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Synthesis of Networked Switching Linear Decentralized Controllers

D. Barcelli, D. Bernardini and A. Bemporad

***Abstract*— This paper proposes an approach based on linear matrix inequalities for synthesizing a set of decentralized regulators for discrete-time linear systems subject to input and state constraints. Measurements and command signals are exchanged over a sensor/actuator network, in which some links are subject to packet dropout. The resulting closed-loop system is guaranteed to asymptotically reach the origin, even if every local actuator can exploit only a (possibly time-varying) subset of state measurements. A model of packet dropout based on a finite-state Markov chain is also considered to exploit available knowledge about the stochastic nature of the network. For such model, a set of decentralized switching linear controllers is synthesized that guarantees mean-square stability of the overall controlled process under packet dropout and soft input and state constraints.**

I. INTRODUCTION

Networked control systems (NCSs) are characterized by a topological distribution over the physical space that sometimes prevents the use of centralized control solutions. In fact, the set of measurements might not be available at each control instant, due for instance to temporarily or permanently faulty sensors connections. A natural workaround is to define a set of controllers, each one in charge of commanding only a subset of actuators. The underlying idea is that the information provided by a subset of sensor measurements might be enough to control a subset of actuators satisfactorily. In this case, a decentralized control scheme clearly reduces the communication traffic over the network, allowing for a simpler network structure.

These considerations led, since the 70's, to look with interest to decentralized control, mainly investigating stability properties [1]. In the 90's, the rise of convex optimization techniques allowed for convex formulations of decentralized control problems [2], [3]. Decentralized estimation and control schemes based on distributed convex optimization ideas have been proposed recently by means of Lagrangian relaxations [4], [5], where global solutions are achieved after a (possibly large) number of inter-agent communications. Hence, looking at a real implementation, the sample time must be set conservatively high in order to let all the agreements to conclude without having consequences on the control action. Moreover, the need of mutual exchange of information between network agents produces an overhead in the communication channel which must be taken into account when dealing with network-related issues such as delay and packet loss.

Davide Barcelli and Daniele Bernardini are with the Department of Information Engineering, University of Siena, Italy. Email: {barcelli,bernardini}@dii.unisi.it.

Alberto Bemporad is with the Department of Mechanical Engineering, University of Trento, Italy. Email: bemporad@ing.unitn.it.

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In this paper we present an approach for the off-line synthesis of a set of decentralized linear regulators for discrete-time linear systems subject to input and state constraints. Measurements are provided by a distributed set of sensors to a distributed set of actuators through a network connection, in which some of the links are subject to random packet dropout. We aim at enforcing stability of the closed-loop system for every possible combination of packet losses that can occur in the network at every time step. Conservativeness of the resulting control law is reduced by using a different set of local control laws for every possible network configuration, without the need of communication among different controllers. Moreover, we take into account a model of packet dropouts based on a finite-state Markov chain, in order to exploit available knowledge about the stochastic nature of the network, and improve the closed-loop performance.

In the last years, mean-square stability of networked control systems (NCSs) has been often analyzed in literature. For example, in [6] a stabilizing controller for linear systems subject to random but bounded delays in the feedback loop is designed by augmenting the state vector and modeling the overall process as a Markov jump linear system. A NCS subject to communication constraints is studied in [7], where a Markov model is used to represent the dynamics of the transmission update times, and stability is guaranteed by means of a stochastic quadratic Lyapunov function. More recently, a framework to analyze stability of stochastic linear NCSs subject to time-varying transmission intervals, delays, packet dropouts and communication constraints by means of overapproximation methods has been proposed in [8]. Most of these works (if not all, and this paper makes no exception) rely on convex optimization, and more specifically on the formulation of optimization problems constrained by a set of linear matrix inequalities (LMIs) [9], [10].

II. CONTROL OVER IDEAL NETWORKS

Consider the discrete-time time-invariant linear system

$$x(t+1) = Ax(t) + Bu(t), \tag{1}$$

where $x = [x_1, \dots, x_n]' \in \mathbb{R}^n$ is the state, $u = [u_1, \dots, u_m]' \in \mathbb{R}^m$ is the input, $t \in \mathbb{N}_0$ is the time index, and the matrices $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$. We assume that states and inputs are subject to the constraints[1]

$$\|x(t)\|_2 \le x_{\max},\ \|u(t)\|_2 \le u_{\max},\ \forall t \in \mathbb{N}_0. \tag{2}$$

The process we consider is a networked control system, where spatially distributed sensor nodes provide measurements of the system state, and spatially distributed actuator nodes implement the control action. More in detail, at every

[1]Other kinds of constraints, such as element-wise bounds, can be considered in a similar fashion (see, e.g., [11]).

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time step t every sensor $s_1,\ldots,s_n$ measures a component $x_i(t)$ of the state vector, $i=1,\ldots,n$. Then, measurements are sent to actuators $a_1,\ldots,a_m$ through a user-defined networked connection. Given a process of the form (1) we define its *network topology* by means of an adjacency matrix $\Lambda\in\{0,1\}^{m\times n}$ with elements

$$\lambda_{ij}=\begin{cases}1 & \text{if sensor } s_j \text{ is linked to actuator } a_i,\\ 0 & \text{otherwise.}\end{cases} \tag{3}$$

for $i=1,\ldots,m$ and $j=1,\ldots,n$. In other words, $\lambda_{ij}=1$ if and only if the measurement $x_j(t)$ can be exploited to compute the input signal $u_i(t)$, $\forall t\in\mathbb{N}_0$. We assume here that all the network links are *ideal* (no packet dropout, delays, etc.); the decentralized control problem is extended to consider packet dropouts in Section III.

