

Stability Analysis of Discrete-Time Piecewise-Affine Systems Over Non-Invariant Domains

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Abstract—This paper analyzes stability of discrete-time piecewise-affine systems defined on non-invariant domains. An algorithm based on linear programming is proposed, in order to prove the exponential stability of the origin and to find a positively invariant estimate of the region of attraction. The theoretical results are based on the definition of a piecewise-affine, possibly discontinuous, Lyapunov function. The proposed method presents a relatively low computational burden, and is proven to lead to feasible solutions in a broader range of cases with respect to a previously proposed approach.

I. INTRODUCTION

The interest in piecewise-affine (PWA) systems, first defined in the seminal work of Sontag [16], has gained a lot of attention in the last years. PWA systems are one of the most common forms of hybrid systems, together with the mixed logical dynamical systems [2].

Different techniques have been recently proposed for stability analysis of PWA systems. Most of these techniques are based on the computation, through semi-definite programming, of a piecewise quadratic Lyapunov function [7], [8]. Other methods are based on piecewise-polynomial Lyapunov functions [13], and on PWA Lyapunov functions [9].

In particular, PWA Lyapunov functions are obtained through linear programming (LP), imposing positive-definiteness and decay conditions at the vertices of the polytopes that compose the (bounded) domain, therefore enforcing the same properties for all the points of interest. Usually, the set (henceforth referred to as \mathcal{X}) where the PWA dynamics is defined is a positively invariant set, because the notion of stability has no practical relevance if the state trajectory can exit the domain where the dynamics is defined [5]. However, there are many cases when the PWA system to be analyzed is not defined in a positively invariant set. A typical example is when an explicit model predictive control (MPC) control law [3] is synthesized for a linear system without a-priori guarantees of stability and invariance for the closed-loop system. This can occur, for instance, when approximations of the optimal control law are introduced to obtain low-complexity solutions [1], [4], [9].

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In case of a non-invariant domain, a possible approach is to perform an extensive reachability analysis to find, through a recursive procedure, the maximum positively invariant set included in \mathcal{X} (see [14], [6, Chap. 4-5] and references therein). Then, the Lyapunov stability analysis can be carried out on the maximum positively invariant set. However, it often happens that this latter one is not a domain of attraction for the origin, because the domain of attraction is a proper subset of it. The domain of attraction (or a positively invariant subset of it) must then be determined in order to get a feasible solution applying one of the previously mentioned methods [7]–[9], [13]. However, this procedure, when applied to PWA systems, can lead to computationally intractable solutions due to the exponential complexity of reachability analysis. Moreover, in many cases, searching for the maximum invariant set is an undecidable problem.

An alternative solution is proposed in [15], where an invariant set is determined a-posteriori by defining a fictitious dynamics which extends the actual dynamics of the system defined in \mathcal{X} . In this way, a larger domain \mathcal{X}_e is considered, which is positively invariant for the extended system. If a PWA Lyapunov function can be determined for the extended system, a positively invariant (not necessarily maximum) set \mathcal{P}_e included in \mathcal{X} is determined for the actual system. This solution is usually computationally much simpler than the extensive reachability analysis. However, its main drawback consists in the arbitrariness in the definition of the fictitious dynamics. This latter one can lead, for instance, to the artificial introduction of limit cycles on the extended dynamics, which would make it impossible to find a PWA Lyapunov function.

This paper proposes a method based on PWA Lyapunov functions to assess exponential stability of the origin of a PWA system defined on a non-invariant domain \mathcal{X} , and determines a region of attraction \mathcal{P} included in \mathcal{X} . Even though the goal is the same as in [15], no fictitious dynamics is required, and the PWA Lyapunov function (directly defined in \mathcal{P}) and the set \mathcal{P} itself are found simultaneously. Also, discontinuities on the boundaries of the polytopic sets are allowed for both the system dynamics and the PWA Lyapunov function. Moreover, it is proven that the proposed method leads to a feasible solution whenever there exists a fictitious dynamics leading to a feasible solution by applying the method in [15]. On the other hand, there are cases when a method based on a fictitious dynamics would fail, while the method proposed in this paper leads to a feasible solution.

The paper is organized as follows. Section II introduces the basic notations and definitions used throughout the paper.

Section III introduces the stability analysis problem, and Section IV analyzes the possible transitions that can occur between the different subsets of \mathcal{X} . The Lyapunov stability analysis is presented in Section V. Section VI presents the result on the comparison with the approach proposed in [15]. A simulation example is presented in Section VII, and conclusions are drawn in Section VIII. For the sake of readability, the proofs of the theoretical results are moved to the Appendix section.

