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Further Results on Multiparametric Quadratic Programming

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***Abstract*— In this paper we extend recent results on strictly convex multiparametric quadratic programming (mpQP) to the convex case. An efficient method for computing the mpQP solution is provided. We give a fairly complete description of the mpQP solver, focusing on implementational issues such as degeneracy handling.**

I. Introduction

Parametric programming amounts to explicitly representing the solution to an optimization problem for a range of parameter values. In particular, in multiparametric programming, a vector of parameters is considered. Much of the recent interest in multiparametric programming has been motivated by the need for technology to implement constrained optimal feedback control with a minimal amount of real-time computations. Parametric programming solutions allow explicit nonlinear (typically piecewise affine (PWA)) feedback control laws to replace computationally expensive real-time numerical optimization algorithms.

A thorough treatment of multiparametric LP (mpLP) with an algorithm to solve such problems is given in [8]. Strictly convex multiparametric QP (mpQP) was treated in [1], in which also a geometric algorithm to solve the problem is presented. These ideas were modified to an mpLP algorithm in [2]. An alternative strategy for mpQP was used in [11]. In [14] a more efficient mpQP solver was developed, extending the main ideas of [8] to the strictly convex mpQP case.

The main contribution of this paper is to combine the efficiency of the active set mpQP solver [14], which handles only strictly convex problems, with the simple degeneracy handling of the geometric mpQP solver [1], in order to solve mpQPs which has a positive semi-definite projection of the Hessian onto the subspace defined by the active constraints. Moreover, we discuss how primal degeneracies can best be treated in the mpQP solver, and include mpQPs with equality constraints. Note that the algorithm suggested in this paper can also be used for mpLP, as this is a special case of convex mpQP.

II. Basic Results

We will consider the following class of problems

$$V(\theta) = \min_{x\in\mathbb{R}^n} \frac{1}{2} x^T H x + \theta^T F^T x + c^T x, \tag{1}$$

$$A_i x = b_i + S_i\theta, \quad i \in \mathcal{E}, \tag{2}$$

$$A_i x \le b_i + S_i\theta, \quad i \in \mathcal{I}, \tag{3}$$

where $\theta \in \mathbb{R}^p$ is a parameter of the optimization problem, and the vector $x \in \mathbb{R}^n$ is to be optimized for all values of $\theta \in \Theta$, where $\Theta \subseteq \mathbb{R}^p$ is some polyhedral set. Moreover, $H = H^T \in \mathbb{R}^{n\times n}$, $F \in \mathbb{R}^{n\times p}$, $A \in \mathbb{R}^{q\times n}$, $b \in \mathbb{R}^{q\times 1}$ and $S \in \mathbb{R}^{q\times p}$ are matrices, and $\mathcal{E}$ and $\mathcal{I}$ are sets of indices such that $\mathcal{E} \cup \mathcal{I} = \{1, ..., q\}$ and $\mathcal{E} \cap \mathcal{I} = \emptyset$. Even if the mpQP is solved with equality constraints in this paper, note that another possibility would be to eliminate the equalities apriori to solving the problem, by using standard techniques, see e.g. [10, Chapter 15]. Moreover, if the set of feasible parameter values is not full-dimensional, the technique described in [2] should be applied to obtain a full-dimensional set of feasible parameters. Let the subscript index denote a subset of the rows of a matrix or vector, and $|\cdot|$ denote the number of elements in a set.

A. Solving the MpQP

Definition 1: Let x be a feasible solution to (1)–(3) for a given θ. We define **active constraints** the set of constraints with $A_i x - b_i - S_i\theta = 0$, and **inactive constraints** the set of constraints with $A_i x - b_i - S_i\theta < 0$. The **active set** $\mathcal{A}(x,\theta)$ is the set of indices of the active constraints, that is,

$$\mathcal{A}(x,\theta) = \{i \in \{1, ..., q\} \mid A_i x = b_i + S_i\theta\}.$$

Moreover, let $\mathcal{N}(x,\theta)$ denote the set of inactive constraints, that is, $\mathcal{N}(x,\theta) = \{1, ..., q\} \setminus \mathcal{A}(x,\theta)$.

Definition 2: Let θ be given. $X^*(\theta)$ is the set of optimal solutions to (1)–(3).

Definition 3: Let θ be given. Let the optimal active set $\mathcal{A}^*(\theta)$ be the set of constraints which are active for all $x \in X^*(\theta)$, that is

$$\mathcal{A}^*(\theta) = \{i \mid i \in \mathcal{A}(x,\theta), \forall x \in X^*(\theta)\} = \bigcap_{x\in X^*(\theta)} A(x,\theta).$$

Let $\mathcal{N}^*(\theta) = \{1, ..., q\} \setminus \mathcal{A}^*(\theta)$.

When the mpQP is strictly convex, the optimal solution x^* is unique (see Theorem 2) and the active set $\mathcal{A}(x^*(\theta),\theta)$ is the unique active set for the optimal solution. However, Definition 3 gives a unique optimal active set also in the convex case, when the solution is not unique.

Assume for the moment that we know the set $\mathcal{A}$ of active constraints at the optimum for a given θ. We can now form matrices $A_{\mathcal{A}}$, $b_{\mathcal{A}}$ and $S_{\mathcal{A}}$ as the rows of A, b and S corresponding to this optimal active set $\mathcal{A}$.

Definition 4: For an active set $\mathcal{A}$, we say that the **linear independence constraint qualification (LICQ)** holds if the set of active constraint gradients are linearly independent, i.e., $A_{\mathcal{A}}$ has full row rank. When LICQ is violated, this is referred to as **primal degeneracy**.