A. Linear controller synthesis

Our goal is to find a gain matrix $K\in\mathbb{R}^{m\times n}$ such that the system (1) in closed-loop with

$$u(t)=Kx(t) \tag{4}$$

is asymptotically stable. The desired control law must be decentralized, i.e., each actuator $a_1,\ldots,a_m$ can only exploit the measurements that are available in accordance with the network topology (3). In other words, each row i of K can only have non-zero elements in correspondence with the state measurements available to actuator a_i, $i=1,\ldots,m$. This imposes the following structure on K:

$$\lambda_{ij}=0 \;\Rightarrow\; k_{ij}=0,\quad i=1,\ldots,m,\; j=1,\ldots,n, \tag{5}$$

where k_{ij} is the (i,j)-th element of K.

Closed-loop stability is enforced through the condition

$$V(x(t+1))-V(x(t))\le -x(t)'Q_x x(t)-u(t)'Q_u u(t), \tag{6}$$

where $V:\mathbb{R}^n\to\mathbb{R}$ is a Lyapunov function of the state x for the closed-loop system given by (1) and (4), and $Q_x\in\mathbb{R}^{n\times n}$, $Q_u\in\mathbb{R}^{m\times m}$ are weight matrices, with $Q_x=Q_x{}'\succ 0$, $Q_u=Q_u{}'\succeq 0$. In the following we consider quadratic Lyapunov functions, and define

$$V(x)\triangleq x'Px, \tag{7}$$

with $P\in\mathbb{R}^{n\times n}$, $P=P'\succ 0$. It is well known that satisfaction of (6) for all time steps $t\in\mathbb{N}_0$ implies asymptotical stability of the closed-loop system (see, e.g., [11]). If (6) is satisfied, then we can show that

$$V(x(t))\ge J_\infty(t)\triangleq \sum_{i=0}^{\infty}\left(x(t+i)'Q_x x(t+i)+u(t+i)'Q_u u(t+i)\right),$$

i.e., $V(x(t))$ is an upper bound of the infinite-horizon quadratic cost-to-go $J_\infty(t)$ defined by Q_x, Q_u [11]. Our goal is to find the smallest scalar $\gamma>0$ such that

$$x(t)'Px(t)\le\gamma,\ \forall t\in\mathbb{N}_0, \tag{8}$$

or, equivalently, $x(t)'Q^{-1}x(t)\le 1$, $\forall t\in\mathbb{N}_0$, by substituting $Q=\gamma P^{-1}$. Clearly, the satisfiability of (8) depends on the initial state $x(0)$. Rather than finding the proper value of γ for a given initial state $x(0)\in\mathbb{R}^n$, we look for a γ which is valid for all $x(0)\in\mathcal{X}_0\subset\mathbb{R}^n$, where $\mathcal{X}_0\triangleq\mathcal{H}(v_1,\ldots,v_{n_v})$ is a polytope with vertices $v_1,\ldots,v_{n_v}$, and $\mathcal{H}(\cdot)$ denotes the convex hull operator, so that the controller K that we are going to synthesize is valid for any initial condition $x(0)\in\mathcal{X}_0$. As noted in [10], by making the standard substitution $K=YQ^{-1}$, $Y\in\mathbb{R}^{m\times n}$, we can obtain any desired structure for K by imposing the same structure on Y and fixing the block-diagonal structure of Q

$$(\lambda_{ij}=0)\Rightarrow y_{ij}=0,\ i=1,\ldots,m,\ j=1\ldots,n, \tag{9a}$$
$$(\lambda_{ij}=0)\wedge(\lambda_{ih}=1)\;\Rightarrow\; q_{hj}=0,\ q_{jh}=0,\quad i=1,\ldots,m,\ j=1,\ldots,n,\ h=1,\ldots,n, \tag{9b}$$

where $\wedge$ denotes logical "and".

Theorem 1: Consider an ideal network with topology $\Lambda\in\{0,1\}^{m\times n}$ and let $P=\gamma Q^{-1}$, $K=YQ^{-1}$ be obtained by solving the semidefinite programming (SDP) problem

$$\min_{\gamma,Q,Y}\ \gamma \tag{10a}$$
$$\text{s.t.}\ \begin{bmatrix} Q & \star & \star & \star\\ AQ+BY & Q & \star & \star\\ Q_x^{1/2}Q & \mathbf{0} & \gamma I_n & \star\\ Q_u^{1/2}Y & \mathbf{0} & \mathbf{0} & \gamma I_m\end{bmatrix}\succeq 0, \tag{10b}$$
$$\begin{bmatrix} Q & \star\\ AQ+BY & x_{\max}^2 I_n\end{bmatrix}\succeq 0, \tag{10c}$$
$$\begin{bmatrix} u_{\max}^2 I_m & \star\\ Y & Q\end{bmatrix}\succeq 0, \tag{10d}$$
$$\begin{bmatrix} 1 & \star\\ v_i & Q\end{bmatrix}\succeq 0,\ i=1,\ldots,n_v, \tag{10e}$$
$$Y\in\mathcal{Y},\ Q\in\mathcal{Q}, \tag{10f}$$

