# Hierarchical Multi-Rate Control Design for Constrained Linear Systems

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Abstract— This paper proposes a hierarchical multi-rate control design approach to linear systems subject to linear constraints on input and output variables. At the lower level, a linear controller stabilizes the open-loop process without considering the constraints. A higher-level controller commands reference signals at a lower sampling frequency so as to enforce linear constraints on the variables of the process. By optimally constraining the magnitude and the rate of variation of the reference signals applied to the lower control layer, we provide quantitative criteria for selecting the ratio between the sampling rates of the upper and lower layers to preserve closed-loop stability without violating the prescribed constraints.

#### I. INTRODUCTION

The increasing demand for automation of large-scale systems requires engineers to develop more complex and scalable control designs, based on multi-layer and possibly decentralized architectures. The control problem becomes particularly difficult when the design must take into account constraints on input and output variables.

Model predictive control (MPC) has been extensively used in the process industries for control and coordination of large-scale systems subject to constraints [1]. Traditionally, MPC is used for generating reference signals to single-loop controllers in order to optimize a global performance and enforce constraints on multiple inputs and outputs. In order to achieve this task, MPC requires a dynamical model of the entire process, used to make predictions over which to optimize the control signals. As a consequence, MPC suffers from the aforementioned scalability and model maintenance issues, exacerbated by the complexity issue of solving a large-scale optimization problem on-line.

Decentralized and hierarchical MPC schemes have been investigated recently to address the complexity issue of centralized MPC. We refer the reader to the excellent recent survey [2]. Reference governors (RG) were also proposed to mitigate the complexity of MPC by separating the stabilization problem from the constraint fulfillment problem [3]– [7]. In the RG approach, a (global) model of the underlying closed-loop system is exploited in a predictive manner to provide a reference signal to the lower-level controller which is as close as possible to the desired one, compatibly with the given constraints. Although providing good computational benefits, reference governors have still the drawback of needing a detailed global dynamical model of the entire underlying closed-loop system for on-line optimization.

In this paper we propose a hierarchical multi-rate control approach that exploits the idea of manipulating reference



Fig. 1. Hierarchical control scheme

signals to enforce constraints. We assume that the openloop process is stabilized by a linear (possibly decentralized) controller with sampling time  $T_L$  without taking care of the constraints, whose reference signals are generated by a higher-level controller running at a larger sampling time  $T_H = NT_L$ . As in [6], the higher level controller bounds the commanded reference signals to prevent violations of the contraints. In this paper, however, constraints are set also on the *variations* of the reference signals. In addition we adopt a multi-rate setting, providing quantitative relations between the maximum allowed reference variations and to ratio  $N = T_H/T_L$  between the sampling times.

Multirate MPC schemes have been addressed in a variety of papers, see e.g. the early work [8], and the application papers [9], [10], where hybrid MPC control is used at the higher level to enforce complex linear and logical constraints. Two main issues arise in hierachical MPC design: the choice of a simple ("as much abstracted as possible, but not too much") prediction model of the underlying subsystem, and the choice of the sampling time  $T_H$ . Rule of thumbs suggest that the latter must be "large enough" to assume that the adopted prediction model is "enough consistent" with the true underlying closed-loop system, but "not too small" to ensure enough reactiveness of the hierarchical scheme to changes of desired references. In this paper we quantify exactly what "large enough" should be, and free the designer from concerns about the choose of the prediction model of the underlying closed-loop system. In fact, safe operations are guaranteed by the resulting magnitude and rate constraints on reference signals, no matter how the performance index (if any) is optimized on top by the higherlevel controller.

The paper is organized as follows. The proposed hierarchical control architecture is described in Section II. In Section III the constraints on reference signals and their dependence on the ratio between sampling times is characterized and optimized, and used in Section IV to the define the general hierarchical control design. A particular design for the upper control layer based on on-line optimization is described in Section V. Simulation results are reported in Section VI.

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#### II. PROBLEM SETUP

Consider the hierarchical control architecture depicted in Figure 1. The open-loop process is stabilized by a lower-level control layer running at a sampling frequency  $\frac{1}{T_L}$  (possibly decentralized, as depicted in Figure 1). At the higher level a supervisor running at a lower sampling frequency  $\frac{1}{T_H}$  decides the reference signals to send to the lower layer, possibly optimizing a performance criterion (such as an economic criterion), so as to make sure that a certain number of linear constraints on input and output variables are satisfied. Hierarchical control arrangements are frequent in industrial automation, because one can separate the concerns related to stabilization and disturbance rejection (taken care by the lower level controller at a high sampling frequency) and to steady-state optimization and constraint handling (taken care by the higher level, usually at a slower pace).