As in [1] we solve the mpQP by formulating the KKT conditions

$$Hx + F\theta + c + A^T\lambda = 0, \quad \lambda \in \mathbb{R}^q, \tag{4}$$

$$\lambda_i (A_i x - b_i - S_i\theta) = 0, \quad \text{for all } i \in \mathcal{I}, \tag{5}$$

$$A_i x - b_i - S_i\theta = 0, \quad \text{for all } i \in \mathcal{E}, \tag{6}$$

$$A_i x - b_i - S_i\theta \le 0, \quad \text{for all } i \in \mathcal{I}, \tag{7}$$

$$\lambda_i \ge 0, \quad \text{for all } i \in \mathcal{I}. \tag{8}$$

The strategy is first to fix the active set $\mathcal{A}$, giving a linear system with equality constraints only. Suppose θ is given and $\mathcal{A}$ is an optimal active set, then (4), (6) and (7) lead to

$$\begin{bmatrix} H & A_{\mathcal{A}}^T \\ A_{\mathcal{A}} & 0 \end{bmatrix} \begin{bmatrix} x^* \\ \lambda^* \end{bmatrix} = \begin{bmatrix} -c \\ b_{\mathcal{A}} \end{bmatrix} + \begin{bmatrix} -F \\ S_{\mathcal{A}} \end{bmatrix} \theta. \tag{9}$$

To solve equation (9), we use the standard null-space method [10]:

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Theorem 1: Consider the mpQP (1)–(3). Consider an arbitrary active set $\mathcal{A}$, let $m = |\mathcal{A}|$, $\mathcal{N} = \{1, ..., q\} \setminus \mathcal{A}$ and assume that $\mathcal{A}$ satisfies LICQ. Let $Z_{\mathcal{A}}$ be an $n \times (n-m)$ matrix whose columns span the null-space of $A_{\mathcal{A}}$, let $Y_{\mathcal{A}}$ be any $n \times m$ matrix such that $[Y_{\mathcal{A}} \quad Z_{\mathcal{A}}]$ is nonsingular, and assume $Z_{\mathcal{A}}^T H Z_{\mathcal{A}} > 0$. For any $\theta \in \Theta$ such that $\mathcal{A}$ is the optimal active set, the optimal solution and Lagrange multipliers are given by the affine functions

$$x^*(\theta) = K_x\theta + k_x, \tag{10}$$

$$\lambda^*_{\mathcal{A}}(\theta) = K_\lambda\theta + k_\lambda, \tag{11}$$

where

$$K_x = Y(A_{\mathcal{A}}Y_{\mathcal{A}})^{-1}S_{\mathcal{A}} - Z_{\mathcal{A}}(Z_{\mathcal{A}}^T H Z_{\mathcal{A}})^{-1} \cdot Z_{\mathcal{A}}^T(F + HY_{\mathcal{A}}(A_{\mathcal{A}}Y_{\mathcal{A}})^{-1}S_{\mathcal{A}}), \tag{12}$$

$$k_x = (Y_{\mathcal{A}} - Z_{\mathcal{A}}(Z_{\mathcal{A}}^T H Z_{\mathcal{A}})^{-1}Z_{\mathcal{A}}^T H Y_{\mathcal{A}})(A_{\mathcal{A}}Y_{\mathcal{A}})^{-1}b_{\mathcal{A}} - Z_{\mathcal{A}}(Z_{\mathcal{A}}^T H Z_{\mathcal{A}})^{-1}Z_{\mathcal{A}}^T c, \tag{13}$$

$$K_\lambda = -(A_{\mathcal{A}}Y_{\mathcal{A}})^{-T}Y^T(HK_x + F), \tag{14}$$

$$k_\lambda = -(A_{\mathcal{A}}Y_{\mathcal{A}})^{-T}Y^T(Hk_x + c). \tag{15}$$

Moreover, the active set $\mathcal{A}$ is the unique optimal active set in the interior of the critical region (CR) given by those $\theta \in \Theta$ that satisfy

$$G\theta \le g \tag{16}$$

where

$$G = \begin{bmatrix} A_{\mathcal{N}}K_x - S_{\mathcal{N}} \\ (K_\lambda)_{\mathcal{I}\cap\mathcal{A}} \end{bmatrix}, g = \begin{bmatrix} b_{\mathcal{N}} - A_{\mathcal{N}}k_x \\ -(k_\theta)_{\mathcal{I}\cap\mathcal{A}} \end{bmatrix}. \tag{17}$$

Proof: Partition the vector x^* (being the solution to (4)–(8) for a given θ) into two components

$$x^* = Y_{\mathcal{A}}x_Y + Z_{\mathcal{A}}x_Z. \tag{18}$$

This means that $Y_{\mathcal{A}}x_Y$ is a particular solution of $A_{\mathcal{A}}x^* = b_{\mathcal{A}} + S_{\mathcal{A}}\theta$, and $Z_{\mathcal{A}}x_{Z_{\mathcal{A}}}$ is a displacement along these constraints. Since $A_{\mathcal{A}}Z_{\mathcal{A}} = 0$ and $A_{\mathcal{A}}Y_{\mathcal{A}}$ is a nonsingular $m \times m$ matrix, we can substitute (18) into the second equation of (9) to obtain

$$x_Y = (A_{\mathcal{A}}Y_{\mathcal{A}})^{-1}(b_{\mathcal{A}} + S_{\mathcal{A}}\theta). \tag{19}$$