where I_n is the identity matrix in $\mathbb{R}^{n\times n}$, $\mathbf{0}$ is a matrix of appropriate dimension with all zero entries,

$$\mathcal{Q}\triangleq\{Q\in\mathbb{R}^{n\times n}: q_{hj}=q_{jh}=0 \text{ if } (\lambda_{ij}=0)\wedge(\lambda_{ih}=1),\ i=1,\ldots,m,\ j=1,\ldots,n,\ h=1,\ldots,n\},$$
$$\mathcal{Y}\triangleq\{Y\in\mathbb{R}^{m\times n}: y_{ij}=0 \text{ if } \lambda_{ij}=0,\ i=1,\ldots,m,\ j=1\ldots,n\},$$

and q_{ij}, y_{ij} are the (i,j)-th elements of Q and Y, respectively. If problem (10) is feasible, then system (1) with initial state $x(0)\in\mathcal{X}_0$ in closed-loop with the decentralized constant feedback control law (4) is asymptotically stable and satisfies the constraints (2).

Proof: In the particular case where $\mathcal{X}_0=\{x_0\}$ is a singleton (i.e., the initial state $x(0)=x_0$ is fixed), and $\Lambda=\mathbf{1}_{m\times n}$ is a matrix with all one entries (i.e., the control law is centralized and we have no constraints on the structure of K), asymptotical stability is a well known result that follows by showing that $V(x(t))=x(t)'Px(t)$ is a Lyapunov function for the closed-loop system (see, e.g., [9], [11]). Substituting $P=\gamma Q^{-1}$, condition (6) is converted by means of Schur complements to the LMI (10b). Using similar arguments, state and input constraints (2) are enforced by (10c) and (10d). It remains to prove (i) that stability is retained for every initial state $x(0)\in\mathcal{X}_0$ when $\mathcal{X}_0$ has dimension greater than 0, and (ii) that the control law $u(t)=Kx(t)$, with $K=YQ^{-1}$, can be implemented in a decentralized way, according to the network topology Λ. We see that (i) follows by convexity of the ellipsoid $\mathcal{E}_Q\triangleq\{x\in\mathbb{R}^n:\ x'Q^{-1}x\le 1\}$. In fact, since $\mathcal{E}_Q$ contains the vertices

v_i of $\mathcal{X}_0$ due to (10e), then $\mathcal{X}_0 \subset \mathcal{E}_Q$. Regarding (ii), as the structure of diagonal blocks is preserved by matrix inversion, Q block diagonal implies that Q^{-1} is also block-diagonal, and hence (9b) implies that $\tilde{q}_{hj} = 0$ and $\tilde{q}_{jh} = 0$, for all i, j, h such that $\lambda_{ij} = 0$ and $\lambda_{ih} = 1$, where $\tilde{q}_{hj}$ is the (h, j)-th element of Q^{-1}. Since $k_{ij} = \sum_{h=1}^{n} y_{ih}\tilde{q}_{hj}$, by (10f) it follows that the decentralized structure (5) is satisfied. ■

III. Control over lossy networks

In this section we consider packet dropouts occurring in some of the links of the communication network, referred to as *lossy* links. To account for the presence of lossy links in the network, we extend the definition of the topology $\Lambda \in \{-1, 0, 1\}^{m\times n}$ as follows

$$\lambda_{ij} = \begin{cases} 1 & \text{if ideal link between } s_j \text{ and } a_i, \\ -1 & \text{if lossy link between } s_j \text{ and } a_i, \\ 0 & \text{if no link between } s_j \text{ and } a_i, \end{cases} \tag{11}$$

for $i = 1, \dots, m,\ j = 1, \dots, n.$ No probabilistic model of packet loss is considered here; this will be introduced in Section IV.

We denote with l_i the number of lossy links connected with actuator a_i, $i = 1, \dots, m$ (i.e., the number of "-1" in the ith row of Λ), and with $L = \sum_{i=1}^{m} l_i$ the total number of lossy links in the network. Then, we can enumerate all the possible combinations of packet dropouts at a given time step t by replacing every "-1" in Λ with either a "1" or a "0". In this way we obtain a set of $\ell = 2^L$ matrices $\tilde{\Lambda}_h \in \{0, 1\}^{m\times n}$, $h = 1, \dots, \ell$, which describes all the possible network configurations. We denote by $\tilde{\Lambda}(t) \in \left\{\tilde{\Lambda}_1, \dots, \tilde{\Lambda}_\ell\right\}$ the network configuration at time $t \in \mathbb{N}_0$.

