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Convex Polyhedral Invariant Sets for Closed-Loop Linear MPC Systems

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***Abstract*— Given an asymptotically stabilizing linear MPC controller, this paper proposes an algorithm to construct invariant polyhedral sets for the closed-loop system. Rather than exploiting an explicit form of the MPC controller, the approach exploits a recently developed DC (Difference of Convex functions) programming technique developed by the authors to construct a polyhedral set in between two convex sets. Here, the inner convex set is any given level set $V(x) \leq \gamma$ of the MPC value function (implicitly defined by the quadratic programming problem associated with MPC or explicitly computed via multiparametric quadratic programming), while the outer convex set is the level set of a the value function of a modified multiparametric quadratic program (implicitly or explicitly defined). The level gamma acts as a tuning parameter for deciding the size of the polyhedral invariant containing the inner set, ranging from the origin ($\gamma = 0$) to the maximum invariant set around the origin where the solution to the unconstrained MPC problem remains feasible, up to the whole domain of definition of the controller (possibly the whole state space $\mathbb{R}^n$) ($\gamma = \inf$). Potential applications of the technique include reachability analysis of MPC systems and generation of constraints to supervisory decision algorithms on top of MPC loops.**

***Index Terms*— Model predictive control, Polyhedral invariant sets.**

I. INTRODUCTION

Positively invariant sets are a useful instrument used in several branches of systems science, for reachability and stability analysis, as well as for the synthesis of control laws. In particular, invariant sets have been used for the design of stabilizing Model Predictive Controllers. In MPC, the stability of the feedback loop is often guaranteed by augmenting the associated optimization problem problem with the so-called "stability constraint", forcing the state vector to reach an invariant set at the end of the prediction horizon. Polyhedral invariant sets are often preferred over ellipsoidal ones for numerical reasons, as ellipsoidal constraints are usually more difficult to handle than linear constraints.

Most of the MPC literature has focused on the computation of invariant sets for (constrained) *open-loop* systems. Significant advances were obtained in [1], [2] and [3] for linear systems affected by additive disturbances and parametric uncertainties, respectively. Important results were also reported into [4], and [5], [6], [7], for linear systems subject to input saturation and hybrid systems.

In this paper we show how to compute polyhedral invariant sets of arbitrary size for stable MPC *closed-loop* systems. To the best of our knowledge, the only known polyhedral invariant sets for closed-loop linear MPC systems are the origin, the maximum invariant set around the origin (where the unconstrained MPC solution remains feasible) and the feasibility domain (possibly the whole state space $\mathbb{R}^n$).

In [8], the authors presented a method for constructing a polyhedral invariant set that lies between two contractive ellipsoidal sets. The computational part of the method is based on the level sets of a PWA function whose graph lies between the graphs of the two quadratic functions (the ellipsoidal sets are level sets of these functions). Here we exploit the fact that in MPC problems based on quadratic costs and linear prediction models (see the survey [9] for an overview) the value function $V(x)$ is a piecewise quadratic Lyapunov function [10], whose level sets are piecewise ellipsoidal invariant sets for the system. We propose a variant of the approach used in [8] to solve the problem of determining invariant sets for closed-loop MPC systems. First, we observe that the property

$$V(x(t+1)) - V(x(t)) \leq -x(t)^\top Qx(t),$$

where $V(\cdot)$ denotes the value function of a stabilizing linear MPC problem with fixed horizon N and quadratic costs, holds for the closed-loop. Then, we compute polyhedral invariant sets for the closed-loop MPC system by inscribing a polyhedron $\mathcal{P}$ between the sublevel sets of two piecewise quadratic functions given by $V(x)$, and $V(x) - x^\top Qx$, respectively. A proof that the function $V(x) - x^\top Qx$ is a convex, continuous and piecewise quadratic function of x is provided, and it is shown that its level sets $\mathcal{E}_m(\gamma)$ attained at a certain level $\gamma \in \mathbb{R}_+$ are piecewise ellipsoidal invariant sets containing the piecewise ellipsoidal level sets of the function $V(x)$. Finally, we construct a polyhedral set $\mathcal{P}$ that lies between these two sets with a variant of the algorithm presented in [8].

A very appealing feature of the approach is that the explicit form of the MPC controller is not required for computing the invariant sets. Therefore, the algorithm is applicable even when the number of constraints in the MPC problem is large and so the number of regions of the explicit controller would be intractable (if at all computable). An alternative to our approach would be to form a PWA system by collecting the linear model and the piecewise affine MPC control

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law, and to design a PWQ Lyapunov function via LMI (as in [11]). However, the obtained PWQ Lyapunov function is not convex in general, so the resulting polyhedral set obtained through the algorithm of [8] may not be convex. The approach of this paper always generates convex polyhedral invariant sets for the closed-loop MPC systems.