We proceed to solve (9) by substituting (18) into the first equation of (9) and multiply by $Z_{\mathcal{A}}^T$, to obtain

$$(Z_{\mathcal{A}}^T H Z_{\mathcal{A}})x_Z = -(Z_{\mathcal{A}}^T H Y_{\mathcal{A}}x_Y + Z_{\mathcal{A}}^T c + Z_{\mathcal{A}}^T F\theta),$$

$$x_Z = -(Z_{\mathcal{A}}^T H Z_{\mathcal{A}})^{-1}(Z_{\mathcal{A}}^T H Y_{\mathcal{A}}x_Y + Z_{\mathcal{A}}^T c + Z_{\mathcal{A}}^T F\theta), \tag{20}$$

and (10) can be verified by substituting (19) and (20) into (18),

$$x^* = Y_{\mathcal{A}}(A_{\mathcal{A}}Y_{\mathcal{A}})^{-1}(b_{\mathcal{A}} + S_{\mathcal{A}}\theta) \tag{21}$$

$$- Z_{\mathcal{A}}(Z_{\mathcal{A}}^T H Z_{\mathcal{A}})^{-1}(Z_{\mathcal{A}}^T H Y_{\mathcal{A}}(A_{\mathcal{A}}Y_{\mathcal{A}})^{-1}(b_{\mathcal{A}} + S_{\mathcal{A}}\theta) + Z_{\mathcal{A}}^T c + Z_{\mathcal{A}}^T F\theta). \tag{22}$$

It is well known that the 2nd order condition $Z^T H Z > 0$ is sufficient for optimality. Finally, we can obtain the Lagrange multipliers by multiplying the first equation of (9) by $Y_{\mathcal{A}}^T$

$$Y_{\mathcal{A}}^T H x^* + Y_{\mathcal{A}}^T A_{\mathcal{A}}^T \lambda^* = -Y_{\mathcal{A}}^T F\theta - Y_{\mathcal{A}}^T c$$

$$\lambda^* = -(A_{\mathcal{A}}Y_{\mathcal{A}})^{-T}Y_{\mathcal{A}}^T(Hx^* + F\theta + c) \tag{23}$$

We have now characterized the solution to (1)–(3) for a given optimal active set $\mathcal{A}$, and a fixed θ. However, as long as $\mathcal{A}$ remains the optimal active set in a neighborhood of θ, it can be argued as in [1] that the solution (10)–(11) remains optimal, when x^* is viewed as a function of θ. Such a neighborhood where $\mathcal{A}$ is optimal is determined by imposing the two last KKT conditions (7)–(8), and noting that (7) is fulfilled by construction for $i \in \mathcal{I} \cap \mathcal{A}$.

$$A_i x^*(\theta) - b_i - S_i\theta \le 0, \quad \text{for all } i \in \mathcal{N} \tag{24}$$

$$(K_\lambda)_i\theta + (k_\lambda)_i \ge 0, \quad \text{for all } i \in \mathcal{I} \cap \mathcal{A} \tag{25}$$

■

The region (16) is commonly referred to as a critical region (CR). This is a polyhedral set, whose open interior represents the largest set of parameters θ such that $\mathcal{A}$ is the unique optimal active set. This means that if we know every active set which is optimal in some full-dimensional region in the parameter space, we can characterize the solution to the mpQP as a PWA function of the parameter vector. The main task of an mpQP solver is therefore to find every such active set.

We would like to choose $Y_{\mathcal{A}}$ in such a way that $A_{\mathcal{A}}Y_{\mathcal{A}}$ is as well conditioned as possible, to make (19) numerically well-conditioned. This can be done by a QR factorization [12] of $A_{\mathcal{A}}^T$, that is,

$$A_{\mathcal{A}}^T\Pi = [Q_1 \quad Q_2]\begin{bmatrix} R \\ \mathbf{0} \end{bmatrix}, \tag{26}$$

and defining

$$[Y_{\mathcal{A}} \quad Z_{\mathcal{A}}] = [Q_1 \quad Q_2], \tag{27}$$

which would result in a condition number for $A_{\mathcal{A}}Y_{\mathcal{A}}$ which is not larger than that of $A_{\mathcal{A}}$. Π is a permutation matrix, and the Q_2 and $\mathbf{0}$ matrices in (26) may be empty. For ease of notation, we explicitly form the inverses to solve the equations in this paper. However, as the matrices $Y_{\mathcal{A}}$ and $Z_{\mathcal{A}}$ are formed by a QR factorization, (19) (and other equations in this paper) will be in triangular form, and the implementation exploits this by solving the equations by substitution rather than forming the inverses. The matrices $Y_{\mathcal{A}}$ and $Z_{\mathcal{A}}$ may also be obtained by a Gaussian elimination, but as the problem sizes of mpQPs usually are sufficiently small to make the orthogonal QR factorization attainable, this would be preferable due to the numerical advantages.

The following theorem characterizes some properties of the primal and dual parametric solutions, and will be useful in the sequel.

Theorem 2: Consider Problem (1)–(3). Let $\Theta \in \mathbb{R}^n$ be a polyhedron, and suppose $Z^T_{\mathcal{A}(x^*(\theta))}HZ_{\mathcal{A}(x^*(\theta))} > 0$ for all $\theta \in \Theta$. Then the solution $x^*(\theta)$ and the Lagrange multipliers $\lambda^*(\theta)$ of the mpQP (1)–(3) are piecewise affine functions of the parameters θ, and $x^*(\theta)$ is continuous and unique. Moreover, if LICQ holds for $\mathcal{A}(x^*(\theta))$ for all $\theta \in \Theta$, $\lambda^*(\theta)$ is also continuous and unique.

Proof: Follows easily from uniqueness (due to $Z^T_{\mathcal{A}(x^*(\theta))}HZ_{\mathcal{A}(x^*(\theta))} > 0$ and LICQ) of $x^*(\theta)$ and $\lambda^*(\theta)$, cf. [1], [6]. ■

B. MpQP Solvers

After having characterized a critical region as in Theorem 1 one needs a method for partitioning the rest of the parameter space. Such a method was proposed in [5], formally proved and applied to mpQP in [1] and applied to mpLP in [2]. The method is briefly summarized in Algorithm 1, and for a given set Θ of parameters to be partitioned, this algorithm should be applied with $Y = \Theta$.