A. Switching controller synthesis

We want to design a set of gains $K_h \in \mathbb{R}^{m\times n}$, $h = 1, \dots, \ell$, to be used in the decentralized switching feedback control law

$$u(t) = \begin{cases} K_1 x(t) & \text{if } \tilde{\Lambda}(t) = \tilde{\Lambda}_1, \\ K_2 x(t) & \text{if } \tilde{\Lambda}(t) = \tilde{\Lambda}_2, \\ \quad\vdots & \quad\vdots \\ K_\ell x(t) & \text{if } \tilde{\Lambda}(t) = \tilde{\Lambda}_\ell. \end{cases} \tag{12}$$

Note that in general the implementation of (12) requires the controllers to be aware of the whole network status $\tilde{\Lambda}(t)$. This hypothesis is obviously not realistic in a decentralized framework. Hence, we impose an appropriate structure of the gains $K_1, \dots, K_\ell$, so that every local actuator a_i, $i = 1, \dots, m$, needs only to know which *local* measurements have been lost, regardless of the links status in the rest of the network. To accomplish this, we need to have a control law which univocally defines $u_i(t)$, $\forall i$, for all the network configurations $\tilde{\Lambda}_h$ that have identical values in their ith row. Namely, $[M]_i$ being the ith row of a generic matrix M, we want to impose

$$[\tilde{\Lambda}_h]_i = [\tilde{\Lambda}_j]_i \quad \Rightarrow \quad [K_h]_i = [K_j]_i, \tag{13}$$

for all $h, j = 1, \dots, \ell$, $i = 1, \dots, m$. This relation greatly reduces the number of variables to be considered in our optimization problem. In fact, we have only 2^{l_i} possible values of $[\tilde{\Lambda}(t)]_i$, $i = 1, \dots, m$. We refer to these row vectors as $\Gamma_1^i, \dots, \Gamma_{2^{l_i}}^i$, where $\Gamma_j^i \in \{0, 1\}^{1\times n}$, $\forall i, j$. Hence, we look for $\sum_{i=1}^{m} 2^{l_i}$ local gains $F_1^i, \dots, F_{2^{l_i}}^i$ which define the set of element-wise feedback control laws

$$u_i(t) = \begin{cases} F_1^i x(t) & \text{if } [\tilde{\Lambda}(t)]_i = \Gamma_1^i, \\ F_2^i x(t) & \text{if } [\tilde{\Lambda}(t)]_i = \Gamma_2^i, \\ \quad\vdots & \quad\vdots \\ F_{2^{l_i}}^i x(t) & \text{if } [\tilde{\Lambda}(t)]_i = \Gamma_{2^{l_i}}^i, \end{cases} \tag{14}$$

for all $i = 1, \dots, m$. These local gains $\{F_j^i\}$ are then combined to obtain the ℓ global gains $\{K_h\}$ used in (12). Our purpose is to guarantee the satisfaction of the stability constraint (6) in the presence of random packet dropouts. We are looking for a robust kind of stability, where no information on the dynamics regulating the evolution in time of $\tilde{\Lambda}(t)$ are exploited. Hence, here we take $V(x)$ in (7) as a common Lyapunov function for the switching closed-loop dynamics $x(t+1) = (A + BK_h)x(t)$, $h = 1, \dots, \ell$. In order to compute $K_1, \dots, K_\ell$ we substitute

$$K_h = Y_h Q^{-1},\ \forall h, \tag{15}$$

allowing a different matrix $Y_h \in \mathbb{R}^{m\times n}$ for every possible network configuration $\tilde{\Lambda}_h$. However, since we have a unique Q, it must hold

$$[Y_h Q^{-1}]_i \ = \ [Y_j Q^{-1}]_i,\ \forall i, j, h, \tag{16}$$

in order to satisfy (13). In other words, the structure of Q needs to preserve the structure of every K_h, $h = 1, \dots, \ell$.

Theorem 2: Consider a network with topology $\Lambda \in \{-1, 0, 1\}^{m\times n}$, and let $K_h = Y_h Q^{-1}$, $h = 1, \dots, \ell$, be obtained by solving the SDP problem

$$\min_{\gamma, Q, \{Y\}} \ \gamma \tag{17a}$$

$$\text{s.t.} \begin{bmatrix} Q & \star & \star & \star \\ AQ+BY_h & Q & \star & \star \\ Q_x^{1/2}Q & \mathbf{0} & \gamma I_n & \star \\ Q_u^{1/2}Y_h & \mathbf{0} & \mathbf{0} & \gamma I_m \end{bmatrix} \succeq 0,\ h = 1, \dots, \ell, \tag{17b}$$

$$\begin{bmatrix} Q & \star \\ AQ+BY_h & x_{\max}^2 I_n \end{bmatrix} \succeq 0,\ h = 1, \dots, \ell, \tag{17c}$$

$$\begin{bmatrix} u_{\max}^2 I_m & \star \\ Y_h & Q \end{bmatrix} \succeq 0,\ h = 1, \dots, \ell, \tag{17d}$$

$$\begin{bmatrix} 1 & \star \\ v_i & Q \end{bmatrix} \succeq 0,\ i = 1, \dots, n_v, \tag{17e}$$

$$[\tilde{\Lambda}_h]_i = [\tilde{\Lambda}_j]_i \ \Rightarrow \ [Y_h]_i = [Y_j]_i,$$
$$h, j = 1, \dots, \ell,\ i = 1, \dots, m, \tag{17f}$$

$$Y_h \in \tilde{\mathcal{Y}}_h,\ h = 1, \dots, \ell, \tag{17g}$$

$$Q \in \tilde{\mathcal{Q}}, \tag{17h}$$

where

$$\tilde{\mathcal{Q}} \triangleq \{Q \in \mathbb{R}^{n\times n} : q_{wj} = q_{jw} = 0 \text{ if } (\tilde{\lambda}_{ij}^h = 0) \wedge (\tilde{\lambda}_{iw}^h = 1),$$
$$i = 1, \dots, m,\ j, w = 1, \dots, n,\ h = 1, \dots, \ell\},$$
$$\tilde{\mathcal{Y}}_h \triangleq \{Y_h \in \mathbb{R}^{m\times n} : y_{ij}^h = 0 \text{ if } \tilde{\lambda}_{ij}^h = 0,$$
$$i = 1, \dots, m,\ j = 1 \dots, n\},$$

y_{ij}^h is the (i, j)-th element of Y_h, and $\tilde{\lambda}_{ij}^h$ is the (i, j)-th element of $\tilde{\Lambda}_h$, for all $h = 1, \dots, \ell$. If problem (17) is feasible, then system (1) with initial state $x(0) \in \mathcal{X}_0$ in closed loop with the decentralized switching feedback control law (12) is asymptotically stable and satisfies the constraints (2) for any possible realization of packet dropout.