II. BASIC NOTATIONS AND DEFINITIONS

Let \mathbb{R} , $\mathbb{R}_{>0}$, $\mathbb{Z}_{>0}$ and $\mathbb{Z}_{\geq 0}$ denote the sets of reals, strictly positive reals, strictly positive integers and non negative integers, respectively. Given a vector $v \in \mathbb{R}^n$, let $\|v\|$ denote any vector norm. Given two matrices $A_1 \in \mathbb{R}^{m_1 \times n}$, $A_2 \in \mathbb{R}^{m_2 \times n}$, $[A_1; A_2]$ denotes the matrix $[A_1' \ A_2']' \in \mathbb{R}^{(m_1+m_2) \times n}$. Given a set $\mathcal{A} \subseteq \mathbb{R}^n$, its interior is denoted by $\text{int}(\mathcal{A})$, its closure by $\bar{\mathcal{A}}$, and its convex hull by $\text{Co}(\mathcal{A})$. Given a finite number of sets \mathcal{A}_i , $i \in \mathcal{I}_a = \{1, \dots, n_a\}$, we say that $\{\mathcal{A}_i\}$ is a *partition* of \mathcal{A} if $\text{int}(\mathcal{A}_i) \neq \emptyset$, $\text{int}(\mathcal{A}_i) \cap \text{int}(\mathcal{A}_j) = \emptyset$, $\forall i, j \in \mathcal{I}_a$ with $i \neq j$, and $\bigcup_{i=1}^{n_a} \mathcal{A}_i = \mathcal{A}$. If $\{\mathcal{A}_i\}$ is a partition with $\mathcal{A}_i \cap \mathcal{A}_j = \emptyset$, it is referred to as *strict partition*. A *polyhedron* is a set given by the intersection of a finite number of (closed or open) half-spaces. A *polytope* \mathcal{A} is a bounded polyhedron, and the set of the vertices of its closure $\bar{\mathcal{A}}$ is denoted by $\text{vert}(\bar{\mathcal{A}})$.

Consider a discrete-time nonlinear system

$$x(k+1) = \varphi(x(k)) \quad (1)$$

where $k \in \mathbb{Z}_{\geq 0}$, $x \in \mathcal{X} \subseteq \mathbb{R}^n$ is the state vector, and \mathcal{X} is a compact set that contains the origin in its interior. We will often use the notation $x^+ = \varphi(x)$ for system (1) in this paper.

Definition 1: A set $\mathcal{B} \subseteq \mathcal{X}$ is called *positively invariant* (PI) with respect to dynamics (1) if, for all $x \in \mathcal{B}$, $\varphi(x) \in \mathcal{B}$. Note that the set \mathcal{X} is not assumed to be PI with respect to dynamics (1), so that some trajectories leave \mathcal{X} and are therefore defined only on a finite interval of time $[0, k_{max}]$.

Definition 2: Consider dynamics (1) and a PI set $\mathcal{B} \subseteq \mathcal{X}$ with $0 \in \mathcal{B}$. System (1) is *exponentially stable* in \mathcal{B} (ES(\mathcal{B})) if there exist $c > 0$ and $\rho \in [0, 1)$ such that

$$\|x(t)\| \leq c \|x_0\| \rho^k, \quad \forall x_0 \in \mathcal{B} \quad \text{and} \quad \forall k \in \mathbb{Z}_{\geq 0}.$$

Theorem 1: Assume that system (1) admits a (possibly discontinuous) function $W : \mathcal{B} \rightarrow \mathbb{R}$, such that

$$\alpha_1 \|x\|^\eta \leq W(x) \leq \alpha_2 \|x\|^\eta \quad (2a)$$

$$W(\varphi(x)) - W(x) \leq -\alpha_3 \|x\|^\eta \quad (2b)$$

where $\eta, \alpha_i \in \mathbb{R}_{>0}$, $i = 1, 2, 3$. Then, system (1) is ES(\mathcal{B}). *Proof:* The reader is referred to [10, Th. 2.2.4], which is a particular case of the more general result in [12, Th. 2.5]. ■ Note that Theorem 1 allows both $W(\cdot)$ (called *uniformly strict Lyapunov (USL) function*) and $\varphi(\cdot)$ to be *discontinuous functions*. Continuity at the origin is implied by condition (2a), but the continuity on a neighborhood of the origin is not required.

In the remainder of the paper, the fact of being able to employ discontinuous USL functions will be exploited to obtain theoretical results that overcome the conservativity which derives from the imposition of additional continuity conditions. Note that, due to the formulation of the decay condition in (2b), the recalled results of the stability analysis exhibit a certain degree of robustness [11].

