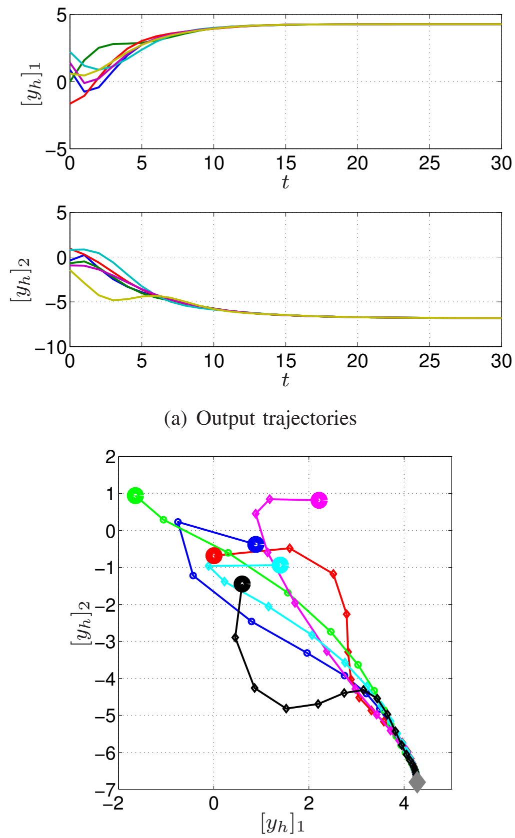
## IV. SIMULATIONS

We present here simulations that confirm the results predicted by Theorems 3 and 4. The examples below are restricted to two-dimensional outputs aligned with the Cartesian axes, for graphical reasons. We consider $N = 6$ systems with $m_h = 2$ inputs, $h \in \mathbb{Z}_{[1,6]}$, $p = 2$ outputs, that satisfy Assumptions 1 and 2. The number of states is $\bar{n}_1 = \bar{n}_2 = \bar{n}_3 = \bar{n}_4 = 4$, $\bar{n}_5 = 3$, $\bar{n}_6 = 5$, and the input incremental form (10) is applied to all. The initial states of the systems (denoted as large circles in the output-plane plots) are chosen according to a normal distribution.

*Case 1*. Some of the systems have unstable poles, but no common eigenvalues to all systems exist, except for the input integrators (9). Thus, according to Theorem 4, the system converges to a steady rendezvous value, and the states remain bounded. An example of this behavior is shown in Figure 1.

*Case 2*. In Figure 2 we show the case where all the systems share the stable eigenvalue $\lambda_\ell^h = 0.8$, $h \in \mathbb{Z}_{[1,6]}$, so that on the corresponding eigenvectors $\phi_\ell^h$, $h \in \mathbb{Z}_{[1,6]}$, the condition in Theorem 4 holds. Thus, the rendezvous occurs and the residual dynamics are stable, hence leading to a steady rendezvous value. Notice how the outputs converge to each other and later stabilize around an equilibrium value.

*Case 3*. The systems have the common unstable eigenvalue $\lambda = 1.125$, and the condition in Theorem 4 holds for the corresponding eigenvectors $\phi_\ell^h$, $h \in \mathbb{Z}_{[1,6]}$. Figure 3 shows that the systems achieve a rendezvous, then the evolution is dictated by the common eigenvalue, causing the network of systems to diverge along the $y_1$-axis.




(a) Output trajectories

(b) Output plane

Fig. 2. Rendezvous simulations by LQ control for 6 systems (7) that share an asymptotically stable eigenvalue with common eigenvalue image.

## V. EXTENSION TO CONSTRAINED SYSTEMS

The results proved in Section III can be used to extend the rendezvous control strategy (11) in several directions. For enabling input and output constraints, designed by (11), a model predictive control (MPC) strategy based on (5) is proposed here.

Given an unconstrained LQR, the Riccati matrix and (possibly) the feedback gain can be used to design an MPC controller that accounts for constraint and that behaves locally as the original LQR [10], [11]. In order to achieve this, the MPC problem is formulated as

$$\min_{U(t)} \quad \|x(N|t))\|_P^2 + \sum_{k=0}^{h_p-1} \|x(k|t)\|_Q^2 + \|u(k|t)\|_R^2 \quad \text{(16a)}$$

$$\text{s.t.} \quad x(k+1|t) = Ax(k|t) + Bu(k|t)) \quad \text{(16b)}$$

$$z(k|t) = C_z x(k|t) + D_z u(k|t) \quad \text{(16c)}$$

$$y_{\min} \le z(k|t) \le y_{\max}, \quad k = 0, \ldots, h_c \quad \text{(16d)}$$

$$u_{\min} \le u(k|t) \le u_{\max}, \quad k = 1, \ldots, h_u \quad \text{(16e)}$$

