Joint 48th IEEE Conference on Decision and Control and
28th Chinese Control Conference
Shanghai, P.R. China, December 16-18, 2009

Scenario-based Model Predictive Control of Stochastic Constrained Linear Systems

Daniele Bernardini† and Alberto Bemporad†

Abstract— **In this paper we propose a stochastic model predictive control (MPC) formulation based on scenario generation for linear systems affected by discrete multiplicative disturbances. By separating the problems of (1) stochastic performance, and (2) stochastic stabilization and robust constraints fulfillment of the closed-loop system, we aim at obtaining a less conservative control action with respect to classical robust MPC schemes, still enforcing convergence and feasibility properties for the controlled system. Stochastic performance is addressed for very general classes of stochastic disturbance processes, although discretized in the probability space, by adopting ideas from multi-stage stochastic optimization. Stochastic stability and recursive feasibility are enforced through linear matrix inequality (LMI) problems, which are solved off-line; stochastic performance is optimized by an on-line MPC problem which is formulated as a convex quadratically constrained quadratic program (QCQP) and solved in a receding horizon fashion. The performance achieved by the proposed approach is shown in simulation and compared to the one obtained by standard robust and deterministic MPC schemes.**

I. INTRODUCTION

Model predictive control (MPC) is a popular strategy which has been widely adopted in industry as an effective means of dealing with multivariable constrained control problems [1], [2]. The idea behind MPC is to obtain the control signal by solving at each sampling time an open-loop finite-horizon optimal control problem based on a given prediction model of the process, by taking the current state of the process as the initial state. The control inputs are implemented in accordance with a receding horizon scheme.

However, classical MPC formulations do not provide a systematic way to deal with model uncertainties and disturbances. Many MPC control schemes have been proposed to guarantee stability and constraint fulfillment in the presence of disturbances. Most works are based on the min-max approach, where the performance index to be minimized is computed over the worst possible disturbance realization [3]–[8]. However, min-max policies are often computationally demanding, and the resulting control law is often too conservative, as no statistical properties about the disturbance are taken into account.

A different approach is addressed by stochastic MPC, where expected values of constraints/performance indices and convergence in probability are considered, by exploiting the available statistical information on the disturbance (see e.g. [9]–[13]). A common assumption when facing uncertainty with values on a continuous domain is to model the disturbance signal as a Gaussian noise, with given mean and covariance matrix [14]–[17]. This allows one to state theoretical results based on the analytical computation of the statistical properties of the controlled process. On the other hand, it can be a restrictive statistical assumption, as often in real processes uncertainty has general, time-varying characteristics which are not satisfactorily modeled by a standard normal distribution.

When dealing with optimization problems in the presence of stochastic data, the approximation of continuous uncertainty to a discrete domain is often used, and constructed in a way to preserve the main statistical properties of the underlying continuous process [18]–[20]. In this framework, the control problem formulation involves the setup of a scenario-based optimization tree, where only the most relevant disturbance patterns are modeled.

In this paper, we propose a stochastic MPC formulation based on scenario generation for linear systems with discrete multiplicative disturbances. Our main goal is to obtain a less conservative control action with respect to standard robust MPC [3], [4] by restricting ourselves to consider stochastic stability of the closed-loop system, and a stochastic performance index. We provide a control scheme for a very general class of discrete disturbances. Finally, with respect to [11] we provide a more flexible optimization tree design by following a maximum likelihood approach in the scenario generation, still enforcing robust constraint fulfillment and recursive feasibility.

The paper is organized as follows. The class of stochastic dynamical models dealt with in the paper is described in Section II. In Section III, two control schemes based on stochastic MPC are proposed, for both the unconstrained and the constrained case. Results of simulation tests are reported in Section IV, and conclusions are drawn in Section V.

II. MODEL DESCRIPTION

Consider the discrete-time linear system

$$x(k+1) = A(w(k))x(k) + B(w(k))u(k) , \qquad (1)$$

where $x(k) \in \mathbb{R}^{n_x}$ is the state, $u(k) \in \mathbb{R}^{n_u}$ is the input, $w(k) \in \mathcal{W}$ is a random disturbance, and $\mathcal{W} = \{w_1, w_2, \ldots, w_s\} \subset \mathbb{R}$ is a known discrete set of reals. By enumerating all the s possible realizations of $w(k)$, (1) can be rewritten as

$$x(k+1) = \begin{cases} A_1 x(k) + B_1(k)u(k) & \text{if } w(k) = w_1 \ , \\ A_2 x(k) + B_2(k)u(k) & \text{if } w(k) = w_2 \ , \\ \vdots & \vdots \\ A_s x(k) + B_s(k)u(k) & \text{if } w(k) = w_s \ , \end{cases} \qquad (2)$$

†Department of Information Engineering, University of Siena, Italy. Email: {bernardini,bemporad}@dii.unisi.it.

where (A_i, B_i), $i = 1, 2, \ldots, s$ are given matrices of appropriate dimensions. To model the evolution of the disturbance w, we introduce the time-varying probability vector $p(k) = [p_1(k),\ p_2(k),\ \ldots,\ p_s(k)]'$, which defines the probability of every disturbance realization at time k, as

$$p_j(k) = \mathbf{Pr}[w(k) = w_j],\ j = 1, 2, \ldots, s, \tag{3}$$

with

$$\sum_{j=1}^{s} p_j(k) = 1,\ \forall k\ . \tag{4}$$

In order to characterize the available information on p we assume the following holds.