Consider the linear time-invariant (LTI) discrete-time model of the lower-level closed-loop system

$$\begin{cases} x(t+1) &= \bar{A}x(t) + \bar{B}u(t) \\ y(t) &= \bar{C}x(t) + \bar{D}u(t) \\ u(t) &= Kx(t) + Er(t) \end{cases}$$
(1)

where  $x(t) \in \mathbb{R}^n$ ,  $y(t) \in \mathbb{R}^{n_y}$ ,  $u(t) \in \mathbb{R}^{n_u}$ , and  $r(t) \in \mathbb{R}^{n_y}$ is the reference signal. We assume that K is an asymptotically stabilizing gain, which could either be a centralized or a decentralized one (see e.g. [11] for LMI-based synthesis of decentralized linear controllers). We also assume that a gain  $E \in \mathbb{R}^{n_u \times n_y}$  exists such that the DC-gain from r to y is the identity,

$$E = ((\bar{C} + \bar{D}K)(I - \bar{A} - \bar{B}K)^{-1}\bar{B} + \bar{D})^{-1} \qquad (2)$$

The closed-loop system (1) can be rewritten as

$$\begin{cases} x(t+1) = Ax(t) + Br(t) \\ y(t) = Cx(t) + Dr(t) \end{cases}$$
(3)

where  $A = \overline{A} + \overline{B}K$ ,  $B = \overline{B}E$ ,  $C = \overline{C} + \overline{D}K$ ,  $D = \overline{D}E$ .

Define the following ratio  $N = T_H/T_L$  between the two sampling times of the control layers, where we assume  $N \in \mathbb{N}$ .

The goal of the higher-level controller is to command the piecewise constant vector of references r(t)

$$r(t) = r_k, \ t = kN, \dots, (k+1)N - 1, \ k = 0, 1 \dots$$
 (4)

to the lower-level controller u(t) = Kx(t) + Er(t) in a way that the output vector y(t) is kept within the admissible output polytope

$$\mathcal{Y} = \{ y \in \mathbb{R}^{n_y} : H_y y \le K_y \}$$
(5)

where  $H_y \in \mathbb{R}^{q \times n_y}$ ,  $K_y \in \mathbb{R}^q$ . Note that input constraints may be also embedded in (5) by augmenting the output vector so that matrix [C D] includes the rows of [K E].

The main goal of this paper is to determine simultaneously the ratio N and restrictions on  $r_k$  and on the set point changes  $\Delta r_k = r_k - r_{k-1}$  so that  $y(t) \in \mathcal{Y}$ . To this end, let the reference vector r(t) be constrained within the tightened set

$$\mathcal{R} = \{ r \in \mathbb{R}^{n_y} : H_y r \le K_y - \Delta K_y \}, \tag{6}$$

where  $\Delta K_y \in \mathbb{R}^q$ ,  $\Delta K_y > 0$  component-wise, and assume set points r(t) are changed in a way that the tracking error  $e(t) \triangleq y(t) - r(t)$  is always within the set

$$\mathcal{E} = \{ e \in \mathbb{R}^{n_y} : H_y e \le \Delta K_y \}$$
(7)

Vector  $\Delta K_y$  is a tuning knob of the proposed approach: the smaller the components of  $\Delta K_y$ , the larger is the set  $\mathcal{R}$  of admissible set points  $r_k$ , but the smaller will be the admissible reference increments  $\Delta r_k$  to maintain tracking errors within  $\mathcal{E}$ .