Algorithm 1

1) Let Y be the polyhedron in parameter space to be investigated.
2) Compute the chebyshev center θ_0 and radius r (see [3]) of Y. If $r \leq 0$, then exit.
3) Solve the QP (1)–(3) for $\theta = \theta_0$ to obtain the active set.
4) Characterize the optimal solution $x(\theta)$, Lagrange multipliers $\lambda(\theta)$ and critical region where this active set is optimal.
5) Divide the parameter space as in Figure 1b) by reversing one by one the hyperplanes defining the critical region.
6) For each new region R_i, let $Y = R_i$, and execute Algorithm 1.

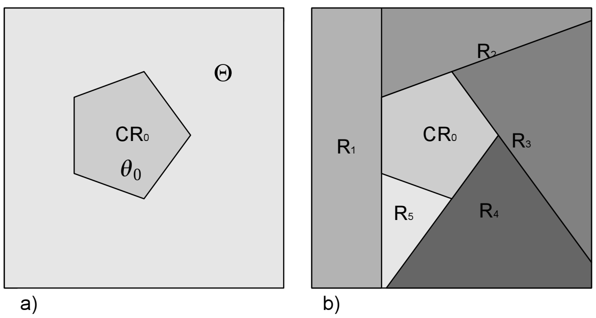

Fig. 1. Parameter space exploration strategy of [1].

One advantage of this way of partitioning the parameter space, is that handling degeneracies is relatively easy. The mpQP solver [14] has a different exploration strategy for dividing the parameter space. Properties of the geometry of the polyhedral partition and its relation to the combination of active constraints at optimum, are used to find the active set in all critical regions which are neighbors of the current one, as in Figure 2. Based on these results, this algorithm avoids unnecessary partitioning of the parameter space, giving significant improvement of efficiency for solving strictly convex mpQPs with respect to the algorithm of [1].

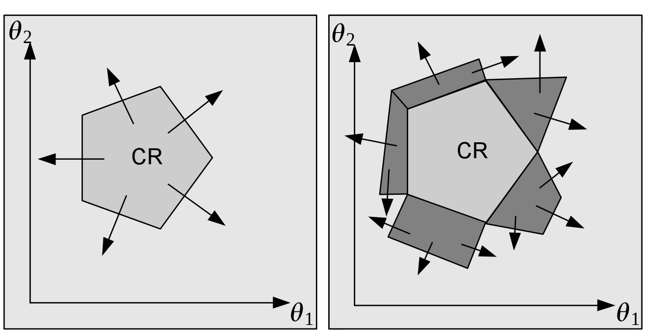

Fig. 2. Parameter space exploration strategy of [14].

C. Primal Degeneracy

When LICQ is violated for an optimal active set, the optimal Lagrange multipliers are no longer unique, giving several optimal combinations of active constraints. In [1] two methods were suggested on how to handle this.

a) To obtain a linear system of full rank by removing some active constraints, and proceed with the new reduced active set.
b) To use a projection in the $[\theta, \lambda]$-space.

It was argued that method b) would be computationally expensive, and consequently, method a) was used. Moreover, method a) is well suited for easy implementation with the algorithm of [1]. However, when using the algorithm of [14], this method leads to overlapping regions, which increases software complexity and degrades the performance of the solver. Example 1 illustrates some problems that arise. When using method b), Definition 3 gives a unique active set for a given θ (and the optimal solution $x^*(\theta)$), avoiding critical regions with mutually overlapping interiors. Moreover, the projection needed is often through one dimension only, which may not be more computationally demanding than characterizing several critical regions as in method a). Hence, we will explore method b) in more detail.

When LICQ is violated, the set of optimal Lagrange multipliers $\lambda^*(\theta)$ can be characterized by a polyhedron in the (λ,θ)-space:

Theorem 3: Consider the same problem as in Theorem 1, however, assume that LICQ is violated. Let $\bar{Z}_{\mathcal{A}}$ span $null(A_{\mathcal{A}}^T)$ and $\bar{Y}_{\mathcal{A}}$ be any matrix such that $\begin{bmatrix}\bar{Y}_{\mathcal{A}} & \bar{Z}_{\mathcal{A}}\end{bmatrix}$ is nonsingular. Let $Y_{\mathcal{A}}$ be as defined in Theorem 1. For any θ such that $\mathcal{A}$ is the optimal active set, optimal Lagrange multipliers are characterized by

$$\lambda^*(\theta) = \bar{K}_\lambda \theta + \bar{k}_\lambda + \bar{Z}_{\mathcal{A}} \lambda_{\bar{Z}}, \tag{28}$$

where

$$\bar{K}_\lambda = -\bar{Y}_{\mathcal{A}}(Y_{\mathcal{A}}^T A_{\mathcal{A}}^T \bar{Y}_{\mathcal{A}})^{-1} Y_{\mathcal{A}}^T (HK_x + F), \tag{29}$$
$$\bar{k}_\lambda = -\bar{Y}_{\mathcal{A}}(Y_{\mathcal{A}}^T A_{\mathcal{A}}^T \bar{Y}_{\mathcal{A}})^{-1} Y_{\mathcal{A}}^T (Hk_x + c), \tag{30}$$

and $\lambda_{\bar{Z}}$ is any vector such that

$$A_{\mathcal{N}} x^*(\theta) - b_{\mathcal{N}} - S_{\mathcal{N}}\theta \leq 0 \tag{31}$$
$$\left(\bar{K}_\lambda\right)_{\mathcal{I}\cap\mathcal{A}} \theta + \left(\bar{k}_\lambda\right)_{\mathcal{I}\cap\mathcal{A}} + \left(\bar{Z}_{\mathcal{A}}\right)_{\mathcal{I}\cap\mathcal{A}} \lambda_{\bar{Z}} \geq 0 \tag{32}$$

The active set $\mathcal{A}$ is optimal in the interior of the projection of (31)–(32) onto the θ-space.