Assumption 1: At time k, the current value of the probability vector $p(k)$ is known. Moreover $p(k) \in \mathcal{P}$, $\forall k \geq 0$, where $\mathcal{P}$ is a convex set, $\mathcal{P} \subseteq \mathcal{D}$, and

$$\mathcal{D} \doteq \left\{ p : 0 \leq p_i \leq 1, i = 1, 2, \ldots, s,\ \sum_{i=1}^{s} p_i(k) = 1 \right\}\ . \tag{5}$$

We denote with $v^1,\ v^2,\ \ldots,\ v^m$ the m vertices of $\mathcal{P}$, with $v^j = [v_1^j,\ v_2^j,\ \ldots, v_s^j]$, $j = 1, 2, \ldots, m$.

Note that, in the absence of *a priori* information on the bounds on $p(k)$, it is always possible to set $\mathcal{P} = \mathcal{D}$. As shown in Section III, the size of $\mathcal{P}$ will affect the conservativeness of the control action from a performance point of view.

In addition to the bounds on p expressed by $\mathcal{P}$, a (time varying) model of the time evolution of $p(k)$ may be available. This model can have any general form, and can be deterministic or subject to unmeasured noise. However, the possible presence of noise in the model of p only influences the optimality of the control action (or, in other words, the closed-loop performance of the controlled process), but it does not affect the stability and feasibility properties that will be given later in the paper. In this sense we claim that stability issues are decoupled from performance optimization. A further discussion on this topic is given in Section III.

Many stochastic dynamic models can be considered to describe the evolution of $p(k)$. A meaningful example is given by Markov chain models, which are used in a wide area of applications, such as physics, statistics, biology and economics (e.g. in dynamic macroeconomics [21]). A distinctive characteristic of Markov chains is that the next state depends only on the current state, and not on the history of transitions that lead to the current state. They are defined by a discrete set of n_M states values $\{z_1, z_2, \ldots, z_{n_M}\}$, a discrete set of m_M output or emissions values $\{y_1, y_2, \ldots, y_{m_M}\}$, a transition probability matrix $T \in \mathbb{R}^{n_M \times n_M}$, and an emission matrix $E \in \mathbb{R}^{n_M \times m_M}$, such that

$$t_{ij} = \mathbf{Pr}[Z_{k+1} = z_j | Z_k = z_i]\ , \tag{6a}$$

$$e_{ij} = \mathbf{Pr}[Y_k = y_j | Z_k = z_i]\ , \tag{6b}$$

where Z_k and Y_k are the state and the output of the Markov chain at time k, and t_{ij} and e_{ij} are the elements of T and E, respectively. If $p(k)$ is modeled by a Markov chain, we can compute $\mathcal{P} = hull(E_1, E_2, \ldots, E_s)$, where E_i is the i-th row of E, and $hull(\cdot)$ denotes the convex hull operator.

III. Stochastic MPC design

Consider the regulation problem of driving the state x to the origin. Our goal is to design a stochastic control scheme for system (1) which solves this problem by exploiting the available information on the disturbance. Stochastic control is intended here with respect to both the performance index to be minimized, and the kind of stability which is guaranteed. In particular, we aim to enforce exponential stability in mean square:

$$\lim_{k \to \infty} \mathbb{E}[x(k)'x(k)] = 0\ . \tag{7}$$

The presence or the absence of state and/or input hard constraints poses different issues. These two cases, which are treated separately in the rest of this section, share the same methodological approach: Off-line, a common stochastic Lyapunov function is obtained to ensure stochastic convergence and recursive feasibility of the controller. On-line, the available information on the state and the disturbance is exploited to build a time-varying optimization tree via (partial) scenario enumeration, and an opportunely tuned control problem is solved in a receding horizon fashion.

A. The unconstrained case

In order to prove stochastic convergence, consider the stochastic contractivity constraint

$$\mathbb{E}[V_x(k+1|k)] - V_x(k|k) \leq -x(k|k)'Lx(k|k)\ , \tag{8}$$

where $V_x : \mathbb{R}^{n_x} \to \mathbb{R}$ is a Lyapunov function, and $L \succ 0$. We restrict ourselves to the case of quadratic Lyapunov functions, and define $V_x = x'Px$, $P \succ 0$. As shown in [22], satisfaction of (8) for all time steps k implies condition (7). Given $p(k) \in \mathcal{P}$, we have

$$\mathbb{E}[V_x(k{+}1|k)] = \sum_{j=1}^{s} p_j(k) x(k{+}1|k,j)'Px(k{+}1|k,j)\ , \tag{9}$$