The paper is organized as follows. The basics of MPC are reviewed first in Section II to derive the problem formulation. In Section III we present some lemmas that are fundamental for assuring the invariance of the polyhedral set $\mathcal{P}$, while the algorithm for computing $\mathcal{P}$ is described in Section IV. An example and the conclusions are reported in Section V and Section VI, respectively.

A. Notation and Basic Definitions

Let $\mathbb{R}$, $\mathbb{R}_+$, $\mathbb{Z}$ and $\mathbb{Z}_+$ denote the field of real numbers, the set of non-negative reals, the set of integer numbers and the set of non-negative integers, respectively. For a set $\mathcal{S} \subseteq \mathbb{R}^n$, we denote by $\partial\mathcal{S}$ the boundary of $\mathcal{S}$, by $\text{int}(\mathcal{S})$ its interior and by $\text{cl}(\mathcal{S})$ its closure. Given vectors $\theta_1, \ldots, \theta_i$, $\text{Co}(\theta_1, \ldots, \theta_i)$ denote their convex hull. For any real $\lambda \geq 0$, the set $\lambda\mathcal{S}$ is defined as

$$\lambda\mathcal{S} \triangleq \{x \in \mathbb{R}^n \mid x = \lambda y \text{ for some } y \in \mathcal{S}\}.$$

A polyhedron (or a polyhedral set) is a set obtained as the intersection of a finite number of open and/or closed half-spaces. A piecewise polyhedral set is the union of a finite number of polyhedra.

Given $(n+1)$ *affinely independent* points $(\theta_0, \ldots, \theta_n)$ of $\mathbb{R}^n$, i.e. $(1\ \theta_0^\top)^\top, \ldots, (1\ \theta_n^\top)^\top$ are linearly independent in $\mathbb{R}^{n+1}$, we define a simplex S as

$$S \triangleq \text{Co}(\theta_0, \ldots, \theta_n) \triangleq \{x \in \mathbb{R}^n \mid x = \sum_{l=0}^{n} \mu_l \theta_l, \sum_{l=0}^{n} \mu_l = 1, \mu_l \geq 0 \text{ for } l = 0, 1, \ldots, n\}.$$

A function $f : \mathbb{R}^n \to \mathbb{R}$ is a *quadratic function* if $f(x) := x^\top P x + Cx + \alpha$ for some $P \in \mathbb{R}^{n\times n}$, $C \in \mathbb{R}^{1\times n}$ and $\alpha \in \mathbb{R}$. A quadratic function f is *strictly convex* if and only if $P > 0$. An ellipsoid (or an ellipsoidal set) $\mathcal{E}$ is a sublevel set (corresponding to some constant level $\gamma \in \mathbb{R}_+$) of a strictly convex quadratic function, i.e.

$$\mathcal{E} \triangleq \{x \in \mathbb{R}^n \mid f(x) \leq \gamma\}.$$

Let $\Omega_1, \ldots, \Omega_N$ denote a polyhedral partition of $\mathbb{R}^n$, i.e. Ω_i is a polyhedron (not necessarily closed) for all $i = 1, \ldots, N$, $\Omega_i \cap \Omega_j = \emptyset$ for $i \neq j$ and $\cup_{i=1,\ldots,N}\Omega_i = \mathbb{R}^n$.

Definition 1: A function $f : \mathbb{R}^n \to \mathbb{R}$ with

$$f(x) = x^\top P_i x + C_i x + \alpha_i \text{ when } x \in \Omega_i,$$

$i = 1, \ldots, N$ is called a *PieceWise Quadratic (PWQ) function*. A function $\bar{f} : \mathbb{R}^n \to \mathbb{R}$ with $\bar{f}(x) = H_i x + a_i$ when $x \in \Omega_i$, for some $H_i \in \mathbb{R}^{1\times n}$, $a_i \in \mathbb{R}$, $i = 1, \ldots, N$ is called a *PieceWise Affine (PWA) function*.

A piecewise ellipsoidal set is a sublevel set of a piecewise quadratic function with matrices $P_i > 0$ for all $i = 1, \ldots, N$.

