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Multiparametric Nonlinear Integer Programming and Explicit Quantized Optimal Control

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Abstract

This paper deals with multiparametric nonlinear integer programming problems where the optimization variables belong to a finite set and where the cost function and the constraints depend in an arbitrary nonlinear fashion on the optimization variables and in a linear fashion on the parameters. We examine the main theoretical properties of the optimizer and of the optimum as a function of the parameters, and propose a solution algorithm. The methodology is employed to investigate properties of quantized optimal control laws and optimal performance, and to obtain their explicit representation as a function of the state vector. [1]

1 Introduction

In several control synthesis problems the number of possible control actions is finite, a situation usually referred to as *quantization* of the input signals. While in most applications the quantization introduced by analog-to-digital converters, finite precision arithmetic units, and digital to analog converters can be safely neglected by treating the control variables as continuous, in some problems this assumption may lead to an unacceptable deterioration of the closed-loop performance. Examples of control problems that must handle quantization range from more traditional mechanical problems (e.g., problems involving stepping motors) and hydraulic problems (e.g., with on/off valves), to new problems in communications, such as the one dealt with in [1], where quantized control is used to coordinate adaptation of multimedia applications and hardware resource, in order to provide user-preferable QoS requirements under resource contention and energy constraints.

It is therefore worthwhile to devise methods that take into account phenomena of quantization, either for the analysis of the effect of quantization of the input signal, or for the synthesis of quantized control laws. Both research topics are currently receiving a growing attention especially in the field of hybrid systems because of the interactions between a continuous dynamical system and a discrete quantized controller (see [2] and references therein).

Among other approaches, receding horizon optimal control ideas were proposed for synthesizing quantized control laws for linear systems with quantized inputs and quadratic optimality criteria. In [2], the authors ensure *practical stability* properties[2], by forcing the terminal state to belong to a special invariant set [4], they deal with state constraints, and propose on-line mixed-integer optimization for the implementation of the control law. In the absence of state-constraints, in [5] the authors show that the control law can be equivalently rewritten as a piecewise affine mapping.

Ideas for solving optimal control problems as an explicit function of the state vector were proposed earlier for linear systems [6], nonlinear systems [7], hybrid systems [8,9], and uncertain linear systems [10]. These approaches rely on *multiparametric programming* [11] to express the optimizer vector (=the optimal input) as a function of a certain number of parameters (=the current states).

Optimal control problems where all decision variables are quantized and where cost function and constraints depend on a real-valued state vector can be handled by *multiparametric integer programming* solvers [12]. A multiparametric integer solver for linear objectives and linear constraints was developed in [13]. The algorithm finds the lexicographic minimum of the set of integer points which lie inside a convex polyhedron that depend linearly on one or more integral parameters, and is based on parameterized Gomory's cuts followed by a parameterized dual simplex method. An alternative method based on a contraction algorithm for multiparametric integer linear programming problems was proposed in [14]. Algorithms for solving a special class of multiparametric nonlinear integer programming problems were investigated in [15].

In this paper we propose a method for solving a quite general class of multiparametric nonlinear integer problems where: (1) the cost function and the constraints depend linearly on a vector of parameters, (2) they depend in an arbitrary nonlinear fashion on the optimization variables, and (3) these are restricted to belong to a finite set. Because of feature (2), the use of relaxation and branching, which is the approach of most multiparametric mixed-integer solvers, would be inappropriate here.

[1] This work was partially supported by the European Project "Computation and Control". We thank E. Triki for pointing out a few typos in the original manuscript.

[2] As underlined in [3], the classical concept of stability must be replaced in a quantized context by "practical" stability

The paper is organized as follows. After examining in Section 2 the main theoretical properties of the optimizer and optimum as a function of the parameters, we propose a solver in Section 3. Multiparametric integer programming is used in Section 4 in the context of quantized optimal control. Numerical results are finally reported in Section 5.