III. STABILITY ANALYSIS PROBLEM

Let $\mathcal{X} \subset \mathbb{R}^n$ be a compact polytope that includes the origin in its interior. Consider a strict partition $\{\mathcal{X}_i\}$ of \mathcal{X} that consists of a finite number s of polytopes

$$\mathcal{X}_i \triangleq \{x : H_i^1 x \leq h_i^1, H_i^2 x < h_i^2\}, \quad i \in \mathcal{I} \triangleq \{1, \dots, s\} \quad (3)$$

where $H_i^1 \in \mathbb{R}^{n \times q_{i,1}}$, $H_i^2 \in \mathbb{R}^{n \times q_{i,2}}$ are constant matrices with $q_{i,1}, q_{i,2} \in \mathbb{Z}_{>0}$, $i \in \mathcal{I}$, and $h_i^1 \in \mathbb{R}^{q_{i,1}}$ and $h_i^2 \in \mathbb{R}^{q_{i,2}}$ are constant vectors. The closure $\bar{\mathcal{X}}_i$ of \mathcal{X}_i is denoted by $\bar{\mathcal{X}}_i = \{x : H_i x \leq h_i\}$, with $H_i = [H_i^1; H_i^2] \in \mathbb{R}^{n \times q_i}$ and $h_i = [h_i^1; h_i^2] \in \mathbb{R}^{q_i}$. The number of vertices of \mathcal{X}_i is denoted by m_i . The subset of indices \mathcal{I}_0 is defined as $\mathcal{I}_0 \triangleq \{i \in \mathcal{I} : 0 \in \bar{\mathcal{X}}_i\}$, and it is assumed without loss of generality that, for all $i \in \mathcal{I}_0$, $0 \in \text{vert}(\bar{\mathcal{X}}_i)$.

Consider the autonomous discrete-time PWA system

$$x^+ = f(x) = A_i x + a_i, \quad x \in \mathcal{X}_i \quad (4)$$

with $x \in \mathcal{X}$, $A_i \in \mathbb{R}^{n \times n}$, $a_i \in \mathbb{R}^n$. Note that dynamics (4) may be discontinuous on the boundaries of the sets \mathcal{X}_i .

Problem statement: Given the PWA system (4), for which \mathcal{X} is not a PI set, prove the exponential stability of the origin and provide an estimate of a PI set $\mathcal{P} \subseteq \mathcal{X}$ contained in the region of attraction.

IV. TRANSITION SETS AND PARTITION OF \mathcal{X}

In order to introduce the Lyapunov stability analysis framework, some sets related to the possible transitions within \mathcal{X} are defined, as follows.

For any pair $(i, j) \in \mathcal{I} \times \mathcal{I}$, define the closed *transition sets*

$$\mathcal{X}_{ij} \triangleq \{x \in \bar{\mathcal{X}}_i : A_i x + a_i \in \bar{\mathcal{X}}_j\} \quad (5)$$

of states that can possibly end up in the polytope $\bar{\mathcal{X}}_j$ in one step from the polytope $\bar{\mathcal{X}}_i$ under dynamics (4). The number of vertices of each region \mathcal{X}_{ij} is denoted by m_{ij} . The sets \mathcal{X}_{ij} are computed as

$$\mathcal{X}_{ij} = \{x \in \mathbb{R}^n : H_i x \leq h_i, H_j(A_i x + a_i) \leq h_j\}. \quad (6)$$

Since \mathcal{X} is not a PI set, then the following inclusion may be strict:

$$\mathcal{X}_I \triangleq \bigcup_{(i,j) \in \mathcal{I} \times \mathcal{I}} \mathcal{X}_{ij} \subset \mathcal{X}.$$

In order to cover the whole set \mathcal{X} , it is necessary to take into account also the points for which $f(x) \notin \mathcal{X}$. To this purpose, for each set \mathcal{X}_i define $\{\Omega_{ip}\}$, $p \in \mathcal{I}_i^\Omega \triangleq \{1, \dots, \tau_i\}$ as a partition of

$$\mathcal{X}_i^- \triangleq \mathcal{X}_i \setminus \bigcup_{j \in \mathcal{I}} \mathcal{X}_{ij}$$

which are in general non-connected sets given by the union of polytopes, which can be divided into convex polyhedral regions as in [3, Th. 3]. Note that the index p does not refer in any way to the region that can be reached in one step. On the contrary, it simply represents an enumeration of the τ_i convex polytopes that compose \mathcal{X}_i^- . The resulting regions Ω_{ip} are clearly such that $\{\mathcal{X}_{ij}, \Omega_{ip}\}$, for all the possible values of the indices i, j, p , is a partition of \mathcal{X} . The number of vertices of each region Ω_{ip} is denoted by μ_{ip} .

V. PWA LYAPUNOV ANALYSIS

To the end of synthesizing an USL function for system (4), define $V_i : \bar{\mathcal{X}}_i \rightarrow \mathbb{R}$, $i \in \mathcal{I}$ as

$$V_i(x) \triangleq F_i x + g_i \quad (7a)$$

where in (7a) $F_i \in \mathbb{R}^{1 \times n}$ and $g_i \in \mathbb{R}$ are coefficients to be determined. Then, define $V : \mathcal{X} \rightarrow \mathbb{R}$ as

$$V(x) = \max_{i \in \mathcal{N}(x)} V_i(x) \quad (7b)$$

where

$$\mathcal{N}(x) \triangleq \{i \in \mathcal{I} : x \in \bar{\mathcal{X}}_i\}. \quad (7c)$$