II. LINEAR MPC ALGORITHM

Consider the linear discrete-time prediction model

$$\begin{cases} x(t+1) = Ax(t) + Bu(t) \\ y(t) = Cx(t) \end{cases} \quad (1)$$

where $x(t) \in \mathbb{R}^n$ is the state vector at time t, $u(t) \in \mathbb{R}^m$ is the vector of manipulated variables, and $y(t) \in \mathbb{R}^m$ is the output vector.

Consider now the finite-time optimal control problem

$$V(x(t)) = \min_U \quad x_N^\top P x_N + \sum_{k=0}^{N-1} \left[x_k^\top Q x_k + u_k^\top R u_k\right] \quad (2a)$$

$$\text{s.t.} \quad x_{k+1} = Ax_k + Bu_k,\ k = 0, \ldots, N-1 \quad (2b)$$

$$y_k = Cx_k,\ k = 1, \ldots, N \quad (2c)$$

$$x_0 = x(t) \quad (2d)$$

$$u_{\min} \leq u_k \leq u_{\max},\ k = 0, \ldots, N_u - 1 \quad (2e)$$

$$y_{\min} \leq y_k \leq y_{\max},\ k = 1, \ldots, N_c \quad (2f)$$

$$u_k = 0,\ k = N_u, \ldots, N-1 \quad (2g)$$

where N is the prediction horizon, $N_u \leq N$ is the input horizon, $N_c \leq N$ the constraint horizon,

$$U \triangleq \left[u_0^\top \ \ldots \ u_{N-1}^\top\right]^\top \in \mathbb{R}^{Nm}$$

is the sequence of manipulated variables to be optimized, $Q = Q^\top \geq 0$, $R = R^\top > 0$, and $P = P^\top \geq 0$ are square weight matrices defining the performance index, $u_{\min}, u_{\max} \in \mathbb{R}^m$, $y_{\min}, y_{\max} \in \mathbb{R}^p$, define constraints on input and output variables, respectively, and "$\leq$" denotes component-wise inequalities.

Problem (2) can be reformulated as the Quadratic Programming (QP) problem

$$U^*(x(t)) \triangleq \arg\min_U J(U, x(t)) \quad (3a)$$

$$\text{s.t. } GU \leq W + Sx(t) \quad (3b)$$

where

$$J(U, x(t)) = \frac{1}{2}U^\top HU + x^\top(t)C^\top U + \frac{1}{2}x^\top(t)Yx(t) \quad (4a)$$

and

$$U^*(x(t)) = \left[u_0^{*\top}(x(t)) \ \ldots \ u_{N-1}^{*\top}(x(t))\right]^\top$$

is the optimal solution, $H = H^\top > 0$ and C, Y, G, W, S are matrices of appropriate dimensions [10], [12].

At each sampling time t, problem (3) is solved for the given measured or estimated current state $x(t)$, therefore getting the optimal sequence $U^*(x(t))$. Only the first optimal move

$$u(t) = u_0^*(x(t)) \quad (5)$$

is applied to the process,

$$x(t+1) = Ax(t) + Bu_0^*(x(t)), \quad (6)$$

the remaining optimal moves are discarded and the optimization is repeated at time $t+1$. We use $\mathcal{X}^*$ to denote the set of all states for which the optimal control problem (3)-(4) is feasible, i.e. it admits a solution that satisfies the imposed constraints.

Several variations to the formulation (2) exist that ensure by construction the asymptotic stability of the closed-loop system (1)-(5). This is usually achieved by adding the constraint $x_N \in \mathcal{X}_f$ in (2), where $\mathcal{X}_f$ is positively invariant under $u(t)$. We mention two examples of such stability constraints:

(i) $\mathcal{X}_f = \{0\}$ and the terminal weight P is set equal to the solution of the discrete-time Lyapunov equation, i.e.

$$P - A^\top PA = Q;$$

(ii) P is set equal to the solution of the algebraic Riccati equation associated with the pair (A, B) and the weights (Q, R)) [10], i.e.

$$P = (A + BK_{\text{LQR}})^\top P(A + BK_{\text{LQR}}) + K_{\text{LQR}}^\top RK_{\text{LQR}} + Q,$$

and the constraint (2g) is replaced with $u_k = K_{\text{LQR}}x_k$, where K_{LQR} is the corresponding LQR gain, i.e.

$$K_{\text{LQR}} = -(R + B^\top PB)^{-1}B^\top PA.$$

The set $\mathcal{X}_f \subseteq \mathcal{X}^*$ is polyhedral and positively invariant under u_k[1].

III. FUNDAMENTAL RESULTS

In this section we derive useful properties of the MPC value function $V(x)$.