Proof: To compute $x^*(\theta)$ one can obtain a reduced set of equations, and proceed as in Section II-A, as $x^*(\theta)$ is still uniquely defined (due to Theorem 2). The Lagrange multipliers can be found by partitioning λ^* as

$$\lambda^* = \bar{Y}_{\mathcal{A}} \lambda_{\bar{Y}} + \bar{Z}_{\mathcal{A}} \lambda_{\bar{Z}}.$$

This means that $\bar{Y}_{\mathcal{A}}\lambda_{\bar{Y}}$ is a particular solution of (9), while $\bar{Z}_{\mathcal{A}}\lambda_{\bar{Z}}$ is a displacement along the constraints. We proceed to find λ^*

$$Hx^*(\theta) + A_{\mathcal{A}}^T \lambda^* = -F\theta - c \tag{33}$$
$$A_{\mathcal{A}}^T(\bar{Y}_{\mathcal{A}}\lambda_{\bar{Y}} + \bar{Z}_{\mathcal{A}}\lambda_{\bar{Z}}) = -Hx^*(\theta) - F\theta - c \tag{34}$$
$$Y_{\mathcal{A}}^T A_{\mathcal{A}}^T \bar{Y}_{\mathcal{A}} \lambda_{\bar{Y}} = -Y_{\mathcal{A}}^T (Hx^*(\theta) + F\theta + c) \tag{35}$$
$$\begin{aligned}\lambda_{\bar{Y}} &= -(Y_{\mathcal{A}}^T A_{\mathcal{A}}^T \bar{Y}_{\mathcal{A}})^{-1} Y_{\mathcal{A}}^T (Hx^*(\theta) \\ &\quad + F\theta + c)\end{aligned} \tag{36}$$
$$\begin{aligned}\lambda^* &= -\bar{Y}_{\mathcal{A}}(Y_{\mathcal{A}}^T A_{\mathcal{A}}^T \bar{Y}_{\mathcal{A}})^{-1} Y_{\mathcal{A}}^T (Hx^*(\theta) \\ &\quad + F\theta + c) + \bar{Z}_{\mathcal{A}}\lambda_{\bar{Z}}\end{aligned} \tag{37}$$
$$= \bar{K}_\lambda \theta + \bar{k}_\lambda + \bar{Z}_{\mathcal{A}}\lambda_{\bar{Z}}, \tag{38}$$

where the last equation defines $\bar{K}_\lambda$, $\bar{k}_\lambda$ and $\bar{Z}_{\mathcal{A}}$. In particular, the transition from (34) to (35) is valid because $Y_{\mathcal{A}}^T$ is full row-rank. The CR (31)–(32) can be obtained by inserting (28) into (7)–(8). ■

In the numerical implementation we use a QR factorization to obtain the matrices $\bar{Z}_{\mathcal{A}}$ and $\bar{Y}_{\mathcal{A}}$ to obtain a well-conditioned system. To obtain a critical region in θ-space from (31)–(32), a projection algorithm is needed, e.g. [7] or a Fourier-Motzkin elimination [4]. The polyhedron to be projected is defined by $|\mathcal{I} \cap \mathcal{A}|$ linear inequalities (only the polyhedron (32) has to be projected), and the projection has to be done through a number of dimensions given by the dimension of the null-space of $A_{\mathcal{A}}$. So even if doing projections is considered computationally expensive in general, we emphasize that the projection needed often is relatively simple, and needed only in degenerate cases. This is due to the fact that when LICQ is violated in a full-dimensional region, the row-rank of $A_{\mathcal{A}}$ is often $|\mathcal{A}| - 1$, and the required projection would be through one dimension only. When using method a) to handle violation of LICQ, several possibly overlapping critical regions will be found instead of the single region found by projection. Avoiding this ambiguity has computational advantages.

Example 1: Consider the following strictly convex mpQP:

$$A = \begin{bmatrix} 1 & 0 & -1 \\ -1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & -1 & -1 \end{bmatrix}, b = -\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, S = \begin{bmatrix} 1 & 0 \\ -1 & 0 \\ 0 & -1 \\ 0 & 1 \end{bmatrix},$$
$$H = I_{3\times3}, F = \mathbf{0}_{3\times2}, c = \mathbf{0}_{3\times1}, \mathcal{I} = \{1, 2, 3, 4\}, \mathcal{E} = \emptyset.$$

The partition obtained from this mpQP is shown in Figure 3. In regions $R1 - R4$, LICQ is violated, and the union of these regions can be obtained by a projection, as explained in Theorem 3. Inside this region, there are four different combinations of active constraints which may be optimal, each of them corresponding to a region $R1, R2, R3$ or $R4$. The problems connected with such overlapping regions when using the mpQP solver of [14], can be as follows: Assume that region $R1$ is found, and the solver shall find the neighboring region of $R1$ in the direction of $R2$. This must be done using the method of [13, Appendix A], since LICQ is violated in $R1$. However, when solving the QP, there is, as far as we can see, no obvious way of preventing that the new combination of active constraints found is the one corresponding to region $R3$ or $R4$. And as shown in Figure 2, the mpQP solver of [14] depends on finding the neighboring region of the current region, to guarantee that there are no holes in the resulting partition. In this simple example one may find methods for handling such problems, but in higher dimensions having higher degrees of degeneracies, we believe that the projection method analyzed in Theorem 3 is the most reliable way of characterizing the solution.