where

$$x(k{+}1|k,j) = A_j x(k|k) + B_j u(k|k),\ j = 1, 2, \ldots, s. \tag{10}$$

Then, by substituting $u(k|k) = Kx(k|k)$, $P = Q^{-1}$, $L = W^{-1}$, $K = YQ^{-1}$, with $Q = Q' \succ 0$, $W = W' \succ 0$, and by using Schur's complements, a sufficient condition for the satisfaction of (8) is given by the satisfaction of the following LMI in the variables Q, W, Y

$$\begin{bmatrix} Q & Q & \sqrt{p_1}(A_1Q{+}B_1Y)' & \cdots & \sqrt{p_s}(A_sQ{+}B_sY)' \\ Q & W & 0 & \cdots & 0 \\ \sqrt{p_1}(A_1Q{+}B_1Y) & 0 & Q & & \\ \vdots & \vdots & & \ddots & \\ \sqrt{p_s}(A_sQ{+}B_sY) & 0 & & & Q \end{bmatrix} \succeq 0 \tag{11}$$

where the dependence on the time index k of p has been omitted for brevity. For ease of notation, we refer to the matrix in (11) as $\mathcal{L}(p(k))$, stressing the dependance of the LMI solution on the value of $p(k)$. If $\mathcal{L}(p(k)) \succeq 0$ admits a solution for all $p(k)$, then the resulting constant feedback control law $u(k) = Kx(k)$, $\forall k \geq 0$, guarantees asymptotical mean-square convergence of the closed-loop state x.

Lemma 1: Let $v^1,\ v^2,\ \ldots,\ v^m$ be the m vertices of $\mathcal{P}$. Define

$$\mathcal{L}(\mathcal{P}) \doteq diag\{\mathcal{L}(v^1),\ \mathcal{L}(v^2),\ \ldots,\ \mathcal{L}(v^m)\} \tag{12}$$

and let (Q^*, W^*, Y^*) be a feasible solution of $\mathcal{L}(\mathcal{P}) \succeq 0$. Then, for any $p(k) \in \mathcal{P}$, $k \geq 0$, (Q^*, W^*, Y^*) is also a solution of $\mathcal{L}(p(k))$.

Proof: Easily follows by convexity arguments, noting that $\forall p \in \mathcal{P}$ there exist $\alpha_i \geq 0$ such that $p = \sum_{i=1}^m \alpha_i v^i$ and $\sum_{i=1}^m \alpha_i = 1$. Since (Q^*, W^*, Y^*) is a solution of $\mathcal{L}(\mathcal{P}) \succeq 0$, we have

$$\sum_{j=1}^{s} v_j^i (A_j + B_j K^*)' P^* (A_j + B_j K^*) \preceq P^* - L^* \ , \tag{13}$$

$i = 1, 2, \ldots, m$, where $P^* = (Q^*)^{-1}$, $L^* = (W^*)^{-1}$, $K^* = Y^*(Q^*)^{-1}$. For (Q^*, W^*, Y^*) to be a solution of $\mathcal{L}(p) \succeq 0$, it must satisfy

$$\sum_{j=1}^{s} p_j^i (A_j + B_j K^*)' P^* (A_j + B_j K^*) \preceq P^* - L^* \ , \tag{14}$$

or, equivalently,

$$\sum_{i=1}^{m} \alpha_i \sum_{j=1}^{s} v_j^i (A_j + B_j K^*)' P^* (A_j + B_j K^*) \preceq P^* - L^* \ , \tag{15}$$

which is satisfied as $\sum_{i=1}^m \alpha_i = 1$ and because of (13). ■

1) Off-line Lyapunov function computation: We can formulate the problem of finding a common stochastic Lyapunov function $x'P^*x$ and a constant feedback control law $u = K^*x$ to enforce (8) as the LMI problem

$$(Q^*, W^*, Y^*) = \arg \min_{Q,W,Y} \quad -\log\det(W) \tag{16a}$$

$$\text{s.t.} \quad W \preceq \varepsilon I_{n_x} \tag{16b}$$

$$\mathcal{L}(\mathcal{P}) \succeq 0 \ , \tag{16c}$$

where $\varepsilon > 0$ is a given parameter which specifies the minimum decrease rate in expected value of V_x, i.e., $L^* \succeq \frac{1}{\varepsilon} I_{n_x}$ in (8), with $P^* = (Q^*)^{-1}$, $L^* = (W^*)^{-1}$, $K^* = Y^*(Q^*)^{-1}$.

Remark 1: If $\mathcal{P} = \mathcal{D}$, problem (16) is a robust control problem (e.g., it is analogous to what proposed in [3]), and its solution enforces robust convergence to the origin for all the possible sequences of $[A(w(k))\ B(w(k))] \in \Omega$, where $\Omega = hull([A_1\ B_1], [A_2\ B_2], \ldots, [A_s\ B_s])$. In other words, in this context robust stability can be seen a special case of stochastic stability, where no *a priori* information on probability bounds is available. On the other hand, when $p(k) = \bar{p}$ is constant over time, we have $\mathcal{P} = \{\bar{p}\}$ and $\mathcal{L}(\mathcal{P}) = \mathcal{L}(\bar{p})$, i.e., constraint (16c) is imposed over a single value of the probability vector p, thus providing a non-conservative solution which enforces stochastic convergence only for the particular $\bar{p}$ considered. Hence, we see that the size of $\mathcal{P}$ affects the conservativeness of the resulting control law, which ranges from purely stochastic to totally robust, depending on the size of $\mathcal{P}$.