Definition 2: For a given $0 \le \lambda \le 1$, a set $\mathcal{P} \subseteq \mathbb{R}^n$ that contains the origin in its interior is called a *λ-contractive set* for the closed-loop system (1)-(5) if for all $x \in \mathcal{P}$ it holds that $Ax + Bu_0^*(x) \in \lambda\mathcal{P}$. For $\lambda = 1$, the set is called *positively invariant*.

Theorem 3 ([16]): The solution to the optimal control problem (3)-(4) is a PWA state feedback control law of the form

$$u(x(k)) = F_i x(k) + g_i \text{ if } x(k) \in P_i \quad (7)$$

where $P_i \triangleq \{x : H_i x \le k_i\}$, $i = 1, .., N_{tot}$ is a finite partition of the polyhedral set $\mathcal{X}^*$ of feasible states. Moreover the optimal value function V is convex, continuous, and piecewise quadratic, and the set $\mathcal{X}^*$ over which V is defined is a convex polyhedron.

Let

$$V_1(x) \triangleq V(x) - x^\top Qx. \quad (8)$$

The following standing assumption can be ensured via the approaches (i), (ii) described above (and via other approaches) to guarantee a-priori asymptotic closed-loop stability:

Assumption 4 (Asymptotic closed-loop stability): For all $t \ge 0$ and all $x(t) \in \mathcal{X}^*$ it holds that:

$$V(x(t+1)) - V(x(t)) \le -x^\top(t)Qx(t), \quad (9)$$

where $V(\cdot)$ is the MPC value function defined in (2a).

[1]For linear systems, the *maximal admissible set* (MAS) is often chosen as the positively invariant set $\mathcal{X}_f$, see ([13]), ([14]) and ([15])

Almost all proofs of stability in the MPC literature are based on the following technique. Consider for simplicity case (i) (Lyapunov matrix), and the shifted sequence U_s obtained by collecting $u_1^*(x(t)), \ldots, u_{N-1}^*(x(t))$, completed with $u_N = 0$. By construction, U_s is feasible at time $t+1$, and therefore its cost J_s given by (3b) is greater or equal than the optimal one $V(x(t+1))$, provided by the optimizer $U^*(x(t+1))$. Since

$$J_s = V(x(t)) - x^\top(t)Qx(t) - u^\top(t)Ru(t),$$

Assumption 4 readily follows. See for example [10] for more details.

Lemma 5: The function $V_1 : \mathcal{X}^* \to \mathbb{R}$ defined as

$$V_1(x) = V(x) - x^\top Qx$$

is a convex, continuous and piecewise quadratic function.

Proof: Consider the function

$$\begin{aligned} J_1(U, x) &\triangleq J(U, x) - x^\top Qx \\ &= \sum_{k=1}^{N-1} x_k^\top Qx_k + \sum_{k=0}^{N-1} u_k^\top Ru_k + x_N^\top Px_N. \end{aligned}$$

Clearly

$$J_1(U, x) \ge 0, \forall x \in \mathbb{R}^n, \forall U \in \mathbb{R}^{Nm}.$$

Since $J_1(U, x)$ is also a quadratic function of $\begin{bmatrix} x & U \end{bmatrix}^\top$ and $P, Q, R > 0$, it follows that $J_1(U, x)$ is convex. By [10], it follows that $V_1 = \min_U J_1(U, x)$ subject to $GU \le W + Sx$ is also convex piecewise quadratic, and continuous, being the value function of a multiparametric convex quadratic problem. ■

Corollary 6: The level sets $\mathcal{E}_M(\gamma) = \{x \in \mathcal{X}^* : V_1(x) \le \gamma\}$ of V_1 obtained at a generic $\gamma \in \mathbb{R}$ are piecewise ellipsoidal sets.

Lemma 7: Let $\gamma \in \mathbb{R}^+$ and $\mathcal{E}_m(\gamma) = \{x \in \mathcal{X}^* : V(x) \le \gamma\}$. Then

1) $\mathcal{E}_m(\gamma) \subseteq \mathcal{E}_M(\gamma)$,
2) $f_{MPC}(\mathcal{E}_M(\gamma)) \subseteq \mathcal{E}_m(\gamma)$, where $f_{MPC}(x) \triangleq Ax + Bu_0^*(x)$,
3) The piecewise ellipsoidal level sets $\mathcal{E}_M(\gamma)$ are positively invariant sets for the closed loop system (1)-(5).