Note that $V(x) = F_i x + g_i$ for $x \in \text{int}(\mathcal{X}_i)$. However, for numerical reasons, closed sets $\bar{\mathcal{X}}_i$ are considered when defining $V(\cdot)$ in (7). As a consequence, $V_i(x)$ and $V_j(x)$ may be different on common boundaries $\bar{\mathcal{X}}_i \cap \bar{\mathcal{X}}_j$ (unless very conservative continuity conditions are imposed). For the states x on the common boundaries all the required conditions on V_i are imposed for all $i \in \mathcal{N}(x)$, although only one value (the max) is taken in (7b), as $V(x)$ must be single-valued. The constraints

$$F_i v_{i,h} + g_i \geq \alpha_1 \|v_{i,h}\| \quad (8a)$$

are imposed for all the m_i vertices $v_{i,h} \in \text{vert}(\bar{\mathcal{X}}_i)$, $i \in \mathcal{I}$, $h = 1, \dots, m_i$, where α_1 is a free parameter such that

$$\alpha_1 > 0. \quad (8b)$$

Conditions (8a)-(8b) will lead to $V(x) \geq \alpha_1 \|x\|$ in \mathcal{X} , as will be formally shown in the remainder of the paper. Also, in order to obtain larger estimates of the domain of attraction, we bound the size of $V(\cdot)$ at the vertices of each set \mathcal{X}_i by imposing

$$F_i v_{i,h} + g_i \leq M \quad (8c)$$

for all the m_i vertices $v_{i,h} \in \text{vert}(\bar{\mathcal{X}}_i)$, $i \in \mathcal{I}$, $h = 1, \dots, m_i$, where $M > 0$ is a fixed parameter. In order to obtain $V(0) = 0$, it is required that

$$g_i = 0, \quad i \in \mathcal{I}_0. \quad (8d)$$

Due to the boundedness of $V(x)$, (8d) will make it possible to prove that there exists $\alpha_2 > 0$ such that $V(x) \leq \alpha_2 \|x\|$ for all $x \in \mathcal{X}$. Also, it is required that, for all $\mathcal{X}_{ij} \neq \emptyset$,

$$F_j(A_i v_{ij,h} + a_i) + g_j - F_i v_{ij,h} - g_i \leq -\alpha_3 \|v_{ij,h}\| \quad (8e)$$

for all $v_{ij,h} \in \text{vert}(\mathcal{X}_{ij})$, with $h = 1, \dots, m_{ij}$, $(i, j) \in \mathcal{I} \times \mathcal{I}$, and

$$\alpha_3 > 0. \quad (8f)$$

Finally, a free parameter $\theta \in \mathbb{R}$ is defined, such that

$$F_i v_{ip,h} + g_i \geq \theta \quad (8g)$$

for all vertices $v_{ip,h} \in \text{vert}(\Omega_{ip})$, with $h = 1, \dots, \mu_{ip}$, $i \in \mathcal{I}$, and $p \in \mathcal{I}_i^\Omega$. The vector of variables to be determined is composed by θ , α_1 , α_3 , and the terms F_i and g_i , with $i \in \mathcal{I}$.

A procedure is now proposed so as to determine a choice for such variables by means of a linear program, defined as

$$\max \theta \quad (9a)$$

$$\text{s. t.} \quad (9b)$$

Note that the requirement for \mathcal{X} not to be a PI set is needed in order to obtain a bounded solution for (9). Indeed, if \mathcal{X} were a PI set, constraints (8g) on θ would not be imposed, leading to the trivial solution $\theta = +\infty$. Once (9) has been solved, the function $V(x)$ is defined for all $x \in \mathcal{X}$. Now, define the set

$$\mathcal{P} \triangleq \{x \in \mathcal{X} : V(x) < \theta\}. \quad (10)$$

Theorem 2: Consider system (4), whose dynamics is defined on \mathcal{X} , and assume that a feasible solution of problem (9) exists. Then, \mathcal{P} in (10) is a PI set, and system (4) is ES(\mathcal{P}).

Proof: See Appendix. \square

Remark 1: In case (8) is infeasible, a possibility is to increase the number of regions \mathcal{X}_i , therefore providing more flexibility in synthesizing the PWA Lyapunov function (7). A possible way is to consider the sets \mathcal{X}_{ij} and Ω_{ip} as the new sets $\bar{\mathcal{X}}_i$ and restart the one-step reachability analysis.

The overall procedure proposed in this paper for exponential stability is summarized in Algorithm 1.

Algorithm 1: Exponential stability analysis

Input: PWA system (4), $\text{max_iter} \in \mathbb{Z}_{>0}$

Output: Estimate of the region of attraction \mathcal{P} and certificate of ES(\mathcal{P}) for system (4)

n_iter:=0;

repeat

 n_iter:=n_iter+1;

if n_iter > max_iter **then**

\mathcal{P} undefined;

 STOP

if n_iter > 1 **then**

$\mathcal{X}_i := \mathcal{X}_{ij}, \Omega_{ip};$

$\mathcal{X} := \bigcup \mathcal{X}_i$

 Compute the sets \mathcal{X}_{ij} in (6) and the sets $\Omega_{ip};$

 Solve the LP (9);

until the LP has a solution ;

Define the estimate of the region of attraction $\mathcal{P} \subseteq \mathcal{X}$ as in (10);

System (4) is ES(\mathcal{P})

Remark 2: The maximization of θ in (9) is not necessary to obtain the result in Theorem 2, which holds for any feasible θ . However, when we can choose among different feasible solutions, a larger value for θ leads in many cases

to a larger size of the set of points x where $V(x) < \theta$, thus enlarging the size of the estimated region \mathcal{P} .