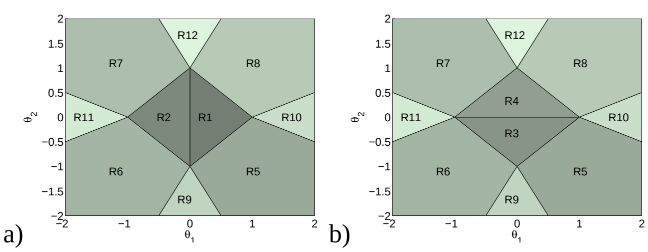

Fig. 3. Partition of parameter space, Example 1.

D. Positive Semi-definite MpQP and Dual Degeneracy

In this section, we explain how to deal with cases when the matrix $Z_{\mathcal{A}}^T H Z_{\mathcal{A}}$ in Theorem 1 has some eigenvalues equal to zero, that is, the positive definiteness assumption is relaxed into positive semi-definiteness. This is referred to as dual degeneracy, which means that the dual of (1)–(3) is primal degenerate. In Section II-C we saw that in the case of primal degeneracy we could obtain a partition with mutually non-overlapping regions by using a projection algorithm. A similar procedure can be applied in the case of dual degeneracy. By again using the null-space method, the optimal solution and Lagrange multipliers can be obtained as affine functions of the parameters and some additional variable p.

$$\begin{bmatrix} x^*(\theta) \\ \lambda^*(\theta) \end{bmatrix} = K_{x\lambda}\theta + k_{x\lambda} + K_p p$$

For a positive semi-definite mpQP, one can (similar to the procedure of section II-C) for an optimal active set $\mathcal{A}^*$ characterize a polyhedron in the (θ, p)-space from the KKT conditions (4)–(8). By construction, $\mathcal{A}^*$ is in the interior of this region the unique optimal active set according to Definition 3. One can further apply a projection algorithm to obtain a critical region in the parameter space.

The mpQP algorithm of [1] can fairly straightforwardly be extended to deal with dual degeneracy and positive semi-definite mpQP. However, we would like to take advantage of the increased execution speed of the algorithm of [14] in parts of the parameter space where the problem is non-degenerate. We therefore suggest to combine these two algorithms to solve such problems efficiently:

1) Partition Θ by using the algorithm of [14], using a projection algorithm in regions in which the solutions are (primal or dual) degenerate, to obtain a partition with mutually non-overlapping regions.
2) Continue partitioning each region in which the solution is dual degenerate, by using the algorithm of [1].

The efficiency of this method lies not only in the algorithm of [14] being used in parts of the parameter space where the problem is non-degenerate, but also in the fact that the main disadvantage of the algorithm of [1] is reduced, as the artificial partitioning induced by the algorithm is limited to smaller parts of the parameter space.

Example 2: An mpLP can be considered a special case of positive semi-definite mpQP, namely the one with $H = \mathbf{0}$. Consider Example 4-2 from [8]:

$$H = \mathbf{0}_{2\times2}, F = \mathbf{0}_{2\times2}, c^T = -\begin{bmatrix} 3 & 8 \end{bmatrix},$$
$$A^T = \begin{bmatrix} 1 & 5 & -8 & 4 & 1 & 0 \\ 1 & -4 & 22 & 1 & 0 & 1 \end{bmatrix},$$
$$b^T = \begin{bmatrix} 13 & 20 & 121 & 8 & 0 & 0 \end{bmatrix},$$
$$S^T = \begin{bmatrix} 0 & 0 & 3 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$\mathcal{I} = \{1, ..., 6\}$ and $\mathcal{E} = \emptyset$. This mpLP was solved using Algorithm 2, giving the partition shown in Figure 4 a). The mpLP is non-degenerate for all the regions in the partition. By introducing the extra inequality constraint

$$\begin{bmatrix} 3 & 8 \end{bmatrix} x \le \theta,$$

the partition of Figure 4 b) is obtained. The solution is dual degenerate in regions $R5, R6$ and $R7$. When using the exploration strategy shown in Figure 2, the union of these 3 regions were obtained by a projection, as explained in Section II-D. Then in the second phase, this larger region was further partitioned with Algorithm 1, to obtain the shown partition.

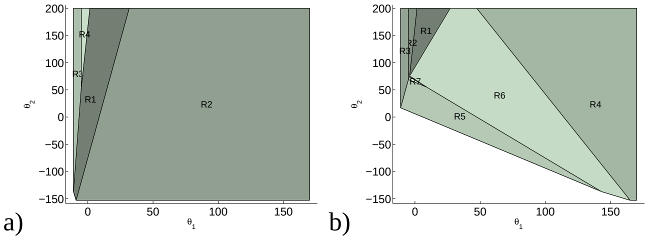

Fig. 4. Partition of parameter space, Example 2.

III. Active Sets in Neighboring Regions

This section will give some results on how to obtain the optimal active set in a neighboring region. Non-degenerate cases are handled in Section III-A, while Section III-B treats the degenerate cases. The results are straightforward extensions of results presented in [14].

A. Non-degenerate Cases

Below, we denote by $x^{*,k}(\theta)$ the affine expression of the PWA function $x^*(\theta)$ restricted to the critical region CR^k, where k is an index enumerating the optimal active sets.