Proof: Given γ and $\tilde{x}(t)$ such that $\tilde{x}(t) \in \mathcal{E}_m(\gamma)$, we have that

$$V(\tilde{x}(t)) \le \gamma \Rightarrow V(\tilde{x}(t)) - \tilde{x}^\top(t)Q\tilde{x}(t) \le \gamma.$$

This proves the first statement. Choosing $\bar{x}(t)$ such that $\bar{x}(t) \in \mathcal{E}_M(\gamma)$, we have that

$$V_1(\bar{x}(t)) = V(\bar{x}(t)) - \bar{x}^\top(t)Q\bar{x}(t) \le \gamma.$$

Then, by Assumption 4 it holds that

$$\begin{aligned} V(\bar{x}(t+1)) &= V(A\bar{x}(t) + Bu(\bar{x}(t))) \\ &\le V(\bar{x}(t)) - \bar{x}^\top(t)Q\bar{x}(t) \\ &\le \gamma \end{aligned} \quad (10)$$

Note that this proves that

$$f_{MPC}(\mathcal{E}_M(\gamma)) \subseteq \mathcal{E}_m(\gamma). \tag{11}$$

Since $Q \geq 0$, we have that

$$V_1(\bar{x}(t+1)) = V(A\bar{x}(t) + Bu(\bar{x}(t))) - (A\bar{x}(t) + Bu(\bar{x}(t)))^\top Q(A\bar{x}(t) + Bu(\bar{x}(t))) \leq \gamma.$$

Therefore, $\mathcal{E}_M(\gamma)$ is a positively invariant set for the closed-loop system (1)-(5). ■

We proved that the piecewise ellipsoidal sets $\mathcal{E}_M(\gamma)$ are invariant for the MPC problem in (1)-(5), and that these sets contain $\mathcal{E}_m(\gamma)$, for the same γ. Both sets are convex, since the two functions V_1, V are convex in $\mathcal{X}^*$(cf. [10] and Lemma 5).

Lemma 8: Given $\gamma \in \mathbb{R}^+$, let $\mathcal{P}$ be any polyhedron such that $\mathcal{P} \subseteq \mathcal{X}^*$ and $\mathcal{E}_m(\gamma) \subseteq \mathcal{P} \subseteq \mathcal{E}_M(\gamma)$. Then $\mathcal{P}$ is positively invariant for the closed-loop system (1)-(5).

Proof: Let $x(t) \in \mathcal{P} \subseteq \mathcal{E}_M(\gamma)$. From Lemma 7 $f_{MPC}(\mathcal{E}_M(\gamma)) \subseteq \mathcal{E}_m(\gamma), \forall x \in \mathcal{E}_M(\gamma)$. It follows that $x(t+1) \in \mathcal{E}_m(\gamma) \subseteq \mathcal{P}$. Therefore, $\mathcal{P}$ is a polyhedral invariant set for the closed loop system $(1) - (5)$. ■

In the next section we present an algorithm for computing polyhedral invariant sets $\mathcal{P}$ satisfying the hypothesis of Lemma 8.

IV. COMPUTATION OF THE POLYHEDRAL INVARIANT SET

In this section we present a solution to the problem of fitting a polyhedral set $\mathcal{P}$ in between two piecewise ellipsoidal sets $\mathcal{E}_m(\gamma) := \{x \in \mathcal{X}^* \mid V(x) \leq \gamma\}$ and $\mathcal{E}_m(\gamma) := \{x \in \mathcal{X}^* \mid V_1(x) \leq \gamma\}$. To avoid degenerate situations, we assume that $\mathcal{E}_m(\gamma)$ is contained in the interior of $\mathcal{E}_M(\gamma)$. The two sets are sublevel sets of two piecewise quadratic functions (with strictly convex pieces), $V(x)$ and $V_1(x)$, respectively, that correspond to a certain constant (level) $\gamma \in \mathbb{R}_+$. Then, we compute a PWA function $\bar{f}$ that satisfies $V(x) > \bar{f}(x) \geq V_1(x)$ for all $x \in \mathbb{R}^n$, and we consider the piecewise polyhedral set $\bar{\mathcal{P}} := \{x \in \mathbb{R}^n \mid \bar{f}(x) \leq \gamma\}$.

The desired polyhedral set $\mathcal{P}$ is the convex hull of $\overline{\mathcal{P}}$, where clearly $\overline{\mathcal{P}} = \mathcal{P}$ if $\overline{\mathcal{P}}$ is a polyhedron. The parameter γ is used as a tuning knob of the procedure.