2 Multiparametric Nonlinear Integer Programming

We consider the following multiparametric optimization problem:

$$V^*(\theta) \triangleq \min_{x\in\mathcal{Q}} \; f_1(x) + f_2'(x)\theta \quad \text{s.t.} \; g_1(x) \le g_2'(x)\theta, \; \theta\in\Theta \tag{1}$$

where: $x \in \mathbb{R}^n$ is the optimization vector, which is constrained to belong to the finite set of values $\mathcal{Q} = \{q_1, \ldots, q_N\}$, $q_i \in \mathbb{R}^n$, $\forall i = 1, \ldots, N$; $\theta \in \mathbb{R}^m$ is a vector of parameters, lying in the polyhedron $\Theta = \{\theta \in \mathbb{R}^m : \; T\theta \le S\} \subseteq \mathbb{R}^m$; $f_1 : \mathbb{R}^n \mapsto \mathbb{R}$, $f_2 : \mathbb{R}^n \mapsto \mathbb{R}^m$, $g_1 : \mathbb{R}^n \mapsto \mathbb{R}^p$, $g_2 : \mathbb{R}^n \mapsto \mathbb{R}^{m\times p}$ are generic nonlinear functions of the optimization variables.

A typical instance of $\mathcal{Q}$ is given when each component $x^{\{j\}}$ of x is restricted to a finite set $\Phi_j = \{\phi_{j1}, \ldots, \phi_{jN_j}\}$, $j = 1, \ldots, n$, so that $\mathcal{Q}$ is the Cartesian product $\Phi_1 \times \ldots \times \Phi_n$, and its cardinality $N = \prod_{j=1}^n N_j$.

A solution to the multiparametric program (1) is defined as follows. The *feasible parameter set* Θ^* is the set of all $\theta \in \Theta$ for which there is a vector $x \in \mathcal{Q}$ such that $g_1(x) \le g_2'(x)\theta$. The *value function* $V^* : \Theta^* \mapsto \mathbb{R}$ is the function that associates to a parameter vector $\theta \in \Theta^*$ the corresponding optimum $V^*(\theta)$ of problem (1). The *optimizer set function* $X^* : \Theta^* \mapsto 2^{\mathcal{Q}}$ is the function that associates to a parameter vector $\theta \in \Theta^*$ the corresponding set of optimizers $X^*(\theta) = \{x \in \mathcal{Q} : \; f_1(x) + f_2'(x)\theta = V^*(\theta)\}$ of problem (1). The *optimizer function* $x^* : \Theta^* \mapsto \mathcal{Q}$ is the function that associates to a parameter vector $\theta \in \Theta^*$ the lexicographic[3] minimum $x^*(\theta)$ of $X^*(\theta)$.

The following Lemma 1 and Theorem 1 establish the main properties of the multiparametric solution to problem (1).

Lemma 1 ([16]) *Consider problem (1) without inequality constraints. Then $V^* : \Theta \mapsto \mathbb{R}$ is a concave piecewise affine function, and $x^* : \Theta \mapsto \mathbb{R}^n$ is a piecewise constant function.*

Theorem 1 *Let $\Theta^* \subseteq \Theta \subseteq \mathbb{R}^m$ be the feasible parameter set of (1), and let $V^* : \Theta \mapsto \mathbb{R}$, $x^* : \Theta \mapsto \mathcal{Q}$ the corresponding value function and optimizer function, respectively. Then Θ^* is the (possibly nonconvex[4]) union of at most N convex polyhedra, and V^*, x^* are a piecewise affine and a piecewise constant function, respectively, of the parameters over a partition of Θ^* in at most $2^N - 1$ (possibly nonconvex) polyhedra.*

[3] The lexicographic order is referred to the order of the elements of Q. For example, if $X^*(\theta) = \{q_i, q_j\} \subseteq Q$ and $i < j$, then $x^*(\theta) = q_i$.