Remark 3: The value chosen for M in (8c) does not influence the feasibility of problem (9). As a matter of fact, M is only used to ensure that the Lyapunov function parameters be uniformly upper bounded and the optimization (9) be well defined. Indeed, if a given solution to (8) is obtained with $M = M_1$, then for any $\lambda > 0$, using $M = M_2 = \lambda M_1$, that same solution scaled by λ is feasible for (8).

Remark 4: Since, as mentioned above, the variables of the LP (10) are $\theta, \alpha_1, \alpha_3, F_i$ and g_i , with $i \in \mathcal{I}$, the total number of variables is $n_v = 3 + s(n+1)$, while the total number of constraints is

$$n_c = 3 + \text{card}(\mathcal{I}_0) + \sum_{i=1}^s \left(2m_i + \sum_{j=1}^s m_{ij} \right).$$

VI. COMPARISON TO A PREVIOUS APPROACH

In [15], the same problem considered here has been addressed by proposing a different solution. The idea of [15] is to define a compact polytope \mathcal{X}_e such that

$$\mathcal{X}_e \supseteq \mathcal{X} \cup \mathcal{R}(\mathcal{X}) \quad (11)$$

where $\mathcal{R}(\mathcal{X})$ is the *one-step reachable set* from \mathcal{X} , defined as $\mathcal{R}(\mathcal{X}) \triangleq \{y \in \mathbb{R}^n : y = f(x), \forall x \in \mathcal{X}\}$. As a next step, a contractive fictitious dynamics $\phi(x)$ is defined a priori in the polytopic regions \mathcal{X}_i , $i = s+1, \dots, s_e$ (which define a partition of $\mathcal{X}_e \setminus \mathcal{X}$), as

$$x^+ = \phi(x) = \phi_i(x), \quad x \in \mathcal{X}_i, \quad i = s+1, \dots, s_e \quad (12)$$

such that \mathcal{X}_e is a PI set for the so-called *extended system*, defined as

$$x^+ = \begin{cases} f(x), & x \in \mathcal{X} \\ \phi(x), & x \in \mathcal{X}_e \setminus \mathcal{X}. \end{cases} \quad (13)$$

Moreover, we refer to the newly obtained set of indices as $\mathcal{I}_e \triangleq 1, \dots, s_e$. Then, a PWA Lyapunov function $V^e : \mathcal{X}_e \rightarrow \mathbb{R}$ is determined by linear programming, in order to satisfy equations (2) with $\eta = 1$ for all $x \in \mathcal{X}_e$. More precisely, we have

$$V_i^e(x) \triangleq F_i^e x + g_i^e, \quad i \in \mathcal{I}_e \quad (14a)$$

with $F_i^e \in \mathbb{R}^{1 \times n}$ and $g_i^e \in \mathbb{R}$, and then we define V^e as

$$V^e(x) = \max_{i \in \mathcal{N}_e(x)} V_i^e(x) \quad (14b)$$

with

$$\mathcal{N}_e(x) \triangleq \{i \in \mathcal{I}_e : x \in \bar{\mathcal{X}}_i\}. \quad (14c)$$

If a feasible realization of function $V^e(x)$ could be determined, the set \mathcal{P}^e was defined as

$$\mathcal{P}^e \triangleq \left\{ x : V^e(x) \leq \inf_{x \in \mathcal{X}_e \setminus \mathcal{X}} V^e(x) \right\} \quad (15)$$

and system (4) was proved to be asymptotically stable in \mathcal{P}^e . In this way, the stability properties of the system defined over a non invariant domain could be studied without relying on

extensive reachability analysis, which, as already mentioned, could easily turn out to be computationally intractable.

The fictitious dynamics provided an additional degree of freedom, but it was in general very hard if not impossible to know (except perhaps for very simple examples) which choice of the fictitious dynamics would lead to a larger set \mathcal{P}_e , or even if there existed a realization of $\phi(x)$ such that a set $\mathcal{P}_e \neq \emptyset$ could be determined. Therefore, in some cases, a wrong choice of the fictitious dynamics could prevent the extended system to converge to the origin, even if the origin of (4) was an asymptotically stable equilibrium point with a nonempty region of attraction included in \mathcal{X} .

The approach of the present paper overcomes this problem, since no dynamics are defined out of the set \mathcal{X} , and the Lyapunov function is defined only for $x \in \mathcal{X}$.