2) Design of optimization tree: We assume here that an exact model of the time evolution of the probability vector $p(k)$ is available. The convergence properties granted by solving problem (16) do not involve any model of the evolution of $p(k)$ (only the bounds described by $\mathcal{P}$). On the other hand, such a model can be exploited to improve the closed-loop performance properties of the controlled system.

We propose a tree design scheme based on a maximum likelihood approach, where at every time step the optimization tree is re-build using the updated information on the state and the disturbance. Each node of the tree represents a predicted state which will be taken into account in the optimization problem. Starting from the root node, a list of possible candidates is evaluated, and the node with largest realization probability is added to the tree. This algorithm is repeated until a desired number of nodes n_{max} is reached. Let us introduce the following quantities to formally define the proposed design scheme:

- $\mathcal{T} = \{\mathcal{T}_1, \mathcal{T}_2, \ldots, \mathcal{T}_n\}$: the set of the tree nodes. Nodes are indexed progressively as they are added to the tree (i.e., $\mathcal{T}_1$ is the root node and $\mathcal{T}_n$ is the last node added).
- $x_{\mathcal{N}}$: the state associated with node $\mathcal{N}$.
- $u_{\mathcal{N}}$: the input associated with node $\mathcal{N}$.
- $\pi_{\mathcal{N}}$: the probability of reaching node $\mathcal{N}$ from $\mathcal{T}_1$.
- $m(\mathcal{N}) \in \{1, 2, \ldots, s\}$: the mode leading to node $\mathcal{N}$.
- $pre(\mathcal{N})$: the predecessor of node $\mathcal{N}$.
- $succ(\mathcal{N}, j)$: the successor of node $\mathcal{N}$ with mode j.
- $\mathcal{C} = \{\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_c\}$: the set of candidate nodes, defined as $\mathcal{C} \doteq \{succ(\mathcal{T}_i, j) : succ(\mathcal{T}_i, j) \notin \mathcal{T},\ i = 1, 2, \ldots, n,\ j = 1, 2, \ldots, s\}$.
- $\mathcal{S} \subset \mathcal{T}$: the set of the leaf nodes, defined as $\mathcal{S} \doteq \{\mathcal{T}_i : succ(\mathcal{T}_i, j) \notin \mathcal{T}, i = 1, 2, \ldots, n,\ j = 1, 2, \ldots, s\}$.

The tree design procedure is listed in Algorithm 1.

Algorithm 1 Design of Optimization Tree

```
set π_{T_1} = 1, n = 1, C = {succ(T_1, j), j = 1, ..., s};
while n < n_max,
    for all C_i ∈ C,
        compute π_{C_i} according to the dynamic model of p(k);
    end
    set i* = arg max_{(i: C_i ∈ C)} π_{C_i};
    set T_{n+1} = C_{i*};
    set T = T ∪ {T_{n+1}};
    set π_{T_{n+1}} = π_{C_{i*}};
    set C = C \ T_{n+1} ∪ {succ(T_{n+1}, j), j = 1, 2, ..., s};
    set n = n + 1;
end
```

3) Control problem formulation: We propose a formulation where the objective function to be minimized relies on an averaged value of the closed-loop performance, evaluated as a quadratic function of state and input. This is an arbitrary choice, however, and other types of objective functions could also be considered, e.g., by optimizing over higher order moments. For ease of notation, in the following the abbreviate forms x_i, u_i, π_i, $m(i)$, $pre(i)$, will be used to denote $x_{\mathcal{T}_i}$, $u_{\mathcal{T}_i}$, $\pi_{\mathcal{T}_i}$, $m(\mathcal{T}_i)$, $pre(\mathcal{T}_i)$, respectively. Let $x(k|k)$ be the measured state and $p(k)$ the known probability vector at time k. Then, the unconstrained stochastic MPC problem can be formulated as

$$\min_{u} \sum_{i\in\mathcal{T}\setminus\{\mathcal{T}_1\}} \pi_i x_i' Q_x x_i + \sum_{j\in\mathcal{T}\setminus\mathcal{S}} \pi_j u_j' Q_u u_j \tag{17a}$$

$$\text{s.t. } x_1 = x(k|k) \tag{17b}$$

$$x_i = A_{m(i)} x_{pre(i)} + B_{m(i)} u_{pre(i)}, \forall i \in \mathcal{T}\setminus\{\mathcal{T}_1\} \tag{17c}$$

$$\sum_{j=1}^{s} p_j(k)(A_j x_1 + B_j u_1)' P^*(A_j x_1 + B_j u_1) \le x_1'(P^* - L^*)x_1 \ . \tag{17d}$$

Problem (17) is a quadratically constrained quadratic problem (QCQP).