Proof: Constraints (17b), obtained by using (12), substituting $K_h = Y_h Q^{-1}$, $h = 1, \dots, \ell$, and taking a Schur complement, are a sufficient condition to the satisfaction of (6) for every $\tilde{\Lambda}(t) \in \{\tilde{\Lambda}_1, \dots, \tilde{\Lambda}_\ell\}$, $\forall t$. Hence robust asymptotical stability is provided for every possible realization of packet dropout. Fulfillment of state and input constraints (2) and robustness with respect to the initial state $x(0) \in \mathcal{X}_0$, which are respectively enforced by (17c)–(17d) and (17e), follow by similar arguments of Theorem 1. It remains to prove that the resulting control law (12) can be implemented in a decentralized way as a combination of (14), $\forall i$. We fix the structure of Y_h to be equal to the structure of K_h, $\forall h$, by means of (17g). Since $K_h = Y_h Q^{-1}$, we have $k_{ij}^h = \sum_{w=1}^n y_{iw}^h \tilde{q}_{wj}$ and we must enforce the counterpart of (5) for the case of a switching control law, i.e., $\tilde{\lambda}_{ih}^h = 0 \;\Rightarrow\; k_{ij}^h = 0$, or equivalently

$$\tilde{\lambda}_{ij}^h = 0 \;\Rightarrow\; \sum_{w=1}^{n} y_{iw}^h \tilde{q}_{wj} = 0, \; \forall i, j, h. \tag{18}$$

Being the structure of Y_h assigned for a fixed h, a sufficient condition for the satisfaction of (18) is given by

$$(\tilde{\lambda}_{ij}^h = 0) \wedge (\tilde{\lambda}_{iw}^h = 1) \;\Rightarrow\; \tilde{q}_{wj} = 0, \; \forall i, j, w, h,$$

which is enforced by (17h) noting that Q is symmetric and block-diagonal. Finally, constraint (17f) together with (15) implies satisfaction of (16), which is a sufficient condition for (13) to hold. This ensures the uniqueness of the local control law (14) to be implemented given $[\tilde{\Lambda}(t)]_i$, $\forall i$, regardless of the global value of $\tilde{\Lambda}(t)$, and proves (12) to be a decentralized control law with the requested structure. ■

IV. Stochastic control under packet dropout

The robust approach undertaken in the previous section can be conservative in some cases, as it requires the existence of a common Lyapunov function which must be decreasing at every time step for every possible network configuration. In this section we pursue a relaxed stability condition by introducing a probabilistic model of the network and exploiting the possibly available stochastic information on packet dropout. We consider stability in the mean-square sense, which in this framework is equivalent to

$$\lim_{t \to \infty} \mathbb{E}\left[\|x(t)\|^2\right] = 0. \tag{19}$$

In other words, we allow the closed-loop Lyapunov function to occasionally increase from one step to another, as long as a decreasing condition of the form (6) is guaranteed to hold on average. The expectation is taken with respect to the realizations of $\tilde{\Lambda}(t)$, which is now modeled as a stochastic process.

A. Stochastic network model

Following the model proposed in [12], we assume that the probability distribution of the network configurations $\{\tilde{\Lambda}_h\}$ is modeled by a finite-state Markov chain with 2 states[2], called Z_1 and Z_2.

[2]More complex Markov chain models of packet loss could be considered here, such as the one used in [13].

The dynamics of the Markov chain are defined by a transition matrix

$$T = \begin{bmatrix} q_1 & 1 - q_1 \\ 1 - q_2 & q_2 \end{bmatrix} \tag{20}$$

such that $t_{ij} = \Pr[z(t+1) = Z_j \,|\, z(t) = Z_i]$, and by an emission matrix $E \in \mathbb{R}^{2 \times 2^L}$ such that $e_{ij} = \Pr[\tilde{\Lambda}(t) = \tilde{\Lambda}_j \,|\, z(t) = Z_i]$, being t_{ij} and e_{ij} the (i,j)-th element of T and E, respectively. In order to define the values in E we need to compute the probabilities of occurrence of $\tilde{\Lambda}_h$, $\forall h$. We assume that the occurrence of a packet dropout at a time step t in a given network link is an i.i.d. random variable, for every state of the Markov chain. In particular, we denote with d_1 and d_2, $0 < d_1 < d_2 < 1$, the probabilities of losing a packet at time t if $z(t) = Z_1$ and $z(t) = Z_2$, respectively (for example, in Z_1 we have “few” dropouts, and in Z_2 we have “many”, according to Gilbert’s model). Moreover, let $s_{1,h}$ and $s_{0,h}$ be the total number of lossy links in Λ which are mapped as ideal links and as no links in $\tilde{\Lambda}_h$, respectively, i.e.,