As a simple example where the new approach leads to better results, consider the following PWA system:

- $f(x) = -0.5x$ for $x \in \mathcal{X}_1 \triangleq [-1, 0)$;
- $f(x) = -0.3x$ for $x \in \mathcal{X}_2 \triangleq [0, 1]$;
- $f(x) = 2x$ for $x \in \mathcal{X}_3 \triangleq (1, 2]$;

Using the approach in [15], a new region would be defined as $\mathcal{X}_4 = (2, 4]$ (including all the points reachable in one step from \mathcal{X}). The fictitious dynamics in \mathcal{X}_4 would be defined a priori as a contractive one. For instance, it could be defined as $x^+ = 0.5x$, leading to a positively invariant set $\mathcal{X}_e = [-1, 4]$. For any initial condition $x(0) \in (1, 4]$, the state evolution would exhibit a limit cycle, therefore preventing to prove the (in this simple case, apparent) fact that the origin is an exponentially stable equilibrium point with domain of attraction $\mathcal{X}_1 \cup \mathcal{X}_2$. This result is instead obtained with the approach presented in this paper.

The next result on the comparison between the two methods is also introduced:

Theorem 3: Given a PWA system in form (4), whose dynamics is defined on \mathcal{X} , define the extended set \mathcal{X}_e as in (11), such that \mathcal{X}_e is a PI set for the extended system (13). Assume that a PWA USL function $V^e(x)$ has been determined for all $x \in \mathcal{X}_e$. Then, for system (4) defined on \mathcal{X} , there exists a feasible solution of problem (9) and a scalar $\beta \in \mathbb{R}_{>0}$ such that $V(x) = \beta V^e(x)$ for all $x \in \mathcal{X}$.

Proof: See Appendix. \square

In conclusion, it is always possible to obtain \mathcal{P} solving (9) when \mathcal{P}_e can be found with the method described in [15]. On the other hand, there are systems for which the method in [15] gives no solution (due to the specific choice of the fictitious dynamics), while it is possible to find a set \mathcal{P} through (9).

VII. SIMULATION EXAMPLE

We consider the application of the PWA control law of [3] to the following plant

$$x^+ = Ax + Bu \quad (16)$$

where

$$A = \begin{bmatrix} 0.9 & 0.6 \\ 0 & 0.9 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

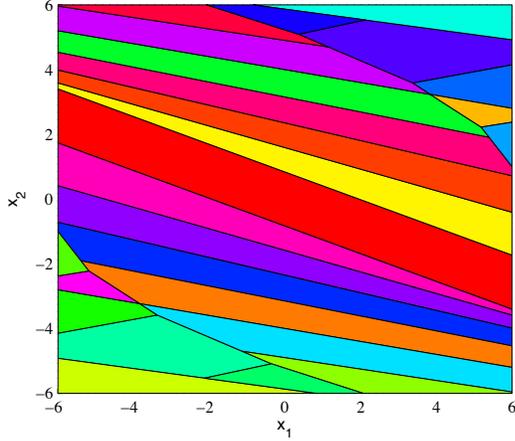


Fig. 1. Graphical representation of the regions \mathcal{X}_i of the considered closed-loop system

and where the goal is to optimally stabilize the origin satisfying the state and control constraints $x(k) \in \mathcal{X}$ and $u(k) \in \mathcal{U}$ for all $k \in \mathbb{Z}_{\geq 0}$, with $\mathcal{X} \triangleq \{x \in \mathbb{R}^2 : \|x\|_{\infty} \leq 6\}$ and $\mathcal{U} \triangleq \{u \in \mathbb{R} : |u| \leq 1\}$. In particular, following [3], we seek for an optimal LQ stabilizer over a prediction horizon $N = 8$, with weight matrices

$$Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad R = 0.1$$

on the state and control variables, respectively, and terminal weight matrix $P \in \mathbb{R}^2$, found as the solution of the Lyapunov equation $A'PA - P = -Q$.

To this end, we design an explicit MPC law, imposing the state constraints as *soft constraints*, and not imposing any terminal constraint, which leads to no a-priori guarantee of stability for the closed loop system.

The arising explicit MPC control law is defined as a PWA function for all $x \in \mathbb{R}^2$. However, the corresponding multiparametric program is solved only for $x \in \mathcal{X}$, obtaining 25 different regions, and the predictions $x(k+t)$, $t = 1, \dots, N$, are allowed to exit \mathcal{X} .

After synthesizing the control law, we obtain a PWA closed-loop system in the form (4) defined over \mathcal{X} , whose regions \mathcal{X}_i are shown in Fig. 1. One can check that the set \mathcal{X} is not a PI set for the closed-loop system. Therefore, we are interested in finding a set $\mathcal{P} \subset \mathcal{X}$ which is PI and contained in the region of attraction. According to Algorithm 1, we find the transition sets \mathcal{X}_{ij} and Ω_{ip} , and formulate the LP (9), setting $M = 100$. The LP is infeasible, and then a further refining of the regions is required, leading to a feasible LP, which proves that the origin of the closed-loop system is ES(\mathcal{P}) for the set \mathcal{P} shown in Fig. 2, with $\alpha_1 = 12.94$, $\alpha_3 = 10^{-5}$, and $\theta = 99.99$. The LP comprises 517 constraints and 87 variables, and solved using GLPK in 82 ms on a 2.4 GHz processor. The computation of the sets \mathcal{X}_{ij} and Ω_{ip} , together with all the other required operation apart from the LP, took about 12.6 s.