Theorem 1: Let Assumption 1 be satisfied and problem (16) admit a feasible solution (Q^*, W^*, Y^*). Then, the state x of (1), controlled by the optimal input u^* given by the receding horizon solution of problem (17), is exponentially stable in mean square.

Proof: Recursive feasibility of the receding horizon control scheme is given by Lemma 1. In particular, it is always possible to fulfill (17d) by imposing $u_1 = K^* x_1$, where $K^* = Y^*(Q^*)^{-1}$. Constraint (17d) enforces (8) for all time steps, hence (7) is guaranteed. ■

Remark 2: In the borderline case where $\mathcal{T}$ is a complete tree, i.e., a s-ary tree in which all the leaf nodes are at a some depth N and all nodes but the leaf nodes have exactly s successors, we have that the performance index (17a) is equivalent to $\mathbf{E}\left[\sum_{j=0}^{N-1}\left(x_{k+j|k}' Q_x x_{k+j|k} + u_{k+j|k}' Q_u u_{k+j|k}\right)\right]$, where $x_{k+j|k}$ and $u_{k+j|k}$ are the predicted values of the state and input at time $k+j$, respectively, given the measurements available at time k.

B. The constrained case

Let us take into account hard constraints on both state and input vectors[1]. We consider component-wise bounds on state and input

$$x \in \mathbf{X} \doteq \{x : |x_i| \le x_i^+,\ i = 1, 2, \dots, n_x\}\ , \tag{18a}$$

$$u \in \mathbf{U} \doteq \{u : |u_i| \le u_i^+,\ i = 1, 2, \dots, n_u\}\ . \tag{18b}$$

While the optimization tree design previously described is not affected by the presence of constraints (18), the off-line Lyapunov function computation and the on-line control problem formulation of Section III-A must be modified.

We propose a solution derived from [3]. Our goal here is not to enlarge the feasibility solution set with respect to the robust controller presented in [3], but to exploit the available information on the disturbance in order to provide a less conservative control action. Hence, the main idea is to obtain off-line a common Lyapunov function and a constant feedback control law such that *robust* convergence to the origin is guaranteed, i.e.,

$$V_x(k+1|k) - V_x(k|k) \le -x(k|k)' L x(k|k)\ , \tag{19}$$

[1]The approach of this paper can be extended to other kind of constraints, such as soft or chance constraints. This will be addressed in a future work.

$\forall k$, and to relax (19) to (8) in the on-line control scheme, using the available information on $p(k)$. By using Schur's complements, robust stability condition (19) can be expressed as the LMI

$$\begin{bmatrix} Q & \left(L^{\frac{1}{2}}Q\right)' & (A_jQ+B_jY)' \\ L^{\frac{1}{2}}Q & \gamma I_{n_x} & 0 \\ A_jQ+B_jY & 0 & Q \end{bmatrix} \succeq 0,\ j = 1, \dots, s, \tag{20}$$

in the variables Q, Y, γ, where $P = \gamma Q^{-1}$, $K = YQ^{-1}$, and $L = L' \succ 0$. In order to take into account constraints, we define the ellipsoid

$$\mathcal{E} = \{x : x'Q^{-1}x \le 1\} = \{x : x'Px \le \gamma\}\ , \tag{21}$$

where Q, P, and γ are a solution of (20). Given a state x_0, we can express the condition $x_0 \in \mathcal{E}$ as

$$\begin{bmatrix} 1 & x_0' \\ x_0 & Q \end{bmatrix} \succeq 0\ . \tag{22}$$

Note that $\mathcal{E}$ is an *invariant* ellipsoid for the closed-loop trajectories of system (1) controlled by $u = Kx$, i.e., $x(k) \in \mathcal{E} \Rightarrow x(k+t) \in \mathcal{E}$, $\forall t \ge 0$ (see [3]). A sufficient condition for the satisfaction of (18) is given by the LMIs

$$\begin{bmatrix} Q & \left(I_{n_x}^i(A_jQ+B_jY)\right)' \\ I_{n_x}^i(A_jQ+B_jY) & (x_i^+)^2 \end{bmatrix} \succeq 0\ , \tag{23}$$

$i = 1, \dots, n_x$, $j = 1, \dots, s$, and

$$\begin{bmatrix} X & Y \\ Y' & Q \end{bmatrix} \succeq 0,\ X_{ll} \le \left(u_l^+\right)^2,\ l = 1, \dots, n_u, \tag{24}$$

respectively, where $I_{n_x}^i$ is the i-th row of the $n_x \times n_x$ identity matrix, and X_{ll} are the diagonal elements of the symmetric matrix X, as shown in [3].

1) Off-line Lyapunov function computation: We can now formulate the problem of computing a common Lyapunov function and a constant feedback control law to enforce (19) and (18) as the LMI

$$(\tilde{Q}, \tilde{X}, \tilde{Y}, \tilde{\gamma}) = \arg\min_{Q,X,Y,\gamma}\ -\log\det(Q) \tag{25a}$$

$$\text{s.t. } (20), (22), (23), (24), \tag{25b}$$

where $x_0 = x(0)$ is the initial state, and $\tilde{P} = \tilde{\gamma}(\tilde{Q})^{-1}$, $\tilde{K} = \tilde{Y}(\tilde{Q})^{-1}$. The cost function (25a) is intended to maximize the volume of the ellipsoid $\mathcal{E}$, to enlarge the feasibility set of the on-line control problem.