Let  $x_r \in \mathbb{R}^n$  be the steady-state state corresponding to a reference signal  $r \in \mathbb{R}^{n_y}$ ,  $x_r = Ax_r + Br$ ,  $x_r = G_x r$ ,  $G_x \triangleq (I - A)^{-1}B$ , and define the shift of coordinates  $\Delta x = x - x_r$ . Then, (3) can be rewritten as

$$\begin{cases} \Delta x(t+1) = A\Delta x(t) \\ e(t) = C\Delta x(t) \end{cases}$$
(8)

Let  $\Omega(0) \subseteq \mathbb{R}^n$  be the maximum admissible output set (MOAS) [12] for the closed-loop system (8) under the constraint  $e(t) \in \mathcal{E}$ 

$$\Omega(0) = \{\Delta x \in \mathbb{R}^n : H_y C A^k \Delta x \le \Delta K_y, \ \forall k \ge 0\}$$
  
$$\triangleq \{x \in \mathbb{R}^n : H_0 \Delta x \le K_0\}$$
(9)

where  $H_0 \in \mathbb{R}^{n_0 \times n}$  and  $K_0 \in \mathbb{R}^{n_0}$ , and define the reference-dependent invariant set

$$\Omega(r) = \{ x \in \mathbb{R}^n : H_0(x - G_x r) \le K_0 \}$$
(10)

Lemma 1: Let  $x(0) \in \Omega(r)$  and  $r(t) \equiv r \in \mathcal{R}, \forall t \geq 0$ . Then  $y(t) \in \mathcal{Y}, \forall t \geq 0$ .

*Proof:*  $x(0) \in \Omega(r)$  implies that  $y(t) - r(t) \in \mathcal{E}, \forall t \ge 0$ . Since  $r(t) \in \mathcal{R}$ , it follows that  $H_y y(t) = H_y e(t) + H_y r(t) \le \Delta K_y + K_y - \Delta K_y \le K_y, \forall t \ge 0$ .

The main idea of this paper is the following. Assume that the reference  $r_k \in \mathcal{R}$  is issued at time t = kN, and that N is large enough so that  $x(t + N - 1) \in \Omega(r_k)$ . At time t = (k + 1)N consider the new reference is  $r_{k+1} \in \mathcal{R}$ . If  $\Delta r_{k+1} = r_{k+1} - r_k$  is "small enough", then  $x(t + N) \in$  $\Omega(r_{k+1})$ . The goal of the next section is to quantify the relationship between the maximum reference variation  $\Delta r_k$ , the ratio N between the sampling intervals  $T_H$ ,  $T_L$ , and  $\Delta K_y$ such that every  $T_H = NT_L$  steps the state vector x(t) of the plant is guaranteed to lie in an invariant set  $\Omega(r_k)$ .

### **III. COMPUTATION OF MAXIMUM REFERENCE RATES**

Assume the ratio N between the sampling times of the upper and lower layers of control is given. Consider the problem of determining the initial state  $x(0) \in \Omega(r_1)$  and the minimum reference variation  $\Delta r(N) = r_2 - r_1$  between two reference values  $r_1, r_2 \in \mathcal{R}$  such that the state x(N) is

<sup>&</sup>lt;sup>1</sup>As  $\Delta K_y > 0$  and A is asymptotically stable,  $\Omega(0)$  is generated by a finite number of inequalities, as proved in [12]. We assume that  $(H_0, K_0)$  are a minimal hyperplane representation of  $\Omega(0)$ .

outside the invariant set  $\Omega(r_2)$ :

s.t.

$$\Delta r(N) = \inf_{r_1, r_2, x(0)} ||r_2 - r_1||_{\infty}$$
(11a)

$$r_1, r_2 \in \mathcal{R} \tag{11b}$$

$$x(0) \in \Omega(r_1) \tag{11c}$$

$$x(t+1) = Ax(t) + Br_2$$
 (11d)  
 $t = 0, 1, \dots, N-1$ 

$$x(N) \notin \Omega(r_2) \tag{11e}$$

Because of constraint (11e), the optimization problem (11) is nonconvex. However, it can be conveniently recast as a mixed-integer linear programming (MILP) problem by introducing an auxiliary binary vector  $\delta \in \{0, 1\}^{n_0}$ , satisfying the following constraints

$$\begin{bmatrix} \delta^i = 1 \end{bmatrix} \leftrightarrow \begin{bmatrix} H_0^i(x(N) - G_x r_2) \le K_0^i \end{bmatrix}$$
(12a)

$$\sum_{i=0}^{n_0} \delta^i \le n_0 - 1 \tag{12b}$$

where the superscript i denotes the ith component or row. The logical constraint (12a) can be converted to mixedinteger linear inequalities using the standard "big-M" approach