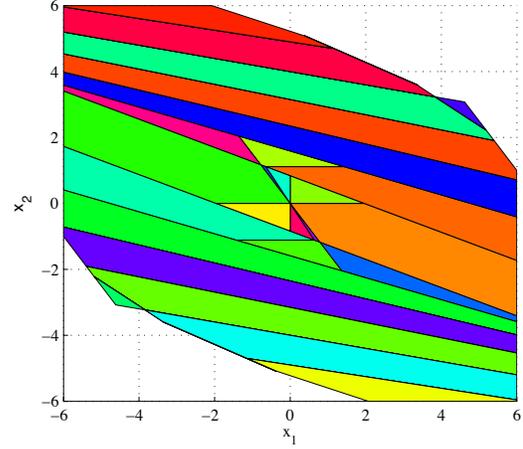


Fig. 2. The PI set \mathcal{P} for the considered example

VIII. CONCLUSIONS

In this paper, a method for stability and invariance analysis based on discontinuous PWA Lyapunov functions has been proposed. The method is shown to lead to a feasible solution in a broader range of cases than the method proposed in [15], thanks to the absence of the fictitious dynamics. The proposed method has also been successfully tested in simulation to analyze a-posteriori the stability of a closed-loop system, where an MPC control law without stability guarantees was applied.

APPENDIX

Proof of Theorem 2. Given $x \in \mathcal{X}_i$, define $\gamma_{i,h} \geq 0$, such that $\sum_{h=1}^{m_i} \gamma_{i,h} = 1$, as a set of coefficients defining x as a convex combination of the vertices of $\bar{\mathcal{X}}_i$. We obtain from (8a)-(8b) that

$$\begin{aligned} \alpha_1 \|x\| &= \alpha_1 \left\| \sum_{h=1}^{m_i} \gamma_{i,h} v_{i,h} \right\| \leq \sum_{h=1}^{m_i} \gamma_{i,h} \alpha_1 \|v_{i,h}\| \\ &\leq \sum_{h=1}^{m_i} \gamma_{i,h} (F_i v_{i,h} + g_i) = F_i x + g_i = V_i(x) \quad (17) \end{aligned}$$

This means that $V_i(x) \geq \alpha_1 \|x\|$, for all i such that $x \in \bar{\mathcal{X}}_i$, namely for all $i \in \mathcal{N}(x)$ (see (14c)). As a consequence, $\alpha_1 \|x\| \leq \max_{i \in \mathcal{N}(x)} \{F_i x + g_i\} = V(x)$. Since $\mathcal{X} = \bigcup_{i \in \mathcal{I}} \mathcal{X}_i$, this implies that, for all $x \in \mathcal{X}$, $V(x) \geq \alpha_1 \|x\|$. With a similar argument one can show that (8c) implies $V(x) \leq M$ for all $x \in \mathcal{X}$, and that (8g) implies $F_i x + g_i \geq \theta$ for all $x \in \Omega_{ip}$, with $i \in \mathcal{I}$ and $p \in \mathcal{I}_i^{\Omega}$. Considering that (8d) implies that $V_i(0) = 0$ for all $i \in \mathcal{I}_0$, and that $V(x) \leq M$ for all $x \in \mathcal{X}$, it is possible to find a scalar $\alpha_2 \in \mathbb{R}_{>0}$ such that $V(x) \leq \alpha_2 \|x\|$ for all $x \in \mathcal{X}$. We conclude that

$$\alpha_1 \|x\| \leq V(x) \leq \alpha_2 \|x\|, \quad \forall x \in \mathcal{X}. \quad (18)$$

Consider now any $x \in \mathcal{X}_I$ and let us assume, without loss of generality, that $x \in \mathcal{X}_{ij}$. Then, using again the convex combination $x = \sum_{h=1}^{m_{ij}} \gamma_{ij,h} v_{ij,h}$ such that $\sum_h \gamma_{ij,h} = 1$, we get

$$\begin{aligned}
V(f(x)) &= g_j + F_j \left[A_i \left(\sum_{h=1}^{m_{ij}} \gamma_{ij,h} v_{ij,h} \right) + a_i \right] \\
&= g_j + \sum_{h=1}^{m_{ij}} \gamma_{ij,h} (F_j (A_i v_{ij,h} + a_i)) \\
&\leq g_j + \sum_{h=1}^{m_{ij}} \gamma_{ij,h} (F_i v_{ij,h} + g_i - \alpha_3 \|v_{ij,h}\| - g_j) \\
&= \left(F_i \sum_{h=1}^{m_{ij}} \gamma_{ij,h} v_{ij,h} + \sum_{h=1}^{m_{ij}} \gamma_{ij,h} g_i - \alpha_3 \sum_{h=1}^{m_{ij}} \gamma_{ij,h} \|v_{ij,h}\| \right) \\
&\leq F_i x + g_i - \alpha_3 \|x\| = V(x) - \alpha_3 \|x\|,
\end{aligned}$$