2) Control problem formulation: With the same notation of Section III-A.3, the constrained stochastic MPC problem at time k can be formulated as

$$\min_u \sum_{i\in\mathcal{T}\setminus\{\mathcal{T}_1\}} \pi_i x_i' Q_x x_i + \sum_{j\in\mathcal{T}\setminus\mathcal{S}} \pi_j u_j' Q_u u_j \tag{26a}$$

$$\text{s.t. } x_1 = x(k|k) \tag{26b}$$

$$x_i = A_{m(i)} x_{pre(i)} + B_{m(i)} u_{pre(i)} \tag{26c}$$

$$x_i \in \mathbf{X},\ \forall i \in \mathcal{T}\setminus\{\mathcal{T}_1\} \tag{26d}$$

$$u_j \in \mathbf{U},\ \forall j \in \mathcal{T}\setminus\mathcal{S} \tag{26e}$$

$$\sum_{j=1}^{s} p_j(k)(A_j x_1 + B_j u_1)' \tilde{P}(A_j x_1 + B_j u_1) \leq x_1'(\tilde{P} - \tilde{L})x_1 \tag{26f}$$

$$(A_l x_1 + B_l u_1) \in \mathbf{X},\ \forall l : succ(\mathcal{T}_1, l) \notin \mathcal{T} \tag{26g}$$

$$(A_h x_1 + B_h u_1)' \tilde{Q}^{-1}(A_h x_1 + B_h u_1) \leq 1,\ \forall h : p_h(k) > 0\ . \tag{26h}$$

In problem (26), constraint (26f) enforces the stochastic stability condition (8). Moreover, (26g) makes sure that the next state $x(k+1)$ satisfies (18a), even if the disturbance realization $w(k)$ at time k is not modeled in the current tree $\mathcal{T}$. Constraint (26h) recursively enforces the state x to lie in $\mathcal{E}$, i.e., $x(k|k) \in \mathcal{E} \Rightarrow x(k+1|k) \in \mathcal{E}$. Constraint (26g) is necessary as, in general, $\mathcal{E} \not\subseteq \mathbf{X}$. Problem (26) is a QCQP.

Theorem 2: Let $(\tilde{Q}, \tilde{X}, \tilde{Y}, \tilde{\gamma})$ be the optimal solution of problem (25), and let $\tilde{P} = \tilde{\gamma}(\tilde{Q})^{-1}$, $\tilde{K} = \tilde{Y}(\tilde{Q})^{-1}$. Then, the state $x(k)$ of (1), with initial condition $x(0) = x_0$ and controlled by the optimal input $u(k) = u^*$ given by the receding horizon solution of problem (26), is exponentially stable in mean square and satisfies constraints (18), $\forall k \geq 0$.

Proof: By construction, $\tilde{P}$ satisfies (19), hence it satisfies also its relaxation (8), equivalent to (26f). The control law $u(k) = \tilde{K}x(k)$ guarantees that constraints (18) are fulfilled, i.e., it satisfies (26d), (26e) and (26g). Recursive feasibility is provided by (26g) and (26h), which enforce the closed-loop state trajectory to lie in $\mathcal{E}\cap\mathbf{X}$. Hence, $u_j = \tilde{K}x_j$, $\forall j \in \mathcal{T}\setminus\mathcal{S}$, is a feasible solution of problem (26) at every time step k. Stochastic convergence is provided by (26f). ■

IV. ILLUSTRATIVE EXAMPLE

In this section we test the performance of the proposed stochastic MPC approach on a simple low-dimensional system. Consider the second-order discrete-time uncertain linear system Σ of the form (1), with $A_i = \begin{bmatrix} 0.8 & 1 \\ 0 & w_i \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $i = 1, 2, 3$, where $w \in \{0.9, 1.2, 2.2\}$. System Σ has one stable mode and two unstable modes. We assume that $p(k)$ is modeled by a time-homogeneous Markov chain with three states, defined by the transition probability matrix $T = \begin{bmatrix} 0.7 & 0.2 & 0.1 \\ 0.3 & 0.4 & 0.3 \\ 0.5 & 0.3 & 0.2 \end{bmatrix}$ and the emission matrix $E = \begin{bmatrix} 0.6 & 0.3 & 0.1 \\ 0.2 & 0.6 & 0.2 \\ 0.1 & 0.3 & 0.6 \end{bmatrix}$, and that the state Z_k of the Markov chain at time k is known, so that the value of $p(k)$ can be derived (and Assumption 1 is satisfied).

The purpose of the control action is to steer the state x of Σ to the origin, while satisfying hard state and input constraints[2] (18), defined by $x^+ = [5\ 1]'$ and $u^+ = 1.5$.

[2]For space reasons, an example of the unconstrained case addressed in Section III-A is not presented here.