$$\begin{aligned} H_0^i(x(N) - G_x r_2) - K_0^i &\leq M_+^i(1 - \delta^i) \quad (13a) \\ H_0^i(x(N) - G_x r_2) - K_0^i &\geq (M_-^i - \sigma)\delta^i + \sigma \, (13b) \end{aligned}$$

where  $i = 1, ..., n_0$ ,  $\sigma > 0$  is a small number (e.g.: the machine precision) and  $M_-, M_+ \in \mathbb{R}^{n_0}$  are vectors of lower and upper bounds obtained by solving the following linear programs

$$M_{-}^{i} = \min_{\left[ \begin{array}{c} x(0) \\ r_{1} \\ r_{2} \end{array} \right]} \left[ H_{0}A^{N} \quad 0 \quad H_{0}R_{G} \right]^{i} \left[ \begin{array}{c} x(0) \\ r_{1} \\ r_{2} \end{array} \right] - K_{0}^{i}$$
(14a)

s.t. 
$$\begin{bmatrix} H_0 & -H_0 G_x & 0\\ 0 & H_y & 0\\ 0 & 0 & H_y \end{bmatrix} \begin{bmatrix} x(0)\\ r_1\\ r_2 \end{bmatrix} \le \begin{bmatrix} K_0\\ K_y - \Delta K_y\\ K_y - \Delta K_y \end{bmatrix}$$
(14b)

where  $R_G \triangleq R_N - G_x$ ,  $R_N \triangleq \left(\sum_{i=0}^{N-1} A^i B\right)$ , and vector  $M_+$  is determined by changing min to max in (14). By introducing an additional variable  $\epsilon \geq ||r_2 - r_1||_{\infty}$ , we address problem (11) by solving the following MILP

$$\Delta r(N) = \min_{\left[x' \ r'_1 \ r'_2 \ \delta' \ \epsilon\right]'} \quad \epsilon \tag{15a}$$

s.t. 
$$\epsilon \ge \pm (r_2^j - r_1^j)$$
 ,  $j = 1, \dots, n_y$  (15b)

$$H_y r_1 \le K_y - \Delta K_y \tag{15c}$$

$$H_y r_2 \le K_y - \Delta K_y \tag{15d}$$

$$H_0(x - G_x r_1) \le K_0 \tag{15e}$$

$$H_0^i(A^N x + R_G r_2) - K_0^i \le M_+^i (1 - \delta^i)$$
(15f)

$$-H_0^i(A^N x + R_G r_2) + K_0^i \le -(M_-^i - \sigma)\delta^i - \sigma$$

$$\sum_{i=0}^{n_o} \delta^i \le n_o - 1 \tag{15g}$$

$$\delta^{i} = \{0, 1\}, \ i = 1, \dots, n_0 \tag{15i}$$

The quantity  $\Delta r(N)$  in (15) is the smallest change of reference vector (expressed in infinity norm) that can be applied to the closed-loop system (3) such that, starting from an invariant set  $\Omega(r_k)$ , the state vector lands outside a new invariant set  $\Omega(r_{k+1})$  after N steps. Or, in other words, for all reference changes  $||r_k - r_{k-1}||_{\infty} \leq \Delta r(N) - \sigma, \forall \sigma > 0$ , the closed-loop system (3) is such that, starting from an invariant set  $\Omega(r_{k+1})$  after N steps. Note that, because of the constraint  $r_1, r_2 \in \mathcal{R}$ , problem (15) may become infeasible for large N, that is any feasible perturbation of the setpoint keeps the state within the invariant set  $\Omega(r_2)$  after N steps. The following lemma shows a monotonicity property of  $\Delta r(N)$  with respect to N, for those values  $N \in \mathbb{N}$  for which  $\Delta r(N)$  is defined.