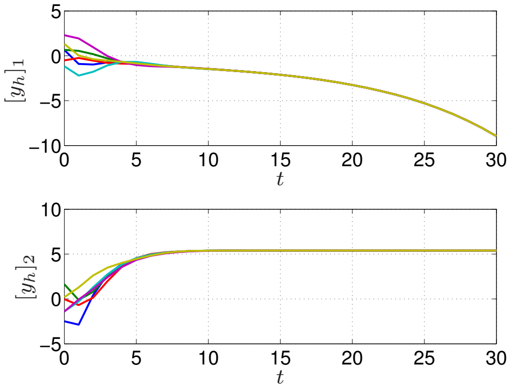
$$u(k|t) = Kx(k|t), \quad k = h_u, \ldots, h_p - 1 \quad \text{(16f)}$$

$$x(0|t) = x(t) \quad \text{(16g)}$$

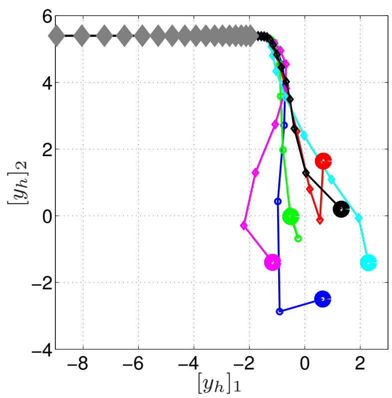
where $U(t) = \{u(1|t), \ldots, u(h_p-1|t)\}$, $z \in \mathbb{R}^q$ is the vector of constrained outputs, generated by (16c), $h_p$, $h_c$, $h_u$ are the prediction, constraint, and control horizons, $Q = C' Q_y C$ and $R$ define the stage cost, $K$ is the terminal controller (5). The terminal cost $P$ is the solution of the Riccati equation solved to design (5), expanded to the dimension of the original state space by zeros (since unobservable states do not contribute to the cost function), and transformed back in the original coordinate frame, i.e., $P = T' \left[\begin{smallmatrix} P_o & 0 \\ 0 & 0 \end{smallmatrix}\right] T$.

When implemented in this way, whenever constraints (16d), (16e) are inactive, the MPC controller (16) generates the same input as the LQR controller (5) designed by (11). Thus, the guaranteed local stability domain of MPC is characterized as the set where the feedback controller (5) designed by (11) satisfies the input and output constraints for all the future time instants, or in other words, the maximum output admissible constraint set for the closed loop dynamics $x(k+1) = (A + BK)x(k)$. The controller can be further extended with a terminal constraint set [12] designed basing on the constraint admissible set, which provides an enlarged guaranteed stability domain.

A simulation example of $N = 5$ systems with $n_h = 4$ states each and no shared eigenvalues (before incremental input augmentation (10)) is shown in Figure 4, along with the input constraints $\|\bar{u}_h\|_\infty \le 1.5$, the constraints on input increments $\|u_h\|_\infty \le 0.75$, $h \in \mathbb{Z}_{[1,N]}$. The rendezvous is achieved, while all the prescribed constraints are enforced.

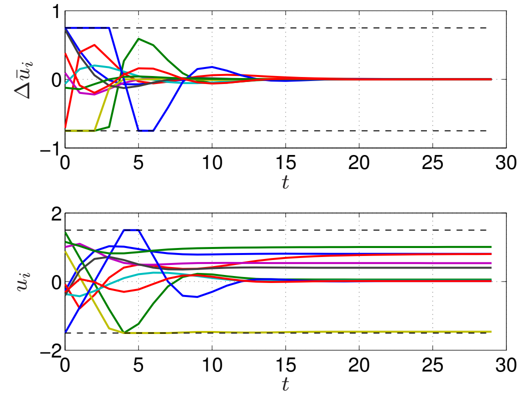



(a) Output trajectories

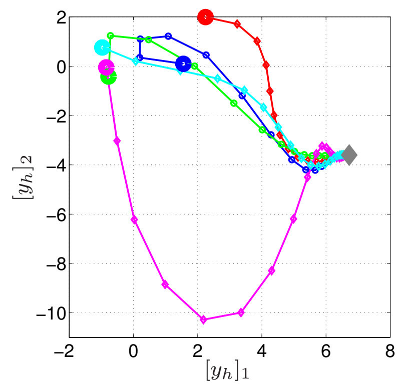



(b) Output plane

Fig. 3. Rendezvous simulations by LQ control for 6 systems (7) that share an unstable eigenvalue with common eigenvalue image




(a) Input and input increment trajectories. Constraints also shown.




(b) Output plane

Fig. 4. Rendezvous simulations by MPC for 5 systems (7) with no shared eigenvalues

## VI. Conclusions

This paper analyzed the rendezvous dynamics of a network of possibly *asymmetric* systems under LQ optimal control. We have shown that a rendezvous is achieved, and that the dynamics governing the rendezvous situation depends on the systems' common eigenvalues and on the corresponding eigenvectors. By using local equivalence results between LQ control and an appropriately designed MPC controller, we have shown how the control design can be extended to enforce constraints. Currently this approach is being extended to rendezvous to a specified formation pattern.

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