which proves that $V_j(f(x)) - V_i(x) \leq -\alpha_3 \|x\|$ for all (i, j) such that $x \in \mathcal{X}_{ij}$. Since this is proven for all such pairs (i, j) , then

$$\max_{j \in \mathcal{N}(f(x))} (F_j(f(x)) + g_j) \leq \max_{i \in \mathcal{N}(x)} (F_i x + g_i) - \alpha_3 \|x\|$$

Since $\{\mathcal{X}_{ij}\}$ is a partition of \mathcal{X}_I , we obtain

$$V(f(x)) - V(x) \leq -\alpha_3 \|x\|, \quad \forall x \in \mathcal{X}_I. \quad (19)$$

Note that, after defining \mathcal{P} as in (10), one has $\mathcal{P} \subseteq \mathcal{X}_I$, because $F_i x + g_i \geq \theta$ for any $x \in \Omega_{ip}$, $i \in \mathcal{I}$, $p \in \mathcal{I}_i^\Omega$. Therefore, conditions (18) and (19) hold for all $x \in \mathcal{P}$, since $\mathcal{P} \subseteq \mathcal{X}_I \subseteq \mathcal{X}$. This fact leads to two conclusions:

- 1) since \mathcal{P} is a sublevel set of $V(x)$, then from (19), \mathcal{P} is a PI set;
- 2) from (18)-(19), applying Theorem 1, system (4) is ES(\mathcal{P}). ■

Proof of Theorem 3. According to the assumptions, for system (13), $V^e(x)$ is defined as a PWA function on \mathcal{X}_e . Moreover, from [15], there exist α_1^e , α_2^e , $\alpha_3^e \in \mathbb{R}_{>0}$ such that

$$\alpha_1^e \|x\| \leq V^e(x) \leq \alpha_2^e \|x\| \quad (20)$$

$$V^e(x^+) - V^e(x) \leq -\alpha_3^e \|x\|. \quad (21)$$

We will show that, starting from $V^e(x)$, we can always find another USL function $\tilde{V}^e(x)$ defined for $x \in \mathcal{X}$, which is a feasible solution of (9).

First of all, define

$$\theta_{\inf} \triangleq \inf_{x \in \mathcal{X}_e \setminus \mathcal{X}} V^e(x) \quad (22)$$

$$V_{\max}^e \triangleq \max_{x \in \mathcal{X}} V^e(x) \quad (23)$$

$$\beta \triangleq M/V_{\max}^e \quad (24)$$

where M is the parameter introduced in the LP (9). Then, define

$$\tilde{V}^e(x) \triangleq \beta V^e(x), \quad x \in \mathcal{X}$$

It is now possible to state that

$$\tilde{V}^e(x) = \max_{i \in \mathcal{N}(x)} \tilde{V}_i^e(x) \quad (25)$$

with

$$\tilde{V}_i^e(x) \triangleq \tilde{F}_i^e x + \tilde{g}_i^e \quad (26)$$

where $\tilde{F}_i^e = \beta F_i^e$ and $\tilde{g}_i^e = \beta g_i^e$.

Note that $V^e(x)$ for $x \in \mathcal{X}$ satisfies, by construction,

$$\tilde{F}_i^e x + \tilde{g}_i^e \geq \beta \alpha_1^e \|x\|, \quad \forall x \in \mathcal{X}_i, \quad i \in \mathcal{I} \quad (27a)$$

$$\tilde{F}_i^e x + \tilde{g}_i^e \leq M, \quad \forall x \in \mathcal{X}_i, \quad i \in \mathcal{I} \quad (27b)$$

$$\tilde{g}_i^e = 0, \quad i \in \mathcal{I}_0. \quad (27c)$$

$$\begin{aligned}
&\tilde{F}_j^e (A_i x + a_i) + \tilde{g}_j^e - \tilde{F}_i^e x - \tilde{g}_i^e \\
&\leq -\beta \alpha_3^e \|x\|, \quad \forall x \in \mathcal{X}_{ij}, \quad (i, j) \in \mathcal{I} \times \mathcal{I} \quad (27d)
\end{aligned}$$

$$\tilde{F}_i^e x + \tilde{g}_i^e \geq \beta \theta_{\inf}, \quad \forall x \in \Omega_{ip}, \quad i \in \mathcal{I}, \quad p \in \mathcal{I}_i^\Omega. \quad (27e)$$

Note that if any of the conditions (27) is satisfied for all x in a given compact set, it is automatically satisfied also on the vertices of that set. Therefore, conditions (27) imply conditions (8), which means that \tilde{V}^e is a feasible solution of problem (9). ■

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