Lemma 2: Let  $\Delta r(N)$  be defined by the optimization problem (15). Then for any  $N_1, N_2 \in N$ ,  $N_1 < N_2$ , such that  $\Delta r(N_1)$ ,  $\Delta r(N_2)$  are defined it holds that

$$\Delta r(N_1) \le \Delta r(N_2) \tag{16}$$

*Proof:* We first prove by contradiction that  $\Delta r(N) \leq \Delta r(N+1)$ ,  $\forall N \in \mathbb{N}$  such that  $\Delta r(N+1)$  is defined. Assume that  $N \in \mathbb{N}$  exists such that  $\Delta r(N+1) < \Delta r(N)$ . This implies that there exists a state x and two references  $r_1, r_2 \in \mathcal{R}$  such that  $\Delta r(N+1) \leq ||r_1 - r_2||_{\infty} < \Delta r(N)$ ,  $x \in \Omega(r_1)$ ,  $A^{N+1}x + \sum_{i=1}^{N} A^i Br_2 \notin \Omega(r_2)$ . Then, also  $A^N x + \sum_{i=1}^{N-1} A^i Br_2 \notin \Omega(r_2)$ , otherwise, by invariance of  $\Omega(r_2)$ , also  $A^{N+1}x + \sum_{i=1}^{N} A^i Br_2$  would belong to  $\Omega(r_2)$ . Hence, the optimality of  $\Delta r(N)$  is violated, a contradiction. The monotonicity condition (16) for generic  $N_1$ ,  $N_2$  easily follows.

## IV. HIERARCHICAL CONTROLLER

Assume that N has been fixed and that the upper control layer commands set-points  $r_k$  under the constraints

$$\|r_k - r_{k-1}\|_{\infty} \le \Delta r(N) - \sigma, \ \forall k = 0, 1, \dots$$
 (17a)  
$$r_k \in \mathcal{R}, \ \forall k = -1, 0, 1, \dots$$
 (17b)

feeding the lower control layer as in (4).

Theorem 1: Let K be a lower-level feedback gain such that  $\overline{A} + \overline{B}K$  is a strictly Schur matrix, and assume that matrix E in (2) exists. Assume a vector  $r_{-1} \in \mathcal{R}$  exists such that the initial state  $x(0) \in \Omega(r_{-1})$ . Let the upper-level controller change the set-points  $r_k$  according to the constraints (17), in which  $\Delta r(N)$  is the solution of (15) and  $\sigma > 0$  is arbitrary small. Then the linear system  $(\overline{A}, \overline{B}, \overline{C}, \overline{D})$  satisfies the constraints  $y(t) \in \mathcal{Y}$  for all  $t \ge 0$ . If in addition  $\lim_{t\to\infty} r(t) = r \in \mathcal{R}$  then  $\lim_{t\to\infty} y(t) = r$ .

*Proof:* Because of (17),  $x(kN) \in \Omega(r_k)$ ,  $\forall k \ge 0$ . By Lemma 1, it follows that  $y(t) \in \mathcal{Y}$ ,  $\forall t = kN, \ldots, k(N + 1) - 1$ ,  $\forall k = 0, 1, \ldots$ , that is  $y(t) \in \mathcal{Y}$ ,  $\forall t \ge 0$ . To prove convergence of y(t) to r when  $\lim_{t\to\infty} r(t) = r$ , similarly to (8) define  $\Delta x(t) = x(t) - G_x r$  and rewrite (3) as

$$\begin{cases} \Delta x(t+1) = A\Delta x(t) + B(r(t) - r) \\ e(t) = C\Delta x(t) + D(r(t) - r) \end{cases}$$
(18)

As (18) is an asymptotically stable linear system it is also input-to-state stable [13], and hence it immediately

follows that  $\lim_{t\to\infty} \Delta x(t) = 0$ , which in turn implies that  $\lim_{t\to\infty} y(t) - r = 0$ .

Theorem 1 shows that any upper-level reference generation strategy satisfying constraints (17) guarantees the fulfillment of output constraints and asymptotic convergence to constant set-points. The MILP (15) provides the supremum  $\Delta r(N)$  of the reference variations  $||r_k - r_{k-1}||_{\infty}$  that the higher-level controller can apply for a given ratio N = $T_H/T_L$  between sampling times. It is worth to investigate the relation between  $\Delta r(N)$  and N further. In fact, the design of the higher control layer could be addressed from a different point of view: given a desired  $\Delta r$ , determine the minimum N such that  $\Delta r < \Delta_r(N)$ . In practical applications N is restricted to a range  $[N_{min}, N_{max}]$  of values: the upper layer is executed at a slower pace than the lower layer ( $N_{\min}$  not too small), but at the same time the upper layer should be reactive enough to adjust set-points ( $N_{\text{max}}$  not too large). Hence, it is worth to solve the MILP (15) only within the restricted range  $N \in [N_{\min}, N_{\max}]$  to characterize  $\Delta r(N)$ that, by Lemma 2, we know increases with N. In particular, it is of interest the ratio  $R(N) = \frac{\Delta r(N)}{N}$  which characterizes the maximum speed of change of the reference signal. In fact, the larger N the larger is the supremum of the variations  $\Delta r$  that the supervisor can issue, but the less frequently such variation happens, that is every  $NT_L$  sampling times. Another issue related to tuning of the upper control layer is the choice  $\Delta K_y$ : from one hand a larger  $\Delta K_y$  tightens the range of admissible references  $\mathcal{R}$ , but on the other hand it enlarges the size of the invariant set  $\Omega(r)$ , and therefore augments the achievable  $\Delta r(N)$ . There is therefore a tradeoff the designer must choose between constraints on reference signals ( $\mathcal{R}$ ) and constraints on reference speed (R(N)).