Consider now an initial polyhedron $\mathcal{P}_0 \subset \mathcal{X}^*$ that contains $\mathcal{E}_m(\gamma)$[2]. Given $(\tilde{\theta}_0, .., \tilde{\theta}_m)$, with $m \geq n$, vertices of $\mathcal{P}_0$ we determine an initial set of simplices $S_1^0, \ldots, S_{l_0}^0$ that contains these points by Delaunay triangulation [17]. Then, for every simplex $S_i^0 := \mathrm{Co}(\theta_{0i}, \ldots, \theta_{ni})$, $i = 1, \ldots, l_0$, the following operations are performed.

Algorithm 9:

1) Let $k = 0$.
2) For every simplex S_i^k, $i = 1, \ldots, l_k$, construct the matrix
$$M_i \triangleq \begin{bmatrix} 1 & 1 & \ldots & 1 \\ \theta_{0i} & \theta_{1i} & \ldots & \theta_{ni} \end{bmatrix}.$$
3) Set
$$v_i \triangleq [\ V_1(\theta_{0i})\ V_1(\theta_{1i})\ \ldots\ V_1(\theta_{ni})\ ]^\top$$
and construct the function
$$\bar{f}_i(x) \triangleq v_i^\top M_i^{-1} \begin{bmatrix} 1 \\ x \end{bmatrix}.$$
4) Solve the QP problem:
$$\min_{x \in S_i^k} \left\{ J_i(x) \triangleq V(x) - \bar{f}_i(x) \right\}, \tag{12}$$
and let
$$x_i^* \triangleq \arg\min_{x \in S_i^k} J_i(x), \quad J_i^* \triangleq J_i(x_i^*).$$
5) If $J_i^* > 0$ for all $i = 1, \ldots, l_k$, then Stop. Otherwise, for all S_i^k, $i = 1, \ldots, l_k$, for which $J_i^* \leq 0$ build new simplices $S_0^i, S_1^i, \ldots, S_n^i$ defined by the vertices $(x_i^*, \theta_{1i}, \ldots, \theta_{ni})$, $(\theta_{0i}, x_i^*, \ldots, \theta_{ni})$, $\ldots$ and $(\theta_{0i}, \theta_{1i}, \ldots, \theta_{ni}, x_i^*)$, respectively.
6) Increment k by one, add the new simplices
$$S_0^i, S_1^i, \ldots, S_n^i$$
to the set of simplices $\{S_i^k\}_{i=1,\ldots,l_k}$ and repeat the algorithm recursively from Step 2.

Algorithm 9 computes a simplicial partition of a given initial polyhedral set $\mathcal{P}_0$ that contains the ellipsoidal set $\mathcal{E}_m(\gamma)$, by splitting a single simplex S_i^k into $n+1$ simplices. This is done by fixing a new vertex x_i^* which is obtained by solving the QP problem (12). Problem (12) can be solved as a QP by optimizing w.r.t. (U, x). A new PWA approximation is then computed over the new set of simplices. The steps of Algorithm 9 are repeated for all resulting simplices, until $J_i^* > 0$ for all simplices. At every iteration k, a tighter PWA approximation of the piecewise quadratic function $V_1(x)$ is obtained. Algorithm 9 proceeds in a typical branch & bound way, i.e. *branching* on a new vertex x_i^*, and *bounding* whenever it finds a simplex S_i^k for which it holds that $J_i^* > 0$.

Suppose Algorithm 9 stops. At the k^*-th iteration[3] for some $k^* \in \mathbb{Z}_+$, the following PWA function is generated:

$$\begin{aligned} \bar{f}(x) &\triangleq \bar{f}^{k^*}(x) \\ &\triangleq \bar{f}_i(x) \text{ when } x \in S_i^{k^*}, i = 1, \ldots, l_{k^*} \\ &\triangleq H_i^{k^*} x + a_i^{k^*} \text{ when } x \in S_i^{k^*}, i = 1, \ldots, l_{k^*}, \end{aligned}$$

where l_{k^*} is the number of simplices obtained at the end of Algorithm 9. The PWA function $\bar{f}$ constructed via Algorithm 9 is such that for $x = \sum_{j=0}^n \mu_j \theta_{ji}$, the corresponding functions $\bar{f}_i$ satisfy:

$$\bar{f}_i(x) = \bar{f}_i\left(\sum_{j=0}^n \mu_j \theta_{ji}\right) = \sum_{j=0}^n \mu_j V_1(\theta_{ji}) \left(\sum_{j=0}^n \mu_j\right)^{-1},$$

which, by strict convexity of V_1, implies that $\bar{f}_i(x) \geq V_1(x)$ for all $x \in S_i^{k^*}$ and all $i = 1, \ldots, l_{k^*}$. Since the stopping

[2] Note that it is always possible to choose $\mathcal{P}_0$ as the entire set of feasible states $\mathcal{X}^*$.