TABLE I
SIMULATION RESULTS

Controller	$\mu(c)$	Constr. violation	CPU time
RMPC	1.17	0%	342 ms
DMPC	1.05	38%	74 ms
SMPC	1	0%	371 ms

The proposed stochastic MPC (SMPC) control scheme has been tested in comparison with the following two controllers:

- The LMI-based robust MPC (RMPC) proposed in [3], which provides robust convergence and hard constraint fulfillment, but does not exploit the available statistical information on the disturbance w.
- A frozen-time deterministic MPC (DMPC) formulation with time-varying system model, where at every step k a nominal MPC control problem is solved based on the dynamics mode $i \in \{1, 2, \ldots, s\}$ which is currently the most probable, i.e., such that $p_i(k) \geq p_j(k)$, $\forall j$. As no a-priori guarantees on recursive constraint satisfaction are imposed, bounds on states and inputs are imposed as soft constraints. This control algorithm does not grant any convergence or hard constraint fulfillment properties, but partially takes into account the knowledge of $p(k)$ to obtain a less conservative control action.

The weight matrices used in simulations are $Q_x = \begin{bmatrix} 1 & 0 \\ 0 & 0.1 \end{bmatrix}$, $Q_u = 0.1$, and $L = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}$. A maximum number of nodes $n_{max} = 10$ has been used to design the optimization tree.

$N_s = 50$ simulations have been run of $T_s = 10$ time steps each. At every iteration, the initial state $x(0)$ is chosen randomly in the region of the state set where problem (25) is feasible, i.e., the ellipsoid $\mathcal{E}$ defined in (21) with $Q = \tilde{Q} = \begin{bmatrix} 41.08 & -1.61 \\ -1.61 & 1.29 \end{bmatrix}$.

To evaluate the performance achieved by the considered control schemes, we define the experimental cost function

$$J(i,c) = \sum_{k=1}^{T_s} \left(x^{i,c}(k)' Q_x x^{i,c}(k) + u^{i,c}(k)' Q_u u^{i,c}(k) \right) \tag{27}$$

where $i = 1, \ldots, N_s$ indexes the values related to the i-th simulation, and $c \in \{\text{RMPC, DMPC, SMPC}\}$ refers to the controller used. Table I shows the simulation results in terms of the mean $\mu(c)$ over all the simulations of the experimental cost function (27) for every controller c, normalized with respect to the related value obtained with the stochastic MPC scheme, i.e.,

$$\mu(c) = \frac{1}{N_s} \sum_{i=1}^{N_s} \frac{J(i,c)}{J(i, \text{SMPC})}\ . \tag{28}$$

The table also reports the frequency of simulations where at least one constraint violation occurred, and the average CPU time needed to solve an iteration of each control problem, obtained on a Macbook 2.4GHz running Matlab 7.6, and CVX [23]. As we can see from the results, the proposed stochastic MPC policy achieves an average improvement in the closed-loop performance of 17% with respect to robust

MPC, and of 5% with respect to deterministic MPC. Both SMPC and RMPC provide hard constraints fulfillment, as expected, while in DMPC more that one third of the simulations showed one or more constraint violations. Finally, for this simple case study, the computation times for SMPC and RMPC are similar, while DMPC provides a smaller CPU burst due to its simpler structure. A comparison between the different closed-loop trajectories is shown in Figure 1.

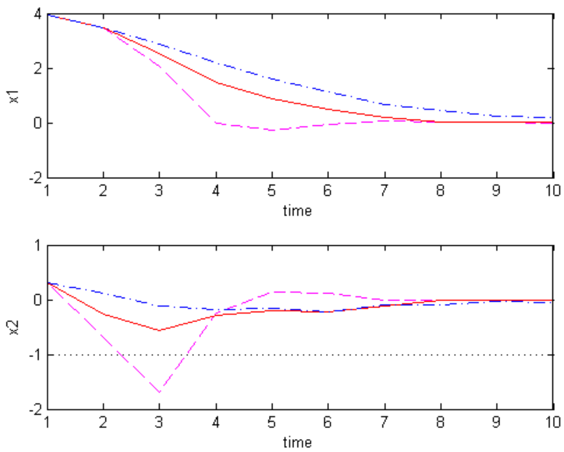

Fig. 1. Example of closed-loop trajectories with stochastic control (red line), deterministic control (pink dashed line), and robust control (blue dot-dashed line) for state components x_1, x_2. The black dotted line denotes the imposed constraint on x_2.

V. Conclusions

In this paper we have presented a stochastic model predictive control formulation based on scenario generation for linear systems affected by discrete multiplicative disturbances. By separating the problems of stochastic performance optimization on one hand, and stochastic convergence to the origin and robust constraint fulfillment on the other, we set up a control scheme which requires the off-line solution of an LMI problem, and the receding horizon implementation of a QCQP problem to obtain the control action. The proposed control algorithm is suitable for application to a wide class of discrete disturbance processes. Simulations on a low dimension system have been run to show the performance of our approach, in comparison with classic robust and deterministic MPC formulations. Extensions to the case of probabilistic constraints on the state will be addressed in future work.