Because of the need of enforcing constraints (17) in the upper control layer, in the next section we propose a model predictive control (MPC) design strategy for such a layer, although any other constraint-handling strategy could be employed, such as static optimization or a rule-based selection.

### V. MPC DESIGN OF UPPER CONTROL LAYER

We introduce an upper-layer MPC strategy, denoted as HiMPC, for generating the reference signal r under constraints (17).

## A. Prediction model

We consider an under-sampled and possibly reduced-order model of the lower-level closed-loop model (3), evolving with sampling time  $T_H = NT_L$ 

$$\begin{cases} x_{k+1}^{H} = A_{H}x_{k}^{H} + B_{H}r_{k} \\ y_{k} = C_{H}x_{k}^{H} + D_{H}r_{k} \end{cases}$$
(19)

where  $y_k = y(kN)$ ,  $x_k^H = Zx(kN)$ , and Z is a matrix mapping the original state x(kN) into the new state  $x_k^H$  (in case the order of the system is not reduced Z = I). Model (19) can be easily obtained by resampling system (3) using standard discretization methods. As a consequence, fastenough modal responses become negligible, which implies that the HiMPC algorithm can exploit only an incomplete information about the underlying closed-loop dynamics. This



Fig. 2. Mass-spring system considered in the simulation example

is a very convenient feature when HiMPC is applied to supervise a decentralized control layer, where maintaining a global detailed dynamical model of the entire lower-level closed-loop process may be a hard task. In the extreme case in which all dynamics are neglected, model (19) becomes the following static model + one-step delay

$$\begin{cases} x_{k+1}^H = r_k \\ y_k = x_k^H \end{cases}$$
(20)

where  $x_k^H = r_{k-1} \in \mathbb{R}^{n_y}$  is a state buffering the reference signal for one step  $T_H$ . Model (19) is particularly appropriate for large values of N. Note that in case model (20) is used, no feedback from the states x of the process is required by the upper control layer.

#### B. Cost function and constraints

The upper-layer MPC controller must embed constraints (17) on the generated references, to ensure stability and constraint satisfaction. It may also embed additional constraints on the reference signals, such as mixed logical/linear constraints (see e.g. [9]).

The MPC controller can optimize virtually any cost function of  $r_k$ ,  $\Delta r_k$ , and  $x_k^H$ , that may be dictated for instance by economic objectives.

Note also that if  $H_y$  is block-diagonal (for example,  $\mathcal{Y}$  is a box), then  $\mathcal{R}$  is also block diagonal, and if performance objectives and possibly other additional constraints are also block diagonal, so that HiMPC based on model (20) can be implemented in a decentralized way.

Finally, note that when HiMPC is based on model (20), a simple static optimization with respect to  $r_k$  can be setup, that possibly leads to a small-scale linear or quadratic programming problem. In this case, multiparametric programming algorithms can be exploited to convert HiMPC into a piecewise affine control law [14].

## VI. SIMULATION EXAMPLE

#### A. Problem description

We test the performance of the proposed HiMPC approach on the multi-mass-spring system depicted in Figure 2. Although the example is academic and relatively low-dimensional, the concepts illustrated in the example are immediately scalable to larger systems.

The process is composed by four masses moving vertically, each one connected by a spring to a fixed ceiling, subject to damping due to viscous friction with the environment, and connected to its neighbor mass by another spring. The values of the parameters of the system are reported in Table I. An input force u [Nm] can be applied to each mass by the lower-level controller. The output of the system is the vector y collecting the vertical positions  $y^1, \ldots, y^4$  of the masses.