Definition 5: Let $x^*(\theta)$ be an optimal solution of (1)–(3) for a given θ, and suppose that the KKT conditions (4)–(8) are satisfied. We say that the i-th inequality constraint is **weakly active** if $i \in \mathcal{A}(x^*(\theta),\theta)$ and $\lambda_i^* = 0$ for all λ^* satisfying (4)–(8). We say that an inequality constraint is **strongly active** if $i \in \mathcal{A}(x^*(\theta),\theta)$ and there exists some $\lambda_i^* > 0$ satisfying (4)–(8).

Definition 6: Let a full-dimensional polyhedron $\Theta \subset \mathbb{R}^p$ be represented by the linear inequalities $G\theta \leq g$. Let the hyperplane $G_i\theta = g_i$ be denoted by $\mathcal{H}$. If $\Theta \cap \mathcal{H}$ is $(p-1)$-dimensional then $\Theta \cap \mathcal{H}$ is called a **facet** of the polyhedron.

Definition 7: Two polyhedra are called **neighboring** if they share a common facet.

Definition 8: Let a polyhedron Θ be represented by $G\theta \leq g$. We say that $G_i\theta \leq g_i$ is **redundant** if $G_j\theta \leq g_j \ \forall j \neq i \Rightarrow G_i\theta \leq g_i$ (i.e., it can be removed from the description of the polyhedron). A representation of a polyhedron is **minimal** if it contains no redundant constraints.

We have seen in the previous section that when we fix the active set, it is fairly straightforward to characterize the optimal solution and Lagrange multipliers corresponding to this active set, and the region in the parameter space in which this active set is optimal. The main task for an mpQP solver is therefore to find every active set which is optimal in some full-dimensional region in the parameter space. We will do this by for each CR we identify, finding the optimal active set in every full-dimensional neighboring CR.

Let us consider a hyperplane defining the common facet between two polyhedra CR^0, CR^i in the optimal partition of the state space. Assuming degeneracies do not occur, there are two different kinds of hyperplanes. The first (Type I) are those described by (24), which represent a non-active inequality constraint that becomes active at the optimum as θ moves from CR^0 to CR^i. As proved in the following theorem, the corresponding constraint will be activated on the other side of the facet defined by this hyperplane. In addition, the corresponding Lagrange multiplier may become positive. The other kind (Type II) of hyperplanes which bound the polyhedra are those described by (25). In this case, the corresponding inequality constraint will be non-active on the other side of the facet defined by this hyperplane.

Theorem 4: Consider an optimal active set $\{i_1, i_2, ..., i_k\}$ and let CR^0 be its corresponding critical region with a representation obtained by removing all non-facet inequalities from (24)–(25). Assume that CR^0 is represented by a minimal representation. Also assume that the mpQP is not primal or dual degenerate for this active set. Let CR^i be a full-dimensional neighboring critical region to CR^0 and assume the problem to be non-degenerate for the optimal active set on their common facet $\mathcal{F} = CR^0 \cap \mathcal{H}$, where $\mathcal{H}$ is the separating hyperplane between CR^0 and CR^i. Moreover, assume that there are no constraints which are weakly active at the optimizer $x^*(\theta)$ for all $\theta \in CR^0$. Then:

Type I If $\mathcal{H}$ is given by $A_{i_{k+1}}x^{*,0}(\theta) = b_{i_{k+1}} + S_{i_{k+1}}\theta$ (where $i_{k+1} \in \mathcal{I} \cap \mathcal{N}$), then the optimal active set in CR^i is $\{i_1, ..., i_k, i_{k+1}\}$.

Type II If $\mathcal{H}$ is given by $\lambda^0_{i_k}(\theta) = 0$ (where $i_k \in \mathcal{I} \cap \mathcal{A}$), then the optimal active set in CR^i is $\{i_1, ..., i_{k-1}\}$.

Proof: Let us prove Type I. In order for some constraint $i_j \in \{i_1, ..., i_k\}$ not to be in the optimal active set in CR^i, by continuity of $\lambda^*(\theta)$ (due to Theorem 2 and LICQ), it follows that $\lambda^*_{i_j}(\theta) = \lambda^0_{i_j}(\theta) = 0$ for all $\theta \in \mathcal{F}$. Since there are no constraints which are weakly active for all $\theta \in CR^0$, this would mean that constraint i_j becomes non-active at $\mathcal{F}$. But this contradicts the assumption of minimality since $\lambda^0_{i_j}(\theta) \geq 0$ and $A_{i_{k+1}}x^{*,0}(\theta) \leq b_{i_{k+1}} + S_{i_{k+1}}\theta$ would be coincident, and thus one of them redundant. On the other hand $\{i_1, ..., i_k\}$ cannot be the optimal active set on CR^i because CR^0 is the largest set of θ's such that $\{i_1, ..., i_k\}$ is the optimal active set. Then, the optimal active set in CR^i is a superset of $\{i_1, ..., i_k\}$. Now assume that another constraint i_{k+2} is active in CR^i. That means $A_{i_{k+2}}x^{*,i}(\theta) = b_{i_{k+2}} + S_{i_{k+2}}\theta$ in CR^i, and by continuity of $x^*(\theta)$, the equality also holds for $\theta \in \mathcal{F}$. However, $A_{i_{k+2}}x^{*,0}(\theta) = b_{i_{k+2}} + S_{i_{k+2}}\theta$ would then coincide with $A_{i_{k+1}}x^{*,0}(\theta) = b_{i_{k+1}} + S_{i_{k+1}}\theta$, which again contradicts the assumption of minimality. Therefore, only $\{i_1, ..., i_k, i_{k+1}\}$ can be the optimal active set in CR^i. The proof for Type II is similar. ■

Corollary 1: Consider the same assumptions as in Theorem 4, except that CR^0 is no longer assumed to be minimal, i.e. two or more hyperplanes can coincide. Let $\mathcal{J} \subset \{i_1, ..., i_k\}$ be the set of indices corresponding to coincident hyperplanes in CR^0.