$$s_{1,h} = \sum_{(i,j) \in \mathcal{I}} \tilde{\lambda}_{ij}^h, \;\; h = 1, \dots, \ell, \tag{21a}$$

$$s_{0,h} = \sum_{(i,j) \in \mathcal{I}} (1 - \tilde{\lambda}_{ij}^h), \;\; h = 1, \dots, \ell, \tag{21b}$$

where $\mathcal{I} \triangleq \{(i,j) \in \{1, \dots, m\} \times \{1, \dots, n\} : \lambda_{ij} = -1\}$. Then, we can define the elements $\{e_{ij}\}$ of E as

$$e_{ij} = {d_i}^{s_{0,j}} (1 - d_i)^{s_{1,j}}, \; i = 1, 2, \; j = 1, \dots, \ell. \tag{22}$$

B. Stochastic switching controller synthesis

Our goal is to design two sets of control gains $K_{1,1}, \dots, K_{1,\ell}$, $K_{2,1}, \dots, K_{2,\ell}$, one for every state of the Markov chain, which define the switching control law

$$u(t) = \begin{cases} K_{1,1} x(t) & \text{if } \; z(t) = Z_1, \; \tilde{\Lambda}(t) = \tilde{\Lambda}_1, \\ \quad \vdots & \qquad\qquad \vdots \\ K_{1,\ell} x(t) & \text{if } \; z(t) = Z_1, \; \tilde{\Lambda}(t) = \tilde{\Lambda}_\ell, \\ K_{2,1} x(t) & \text{if } \; z(t) = Z_2, \; \tilde{\Lambda}(t) = \tilde{\Lambda}_1, \\ \quad \vdots & \qquad\qquad \vdots \\ K_{2,\ell} x(t) & \text{if } \; z(t) = Z_2, \; \tilde{\Lambda}(t) = \tilde{\Lambda}_\ell, \end{cases} \tag{23}$$

so that the closed-loop system is asymptotically stable in mean-square. Consider the stochastic counterpart of the decreasing condition (6)

$$\mathbb{E}\left[V(x(t+1))\right] - V(x(t)) \leq -x(t)' Q_x x(t) - \mathbb{E}[u(t)' Q_u u(t)]. \tag{24}$$

As shown in [14], fulfillment of (24) for all $t \in \mathbb{N}_0$ implies (19). Here $V(x)$ is intended to be a switching stochastic Lyapunov function for the closed-loop system, defined as

$$V(x(t)) \;\triangleq\; \begin{cases} x(t)' P_1 x(t) & \text{if } z(t) = Z_1, \\ x(t)' P_2 x(t) & \text{if } z(t) = Z_2. \end{cases} \tag{25}$$

We assume that the state $z(t) = Z_j$ of the communication network is known[3] at time t, with $j \in \{1, 2\}$. Hence, the expectations in (24) are

[3]In practice, one should estimate the state $z(t)$ of the communication network, see, e.g., [15].

$$\mathbb{E}\left[V(x(t+1))\right] = \sum_{h=1}^{\ell}\sum_{l=1}^{2} e_{jh}t_{jl}x(t)'(A+BK_{j,h})' P_l\left(A+BK_{j,h}\right)x(t), \quad (26a)$$

$$\mathbb{E}\left[u(t)'Q_u u(t)\right] = \sum_{h=1}^{\ell} e_{jh}x(t)'K_{j,h}'Q_u K_{j,h}x(t). \quad (26b)$$

In light of the above considerations, by using (23), (25) and (26), and substituting $P_j = \gamma Q_j{}^{-1}$, $K_{j,h} = Y_{j,h}Q_j{}^{-1}$, $\forall j,h$, we can translate (24) to an appropriate LMI condition with standard methods, as detailed in the following theorem.

Theorem 3: Consider a network with topology $\Lambda \in \{-1,0,1\}^{m\times n}$, where at each time step the packet dropout realizations are driven by the Markov chain defined by (20)–(22), and let $K_{j,h} = Y_{j,h}Q^{-1}$, $j=1,2$, $h=1,\dots,\ell$, be obtained by solving the SDP problem

$$\min_{\gamma,\{Q\},\{Y\}} \quad \gamma \quad (27a)$$

$$\text{s.t.} \begin{bmatrix} Q_j & \star & \star & \star \\ Q_x^{1/2}Q_j & \gamma I_n & \star & \star \\ C_{j,1} & \mathbf{0} & D_{j,1} & \star \\ C_{j,2} & \mathbf{0} & \mathbf{0} & D_{j,2} \end{bmatrix} \succeq 0,\ j=1,2, \quad (27b)$$

$$\begin{bmatrix} Q_j & \star \\ AQ_j+BY_{j,h} & x_{\max}^2 I_n \end{bmatrix} \succeq 0, \quad j=1,2,\ h=1,\dots,\ell, \quad (27c)$$

$$\begin{bmatrix} u_{\max}^2 I_m & \star \\ Y_{j,h} & Q_j \end{bmatrix} \succeq 0, j=1,2,\ h=1,\dots,\ell, \quad (27d)$$

$$\begin{bmatrix} 1 & \star \\ v_i & Q_j \end{bmatrix} \succeq 0,\ i=1,\dots,n_v,\ j=1,2, \quad (27e)$$