Acknowledgment

The authors thank Dr. David Muñoz de la Peña for fruitful discussions and suggestions.

References

[1] E. F. Camacho and C. Bordons, *Model Predictive Control*, 2nd ed., ser. Advanced Textbooks in Control and Signal Processing. London: Springer-Verlag, 2004.

[2] S. Qin and T. Badgwell, "A survey of industrial model predictive control technology," *Control Engineering Practice*, vol. 11, no. 7, pp. 733–764, 2003.

[3] M. Kothare, V. Balakrishnan, and M. Morari, "Robust constrained model predictive control using linear matrix inequalities," *Automatica*, vol. 32, no. 10, pp. 1361–1379, 1996.

[4] P. Scokaert and D. Mayne, "Min-max feedback model predictive control for constrained linear systems," *IEEE Transactions On Automatic Control*, vol. 43, pp. 1136–1142, 1998.

[5] A. Bemporad and M. Morari, "Robust model predictive control: A survey," in *Robustness in Identification and Control*, ser. Lecture Notes in Control and Information Sciences, vol. 245. Springer, 1999, pp. 207–226.

[6] A. Bemporad, F. Borrelli, and M. Morari, "Min-max control of constrained uncertain discrete-time linear systems," *IEEE Transactions On Automatic Control*, vol. 48, no. 9, pp. 1600–1606, 2003.

[7] D. Muñoz de la Peña, T. Alamo, A. Bemporad, and E. Camacho, "A decomposition algorithm for feedback min-max model predictive control," *IEEE Transactions On Automatic Control*, vol. 51, no. 10, pp. 1688–1692, 2006.

[8] T. Alamo, D. Muñoz de la Peña, D. Limon, and E. Camacho, "Constrained minmax predictive control: Modifications of the objective function leading to polynomial complexity," *IEEE Transactions On Automatic Control*, vol. 50, no. 5, pp. 710–714, 2005.

[9] "Stochastic programming community home page (website)," http://www.stoprog.org.

[10] D. van Hessem and O. Bosgra, "A conic reformulation of model predictive control including bounded and stochastic disturbances under state and input constraints," in *Proc. 41th IEEE Conf. on Decision and Control*, Las Vegas, NV, USA, 2002, pp. 4643–4648.

[11] D. Muñoz de la Peña, A. Bemporad, and T. Alamo, "Stochastic programming applied to model predictive control," in *Proc. 44th IEEE Conf. on Decision and Control*, Sevilla, Spain, 2005, pp. 1361–1366.

[12] P. Couchman, M. Cannon, and B. Kouvaritakis, "Stochastic MPC with inequality stability constraints," *Automatica*, vol. 42, pp. 2169–2174, 2006.

[13] J. Primbs, "Stochastic receding horizon control of constrained linear systems with state and control multiplicative noise," in *Proc. American Control Conf.*, New York, NY, USA, 2007, pp. 4470–4475.

[14] D. van Hessem, C. Scherer, and O. Bosgra, "Lmi-based closedloop economic optimization of stochastic process operation under state and input constraints," in *Proc. 40th IEEE Conf. on Decision and Control*, Orlando, FL, USA, 2001, pp. 4228–4233.

[15] M. Ono and B. Williams, "Iterative risk allocation: A new approach to robust model predictive control with a joint chance constraint," in *Proc. 47th IEEE Conf. on Decision and Control*, Cancun, Mexico, 2008, pp. 3427–3432.

[16] F. Oldewurtel, C. Jones, and M. Morari, "A tractable approximation of chance constrained stochastic MPC based on affine disturbance feedback," in *Proc. 47th IEEE Conf. on Decision and Control*, Cancun, Mexico, 2008, pp. 4731–4736.

[17] I. Batina, A. Stoorvogel, and S. Weiland, "Optimal control of linear, stochastic systems with state and input constraints," in *Proc. 41th IEEE Conf. on Decision and Control*, Las vegas, NV, USA, 2002, pp. 1564–1569.

[18] K. Høyland and S. Wallace, "Generating scenario trees for multistage decision problems," *Management Science*, vol. 47, no. 2, pp. 295–307, 2001.

[19] J. Dupačova, G. Consigli, and S. Wallace, "Scenarios for multistage stochastic programs," *Annals of Operations Research*, vol. 100, pp. 25–53, 2004.

[20] G. Goodwin, J. Østergaard, D. Quevedo, and A. Feuer, "A vector quantization approach to scenario generation for stochastic NMPC," in *Int. Workshop on Assessment and Future Directions of Nonlinear Model Predictive Control*, Pavia, Italy, 2008.

[21] M. Brennan and Y. Xia, "Stock price volatility and equity premium," *Journal of Monetary Economics*, vol. 47, pp. 249–283, 2001.

[22] T. Morozan, "Stabilization of some stochastic discrete-time control-systems," *Stochastic Analysis and Applications*, vol. 1, no. 1, pp. 89–116, 1983.

[23] M. Grant and S. Boyd, "CVX: Matlab software for disciplined convex programming (web page and software)," http://stanford.edu/~boyd/cvx, February 2009.