- Every constraint i_j where $i_j \in \{i_1, i_2, ..., i_k\} \setminus \mathcal{J}$ is active in CR^i.
- Every constraint i_j where $i_j \notin \{i_1, i_2, ..., i_k\} \cup \mathcal{J}$ is inactive in CR^i.

When, for instance, two hyperplanes are coincident, by Corollary 1 there are three possible active sets which have to be checked to find the optimal active set in CR^i. One should always a priori remove redundant constraints from (2)–(3). This reduces the complexity of the mpQP, and by this, some degeneracies may also be avoided (see Section III-B). Theorem 4 and Corollary 1 show how to find the optimal active set across a facet only by using the knowledge of which kind of hyperplane the facet corresponds to, except in degenerate cases, which is the topic of the next section.

B. Degeneracy Handling in the MpQP Solver

We assumed in Section III-A when obtaining the active set in a neighboring region, that the the mpQP was not degenerate, neither in the interior of the current critical region nor on its facets. In cases when this assumption is violated, we suggest to obtain the active set in the neighboring region by finding a point θ_0 a small distance into the neighboring region, and solving the QP obtained by inserting $\theta = \theta_0$ into the mpQP (1)–(3), see [13, Appendix A]. This method was in some cases also used in [14]. However, as opposed to in [14], an active set according to Definition 3 must be obtained. This is done by using an interior point method to solve the QP. Primal-dual solvers have the property that when the optimal solution is not unique, a solution in the interior of the optimal facet is obtained, see e.g. [9]. From this solution one can identify the unique active set according to Definition 3.

C. MpQP Algorithm

Based on the results from the previous sections, we finally present an efficient algorithm for the computation of the solution to the mpQP (1)–(3). Generally, there exist active sets which are not optimal anywhere in the parameter space (typically, most active sets are not optimal anywhere). We need an active set which is optimal in a full-dimensional region to start the algorithm. Generally we can do this by choosing a feasible θ, and solve the QP obtained by inserting this θ into the mpQP (1)–(3).
Let L_{cand} be a list of active sets which are found, but not yet explored (i.e., are candidates for optimality) and L_{opt} be the set of active sets which have been explored (i.e., are found to be optimal in some full-dimensional region).

Algorithm 2 Convex mpQP

Choose the initial active set $\mathcal{A}_0$ by finding a feasible θ and solving the corresponding QP; Let $L_{cand} \leftarrow \{\mathcal{A}_0\}$, $L_{opt} \leftarrow \emptyset$;
while $L_{cand} \neq \emptyset$ **do**
  Pick an element $\mathcal{A}$ from L_{cand}. Let $L_{cand} \leftarrow L_{cand} \setminus \{\mathcal{A}\}$;
  Build the matrices $A_{\mathcal{A}}$, $b_{\mathcal{A}}$ and $S_{\mathcal{A}}$ from $\mathcal{A}$.
  if the mpQP is not dual degenerate for the active set $\mathcal{A}$ **then**
    if $A_{\mathcal{A}}$ has full row-rank[1] **then**
      Determine the local Lagrange multipliers $\lambda^*_{\mathcal{A}}(\theta)$, and the solution $x^*(\theta)$ from (11) and (10), and find the CR where $\mathcal{A}$ is optimal from (16)–(17);
    else
      Determine the local Lagrange multipliers $\lambda^*_{\mathcal{A}}(\theta)$, the solution $x^*(\theta)$ and the CR where $\mathcal{A}$ is optimal as in Section II-C;
    end if
  else
    Determine the region in the parameter space where $\mathcal{A}$ is optimal as in Theorem 3. Continue partitioning this region by using Algorithm 1
  end if
  if CR is full-dimensional[2] **then**
    Detect an remove all non-facet hyperplanes from CR by solving LPs;
    $L_{opt} \leftarrow L_{opt} \cup \{\mathcal{A}\}$;
    for each facet $\mathcal{F}$ in CR **do**
      Find the optimal active set on $\mathcal{F}$ by examining the type of hyperplane $\mathcal{F}$ is given by;
      Find any possible optimal active sets in the neighboring region according to Theorem 4, Corollary 1, or if these are not applicable, by solving a QP as in [13, Appendix A], using a primal-dual interior point QP solver;
      For any new active set $\mathcal{A}_{new}$ found, let $L_{cand} \leftarrow L_{cand} \cup \{\mathcal{A}_{new}\}$
    end for
  end if
end while
for each CR^i corresponding to an active set $\mathcal{A}_i \in L_{opt}$ **do**
  if the mpQP is dual degenerate for $\mathcal{A}_i$ **then**
    Partition CR^i further, using Algorithm 1;
  end if
end for

IV. Conclusions

In this paper we have given a detailed description of an approach to solving convex mpQP problems. The approach covers mpQPs with a positive semi-definite projection of the Hessian onto the subspace defined by the active set. Special attention has been given to the handling of degenerate cases. The proposed mpQP algorithm can be considered a generalization of the mpQP/mpLP solvers [1], [14], [2], combining the advantages of the solvers. The result is an algorithm which is more general than the strictly convex mpQP solver of [14] (as it can handle convex mpQPs), and more efficient than Algorithm 1 (which is the convex counterpart of the strictly convex mpQP algorithm in [1]). A few examples, comparing Algorithm 1 and 2 on convex mpQP problems, obtained from model predictive control, can be found in [13, Chapter 3]. We consider this to be an important step towards efficient implementation of constrained optimal feedback control.

V. References

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[1] A numerical tolerance is needed to test the row-rank condition.

[2] This can e.g. be tested by computing the Chebyshev radius of CR, which can be formulated as an LP.