TABLE I
PLANT PARAMETERS

Physical characteristic	symbol	value
mass	M	5 [kg]
viscous friction	$\beta$	0.1 [kg/s]
vertical elastic coefficient	$K_v$	1 [N/m]
lateral elastic coefficient	$K_l$	0.1 [N/m]

The dynamics of the discrete-time model of the system obtained by exact discretization with sampling time  $T_L = .25$  s are described the following matrices

The lower level regulator was designed as a centralized LQR with unit weights on all inputs and outputs, modified by zeroing all extra block-diagonal terms to obtain the decentralized linear gain

$$K = \begin{bmatrix} \begin{smallmatrix} -0.3102 - 2.1343 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -0.2842 - 2.0997 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -0.2842 - 2.0997 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -0.3102 - 2. \end{bmatrix}.$$

It is immediate to verify that the closed-loop matrix  $\overline{A} + \overline{B}K$  is strictly Schur and that matrix E in (2) is well posed. The HiMPC controller is designed to enforce the output constraint  $y(t) \in \mathcal{Y}$ , where  $\mathcal{Y} = \{y \in \mathbb{R}^4 : -0.3 \le y^i \le 1, y^2 \le y^1 + 0.3, i = 1, \dots, 4\}$ , or  $H_y = \begin{bmatrix} I \\ -1 & 1 & 0 \end{bmatrix}^I$ ,  $K_y = \begin{bmatrix} 1 & 1 & 1 & 0.3 & 0.3 & 0.3 & 0.3 & 0.3 \end{bmatrix}^I$ , corresponding to constraining mass positions between -0.3 and 1 m, and by preventing mass #1 to go below mass #2 by more than 0.3 m.

HiMPC adopts a linear MPC formulation based on model (19) or, in alternative, model (20), using the linear MPC setup of the Hybrid Toolbox [15]. The prediction horizon is 2, the control horizon 1, unit weights are used on reference increments and on mass position errors, i.e., on the deviations of  $y_k$  from a user-defined reference position signal p(t). The constraints on control signals  $r_k \in \mathcal{R}$  and on their increments  $|r_k^i - r_{k-1}^i| \leq \Delta r(N) - \sigma$ , i = 1, 2, 3, 4are enforced ( $\sigma$ =machine precision). The quantity

$$\Delta K_y = \Delta_0 \left[ 1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 0.4 \right]'$$

is chosen to restrict the tracking error, where  $\Delta_0$  is a scaling factor. The relation between N,  $\Delta_0$ , and  $\Delta r$  is reported in Table II. A "\*" in the table denotes that (11) has no solution, which means that constraints on  $\Delta r$  become redundant with respect to the condition  $r \in \mathcal{R}$ . The scaling factor  $\Delta_0 = 0.3$ will be used in the following simulations, as it provides a good compromise between the size of the invariant sets  $\Omega(r)$ and the size of the admissible reference set  $\mathcal{R} = \{r \in \mathbb{R}^4 : 0 \le r^i \le 0.7, i = 0, \dots, 4\}.$ 

TABLE II  $\Delta r$  for different values of N and scalings factors  $\Delta_0$ 



Fig. 3.  $\Delta r(N)$  and  $R = \frac{\Delta r(N)}{N}$ , obtained by solving (15) for  $\Delta_0 = 0.3$ 

To give an example of the complexity of the MILP optimization problem (15), this is solved in 141.2 s for N = 49 and  $\Delta_0 = 0.3$  on an iMac Intel Core 2 Duo 2.8GHz running Matlab R2009b under Windows XP. The MPT Toolbox [16] was used for invariant set computations, YALMIP [17] and Cplex 9.0 [18] to formulate and solve, respectively, the MILP (15). Figure 3 shows the resulting  $\Delta r(N)$  and the ratio  $R(N) = \Delta r(N)/N$  as a function of N, on the left and on the right axis respectively. The function depicted with a continuous line in Figure 3 is nondecreasing, in accordance with Lemma 2. Moreover Figure 3 shows the maximum reference rate that can be generated by the upper layer of the proposed hierarchical control scheme, which in this particular case is also increasing with N. By inspecting Figure 3, a good value of N is 49, where both R(N) and  $\Delta r(N)$  are maximized. The resulting reference variations are  $\Delta r(N) = 0.7.$ 