$$[\tilde{\Lambda}_h]_i = [\tilde{\Lambda}_w]_i \ \Rightarrow\ [Y_{j,h}]_i = [Y_{j,w}]_i, \quad j=1,2,\ h,w=1,\dots,\ell,\ i=1,\dots,m, \quad (27f)$$

$$Y_{j,h} \in \tilde{\mathcal{Y}}_{j,h},\ j=1,2,\ h=1,\dots,\ell, \quad (27g)$$

$$Q_j \in \tilde{\mathcal{Q}}_j,\ j=1,2, \quad (27h)$$

where

$$\tilde{\mathcal{Q}}_j \triangleq \{Q_j \in \mathbb{R}^{n\times n} : q_{lw}^j = 0, q_{wl}^j = 0 \text{ if } (\tilde{\lambda}_{iw}^h = 0)\wedge(\tilde{\lambda}_{il}^h = 1),\ w,l=1,\dots,n,\ i=1,\dots,m,\ h=1,\dots,\ell,\},\ j=1,2,$$

$$\tilde{\mathcal{Y}}_{j,h} \triangleq \{Y_{j,h} \in \mathbb{R}^{m\times n} : y_{iw}^{j,h} = 0 \text{ if } \tilde{\lambda}_{iw}^h = 0,\ i=1,\dots,m,\ w=1\dots,n\},\ \text{for all } j=1,2,\ h=1,\dots,\ell,$$

q_{lw}^j is the (l,w)-th element of Q_j, $y_{iw}^{j,h}$ is the (i,w)-th element of $Y_{j,h}$, and

$$C_{j,1} = \begin{bmatrix} \sqrt{e_{j1}t_{j1}}(AQ_j+BY_{j,1}) \\ \vdots \\ \sqrt{e_{j\ell}t_{j1}}(AQ_j+BY_{j,\ell}) \\ \sqrt{e_{j1}t_{j2}}(AQ_j+BY_{j,1}) \\ \vdots \\ \sqrt{e_{j\ell}t_{j2}}(AQ_j+BY_{j,\ell}) \end{bmatrix},\ C_{j,2} = \begin{bmatrix} \sqrt{e_{j1}}(Q_u^{1/2}Y_{j,1}) \\ \vdots \\ \sqrt{e_{j\ell}}(Q_u^{1/2}Y_{j,\ell}) \end{bmatrix},$$

$$D_{j,1} = \text{Blkdiag}\{\underbrace{Q_1,\ \dots,\ Q_1}_{\ell \text{ times}},\ \underbrace{Q_2,\ \dots, Q_2}_{\ell \text{ times}}\},$$

$$D_{j,2} = \text{Blkdiag}\{\underbrace{\gamma I_m,\ \gamma I_m,\ \dots,\ \gamma I_m}_{\ell \text{ times}}\}.$$

If problem (27) is feasible, then system (1) with initial state $x(0) \in \mathcal{X}_0$ in closed-loop with the decentralized switching feedback control law (23) is asymptotically stable in mean-square.

Proof: Constraints (27b), obtained by using (23), substituting $K_{j,h} = Y_{j,h}Q^{-1}$, $j=1,2$, $h=1,\dots,\ell$, and taking a Schur complement, are a sufficient condition to the satisfaction of (24) for every $\tilde{\Lambda}(t) \in \{\tilde{\Lambda}_1,\dots,\tilde{\Lambda}_\ell\}$, $\forall t$, distributed as modeled by (20)–(22). Hence asymptotical closed-loop stability in mean-square is provided. Robustness with respect to the initial state $x(0) \in \mathcal{X}_0$ and desired decentralized structure of the switching feedback control law (23), which are respectively enforced by (27e) and (27g)–(27h), follow by similar reasonings as in Theorems 1–2. The uniqueness of the switching feedback control law (23) is imposed by constraints (27f) for every state $z(t) \in \{Z_1,\ Z_2\}$ of the Markov chain, similarly to what proved in Theorem 2 for the case of a single Q and a single set of gains $\{Y_h\}$. Since by assumption the current Markov chain state $z(t)$ at time t is known to every actuator $a_1,\dots,a_m$, the feedback control law is univocally determined by choosing $u(t) = K_{j,h}x(t)$ if $z(t) = Z_j$, $\tilde{\Lambda}(t) = \tilde{\Lambda}_h$, and this completes the proof. ■

As convergence to the origin provided by Theorem 3 is intended in mean-square sense, we can no more refer to $\mathcal{E}_{Q_1} \triangleq \{x \in \mathbb{R}^n : x'Q_1^{-1}x \le 1\}$ and $\mathcal{E}_{Q_2} \triangleq \{x \in \mathbb{R}^n : x'Q_2^{-1}x \le 1\}$ as invariant ellipsoids for the closed-loop system, as we did in sections II–III. In fact, the decreasing condition (24) only holds in expected value. Hence, even though mean-square stability is retained, we have that $x(t) \in \mathcal{E}_{Q_i} \not\Rightarrow x(t+1) \in \mathcal{E}_{Q_j}$, $\forall t \in \mathbb{N}_0$, $i,j=1,2$. In other words, constraints (27c)–(27d) do not imply fulfillment of (2) at every time step, but only in an averaged sense.