[3] The existence of a finite k^* is proven in [18].

criterion defined in Step 5 of Algorithm 9 assures that at the end of the entire procedure the optimal value J_i^* of the QP problem defined in (12) will be greater than zero in every simplex $S_i^{k^*}$, $i = 1, \dots, l_{k^*}$, it follows that

$$V_1(x) \le \bar{f}(x) < V(x), \quad \forall x \in \cup_{i=1,\dots,l_{k^*}} S_i^{k^*}.$$

Then, the sublevel set of $\bar{f}$ given by

$$\overline{\mathcal{P}} \triangleq \bigcup_{i=1,\dots,l_{k^*}} \{x \in S_i^{k^*} \mid H_i^{k^*} x + a_i^{k^*} \le \gamma\}$$

satisfies $\mathcal{E}_m(\gamma) \subset \overline{\mathcal{P}} \subset \mathcal{E}_M(\gamma)$. Indeed, note that for $x \in \overline{\mathcal{P}}$ it holds that

$$\bar{f}(x) \le \gamma \Rightarrow V_1(x) \le \bar{f}(x) \le \gamma \Rightarrow x \in \mathcal{E}_M(\gamma),$$

and for $x \in \mathcal{E}_m(\gamma)$ it holds that

$$V(x) \le \gamma \Rightarrow \bar{f}(x) < V(x) \le \gamma \Rightarrow x \in \overline{\mathcal{P}}.$$

The desired polyhedral set $\mathcal{P}$ satisfying $\mathcal{E}_m(\gamma) \subset \mathcal{P} \subset \mathcal{E}_M(\gamma)$, is obtained as the convex hull of the vertices of $\overline{\mathcal{P}}$ (note that if $\overline{\mathcal{P}}$ is convex then $\mathcal{P} = \overline{\mathcal{P}}$). Indeed,

$$\mathcal{E}_m(\gamma) \subset \overline{\mathcal{P}} \subset \mathcal{P} \Rightarrow \mathcal{E}_m(\gamma) \subset \mathcal{P}$$

and, by the convexity of $\mathcal{E}_M(\gamma)$, it holds that

$$\mathcal{P} \triangleq \mathrm{Co}(\overline{\mathcal{P}}) \subseteq \mathrm{Co}(\mathcal{E}_M(\gamma)) = \mathcal{E}_M(\gamma).$$

Note that the computation of the vertices of $\overline{\mathcal{P}}$ and of their convex hull can be performed efficiently using, for instance, the Geometric Bounding Toolbox (GBT) [19].

V. EXAMPLE

Consider the following MPC constrained optimization problem:

$$\begin{aligned} V(x(0)) &= \min_U J(U, x(0)) \\ J(U, x(0)) &= x_N^\top P x_N + \sum_{k=0}^{N-1} x_k^\top Q x_k + u_k^\top R u_k \end{aligned} \tag{13a}$$

$$\text{s.t.} \quad \begin{aligned} & x_{k+1} = A x_k + B u_k, \\ & y_k = C x_k, \\ & u_{\min} \le u_k \le u_{\max}, \end{aligned} \tag{13b}$$

where $A = \left[\begin{smallmatrix} 0 & 1 \\ -1 & -1 \end{smallmatrix}\right]$, $B = \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$, $C = \left[\begin{smallmatrix} 1 & 0 \end{smallmatrix}\right]$, $u_{min} = [-5, -5]^\top$, $u_{max} = [5, 5]^\top$, $Q = 10 I_2$, $R = 3 I_2$, P is the solution to the algebraic Riccati equation, the control horizon $N = 1$ and $\gamma = 4000$. The feasible domain $\mathcal{X}^*$ of (13a)-(13b) is clearly $\mathcal{X}^* = \mathbb{R}^n$, as only input constraints are included in (13b).