## B. Simulation results

Denote by HiMPC<sup>49</sup> the upper-layer MPC controller running every  $T_H = 49 \cdot T_L$  seconds, based on prediction model (19), obtained by resampling (3) with sampling time  $T_H$ . Denote by HiMPC<sup>49</sup><sub>DC</sub> the alternative controller based on model (20). To demonstrate the effectiveness of the proposed hierarchical approach, we compare it to other two controllers: "HiNone", where the upper layer is simply bypassed and constraints are ignored by feeding p(t) to the lower level, and "HiQP" that selects  $r_k$  according to the following quadratic program (QP)

$$\min_{r_k} \qquad \|r_k - r_{k-1}\|_2^2 + \|r_k - p(t)\|_2^2 \qquad (21a)$$

s.t. 
$$r^2 - r^1 \le 0.3$$
 (21b)

$$-0.3 \le r_k^i \le 1, \ i = 1, 2, 3, 4$$
 (21c)

which completely ignores the underlying closed-loop dynamics, and therefore gives no guarantee that constraints on y(t) are enforced. We assume that the user-defined reference signal p(t) only varies at common multiples of the sampling times,  $p(t) = p_k, \forall t \in [kN, \dots, (k+1)N - 1], k = 0, 1 \dots$ 



Closed-loop trajectories showing the position of mass #1 Fig. 4. (continuous line) and #2 (dashed line): HiMPC (blue), HiNone (black), HiQP (purple). User defined reference p(t) (dash dotted red) and reference constraints (dotted green) are also shown. On the right zoom of the trajectories showing the violation of the constraint  $y_2 - y_1 \le 0.3$ 



References generated by HiMPC (blue), HiNone (black), HiQP Fig. 5. (purple) for mass #1 (continuous) and mass #2 (dashed)

Figures 4 and 5 show the closed-loop trajectories of the positions of masses #1 and #2 and of the commanded references, respectively, from zero initial condition x(0). The trajectories obtained by using HiMPC<sup>49</sup> and the trajectories of masses #3 and #4 are not reported in the figures. The reasons are that the HiMPC controller based on the resampled model (19) behaves very similarly to the one based on the static+delay model (20), due to the fact that the sampling time  $T_H = NT_l = 12.25$  s is long enough to neglect the closed-loop dynamics; moreover, despite the coupling due to springs, masses #3 and #4 track a constant reference very tightly, even during setpoint variations on the other masses #1, #2.

The unfiltered reference  $p^1 = 0.7$  applied by HiNone during the first sample instant  $[0, T_H]$  makes mass #1 violate the upper limit  $y^1 \leq 1$  between time  $t \approx 3$  s to  $t \approx 6$ s. At time t = 49 s a transition from  $p^1 = 0.7$  to  $p^1 =$ 0.1 is requested, while  $p^2$  is decreased to 0.6. The user is demanding a steady-state infeasible configuration of the masses, since  $p^2 - p^1 \ge 0.3$ . Note the detail evidenced in Figure 4 that HiNone tracks the references violating the constraint. Since HiQP does not tighten enough the constraints by  $\Delta K_y$ , at time  $t \approx 50$  s a violation occurs of the constraint  $y^2 - y^1 \le 0.3$  by 0.03. At time t = 98 s the setpoints are set again to a feasible configuration  $p^1 = 0.4$ and  $p^2 = 0.7$ .

Comparing HiNone, HiQP, HiMPC, it is apparent that HiMPC is the most cautious, as evidenced by the commanded reference signals depicted in Figure 5, but it is also the only one that is able to enforce all constraints correctly.

#### VII. CONCLUSIONS

This paper has proposed an approach to hierarchical multi-rate control design that enforces constraints on the variables of the process and guarantees closed-loop stability. By constraining both the magnitude and the variation of the reference signals applied to the lower control layer, we have provided quantitative guidelines for selecting the ratio between the sampling rates of the upper and lower layers, driven by the idea that the state of the process must always lie in an invariant set at the sampling instants of the higherlevel controller. We believe that the approach provides good insight in the design of hierarchical schemes for decentralized control systems. Future research issues that are worth to address include the extension of the proposed idea to handle mixed logical/linear constraints on the underlying closedloop system.

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