V. Simulation results

In this section the proposed decentralized control schemes are tested on an open-loop unstable system (1) with $n=8$ states and $m=4$ inputs. The matrices A, B in (1) are selected randomly[4] and hence with a high chance that state dynamics are strongly mutually coupled. Measurements are provided from sensors to actuators according to a topology Λ as in (11) with 8 ideal links and 4 lossy links, defined as

$$\Lambda = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & -1 & 0 & 0 \\ -1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & -1 & 1 & 0 & 1 & 1 \end{bmatrix}.$$

The network topology is schematized in Fig. 1. Because of the network structure, the decentralized control law can only exploit a partial knowledge of the state value at each sample time. Since there are $L=4$ unreliable links, the number of possible network configuration is $\ell = 2^L = 16$. Packet dropouts are modeled by a 2-states Markov chain defined by (20) with $q_1 = 0.8$, $q_2 = 0.5$, $d_1 = 0.1$ and $d_2 = 0.5$.

We run $N_{sim} = 50$ simulations of $T_{sim} = 50$ time steps each with constraints (2) defined by $x_{max} = 25$, $u_{max} = 3$, weight matrices $Q_x = I_n$, $Q_u = 10^{-2}I_m$, and a random initial state $x(0) \in \mathcal{X}_0$, with $\mathcal{X}_0 = \{x_c\} + \{x \in \mathbb{R}^8 : \|x\|_\infty \le 2\}$ and $x_c = 7\cdot\mathbf{1}_{8\times 1}$. Fig. 2 shows the behavior of

[4]The MATLAB routine `drss` has been used to obtain the matrices A, B, modified to enforce one eigenvalue of A to be equal to 1.05. Numerical values of A, B are omitted for space reasons.

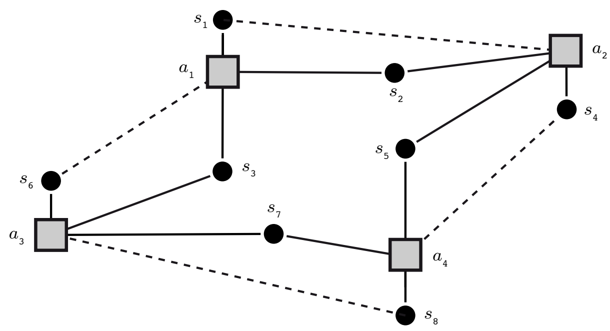

Fig. 1. Network topology used in simulations

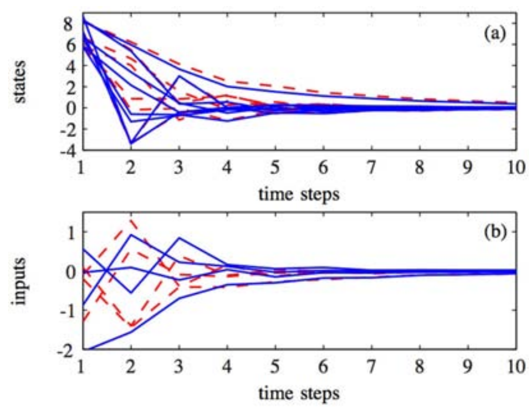

Fig. 2. Total (a) state and (b) input trajectories for robust (dashed line) and stochastic (solid line) decentralized controllers

the entire state and input vectors under decentralized robust and stochastic control.

Table I shows the results obtained by the proposed decentralized techniques in comparison with a centralized controller which implements the control law (4) without any restriction on the structure of K, or, in other words, which considers a topology $\Lambda = \mathbf{1}_{m\times n}$ where every actuator can exploit all the measurements. Performances are evaluated using the cumulated stage cost

$$J_i = \sum_{t=1}^{T_{sim}} \left(\|Q_x x(t)\|_2 + \|Q_u u(t)\|_2\right)$$

over the simulation horizon, where J_i refers to the ith run and $\mu(J_i)$, $\sigma(J_i)$ are the mean and the standard deviation of J_i over all the simulations. We can see that the stochastic decentralized controller achieves a good closed-loop behavior, being less conservative than the robust controller and still providing convergence to the origin. In Table I is also shown the computational time needed to solve the SDP problems off-line on a 2.8GHz Intel processor with the MATLAB modeling language YALMIP. Indeed, the complexity of the stochastic SDP problem (27), due mainly to the size of (27b), requires a CPU time of an order of magnitude larger. However, this increased computational load provides in turn a larger solution set, since the mean-square stability constraint (24) is less stringent than the robust counterpart (6).

VI. Conclusions

This paper has proposed a method based on semidefinite programming (SDP) for synthesizing decentralized linear control laws for networked linear systems, for both ideal and lossy networks. For the latter case, packet loss is modeled as a random process driven by a two-state Markov chain. The SDP problem formulation guarantees that the resulting switching controller enforces mean-square stability of the closed-loop system. Simulation results on a numerical example have shown that the performance deteriorates with respect to an ideal centralized controller, on average, by 15% in the case of stochastic decentralized control, and by 22% in the case of robust decentralized control.

TABLE I
SIMULATION RESULTS

	$\mu(J_i)$	$\sigma(J_i)$	CPU
Ideal network			(off-line time)
Centralized control	41.0	0	2.8 s
Decentralized control	45.1	0	1.2 s
Lossy network			(off-line time)
Dec. robust control	50.0	1.57	8.1 s
Dec. stochastic control	47.1	2.38	59.2 s

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