Algorithm 1 computes the polyhedral set

$$\mathcal{P} = \left\{ x : \begin{bmatrix} -901.7694 & -146.6748 \\ -471.8016 & 80.5390 \\ -262.6671 & 289.6734 \\ 549.3794 & 409.7714 \\ -281.2966 & 229.1337 \\ 574.0301 & 335.6954 \\ -574.0301 & -335.6954 \\ 281.2966 & -229.1337 \\ -549.3794 & -409.7714 \\ 262.6671 & -289.6734 \\ 471.8016 & -80.5390 \\ 901.7694 & 146.6748 \end{bmatrix} x \le 10^4 \begin{bmatrix} 1.1895 \\ 0.6826 \\ 0.6826 \\ 0.9663 \\ 0.5778 \\ 0.8427 \\ 0.8427 \\ 0.5778 \\ 0.9663 \\ 0.6826 \\ 0.6826 \\ 1.1895 \end{bmatrix} \right\} \tag{14}$$

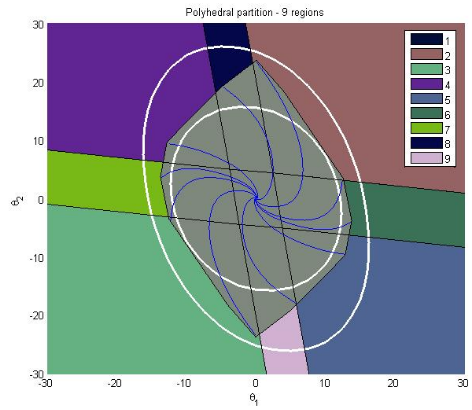

Fig. 1. Polyhedron $\mathcal{P}$ and the piecewise ellipsoidal sets $\mathcal{E}_m$, $\mathcal{E}_M$

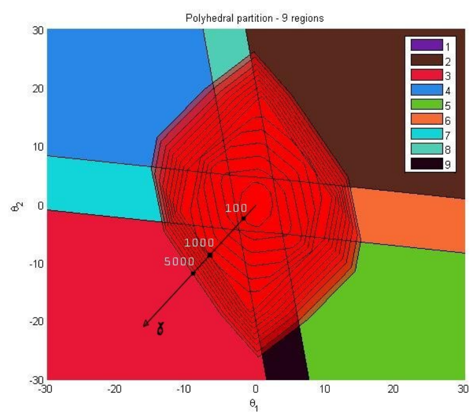

Fig. 2. Various shapes of polyhedron $\mathcal{P}$ as γ varies from 100 to 5000

in 21.7030 seconds on a DELL Workstation Xeon(TM), CPU 3.20GHz, and 1.00 GB of RAM, with MATLAB ver. 7.00, after generating 264 simplices.

Figure 1 shows the desired polyhedron $\mathcal{P}$, which lies between the sets $\mathcal{E}_m(\gamma)$ and $\mathcal{E}_m(\gamma)$. In the same figure the state space partition corresponding to the explicit form of the MPC controller (13) is provided, just to illustrate that $\mathcal{E}_m(\gamma)$ and $\mathcal{E}_m(\gamma)$ are piecewise-ellipsoidal, as they cover multiple partitions. Note that the trajectories starting from states inside $\mathcal{P}$ remain in $\mathcal{P}$, which illustrates the invariance property of $\mathcal{P}$. The partition corresponding to the explicit MPC solution is plotted for completeness, but it is not required by the algorithm. Figure 2 shows the shapes of the polyhedral invariant sets as the level γ varies from 100 to 5000. Note that (Figure 2) the set corresponding to $\gamma = 100$, is contained into the central region corresponding to the inactivity of all input constraints in (13b) ($u_0^*(x) = Kx$).

VI. CONCLUSIONS

In this paper we presented a geometrical procedure for computing convex polyhedral positively invariant sets for discrete-time linear time-invariant systems in closed-loop with MPC controllers. The present work is an extension of [11]. Here we do not use the value function $V(x)$ as a Lyapunov function to get the two sublevel sets $\mathcal{E}_M(\gamma), \mathcal{E}_m(\gamma)$ which are necessary for inscribing the polyhedron in-between. Moreover, the two piecewise ellipsoidal sets $\mathcal{E}_M(\gamma), \mathcal{E}_m(\gamma)$ are not contractive sets, see [11] for further details. For the same reasons, the approach used in [8] cannot be followed in the present work. The polyhedral set determined here is inscribed between two convex piecewise ellipsoidal level sets of two piecewise quadratic functions. The algorithm computes a PWA function whose graph is contained between the graphs of two PWQ functions. The level sets of the resulting PWA function are polyhedral and invariant. The desired polyhedral invariant set can then be simply obtained by choosing an appropriate level γ and retrieving the corresponding level set of the constructed PWA function. An open question that remains to be addressed is the extension of the presented procedure to MPC closed-loop systems with disturbances.

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