[4] We use here the following definition of nonconvex polyhedral set: A set $\Omega \subseteq \mathbb{R}^m$ is a nonconvex polyhedral set if Ω is nonconvex and $\Omega = \bigcup_{i=1}^s \Omega_i$, where each set Ω_i is a convex polyhedron and $\Omega_i \cap \Omega_j$ is not full dimensional, $\forall i, j = 1, \ldots, s$, $i \ne j$.

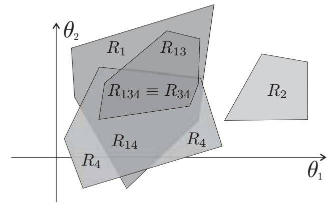


Figure 1: Example of a partition of Θ^* into (possibly nonconvex and disconnected) regions R_I, where R_I is the set of all $\theta \in \Theta$ such that $g_1(q_i) \le g_2'(q_i)\theta$ if and only if $i \in I$

Proof: For each $i \in \{1, \ldots, N\}$ the linear inequality constraints $g_2'(q_i)\theta \ge g_1(q_i)$ and $T\theta \le S$ define a (possibly empty) polyhedron P_i in $\mathbb{R}^m$. Then, $\Theta^* = \bigcup_{i=1}^N P_i$. Consider now the set C of all combinations of indices $I = \{i_1, \ldots, i_K\}$, $i_1 \ge 1$, $i_K \le N$, $K \le N$, $i_j < i_{j+1}$, $\forall j \in \{1, \ldots, K-1\}$, without permutations and repetitions (e.g.: for $N = 3$ the combinations $\{1,2\}$, $\{2,1\}$, $\{1,1,2\}$, $\{1,2,1\}$, $\{1,2,2\}$, $\{2,1,1\}$, $\{2,1,2\}$, $\{2,2,1\}$ are only taken once as $\{1,2\}$). The number of elements of C is $\sum_{k=1}^N \binom{N}{k} = 2^N - 1$. Then, for $K = 1, \ldots, N$ consider the (possibly nonconvex) polyhedral sets $R_{i_1\ldots i_K} = \{\theta \in \mathbb{R}^m : \theta \in P_j, \forall j \in \{i_1, \ldots, i_K\}, \theta \notin P_h, \forall h \notin \{i_1, \ldots, i_K\}\}$ (for instance, for $N = 2$ we have $R_1 = P_1 \setminus (P_1 \cap P_2)$, $R_2 = P_2 \setminus (P_1 \cap P_2)$, $R_{12} = P_1 \cap P_2$; another example is reported in Figure 1, where it can be noticed that R_1, R_4, R_{14} are nonconvex polyhedral sets, and that R_1, R_4 are also disconnected).

Define $\bar{C} \subseteq C$ as the subset of indices I for which R_I is nonempty (although R_I may not be full dimensional). As $\bigcup_{I\in\bar{C}} R_I = \Theta^*$, the sets R_I define a partition of Θ^* into a finite number of (possibly nonconvex) polyhedra.

On each set R_I, we have

$$V^*(\theta) = \min_{i\in I}\{f_1(q_i) + f_2'(q_i)\theta\}, \; \forall\theta \in R_I, \tag{2}$$

and by Lemma 1 we conclude that V^* is a concave piecewise affine function of θ over R_I. Hence, V^* is piecewise affine over Θ^*. For each given $\theta \in R_I$ the corresponding optimizer is defined as $x^*(\theta) = q_j$, where $j = \min\{i \in I : \; f_1(q_i) + f_2'(q_i)\theta = V^*(\theta)\}$, and where minimization is necessary to obtain the lexicographic minimum in case of multiple optima.[5] □

[5] If equality constraints of the form $h_1(x) + h_2'(x)\theta = 0$ are considered in problem (1), the set of feasible parameters Θ^* (or subsets of it) may not be full dimensional. In fact, as the optimizer function $x^*(\theta) \in \mathcal{Q}$ can only assume a finite number N of values, equality constraints $h_1(x^*(\theta)) + h_2'(x^*(\theta))\theta = 0$ would force θ to lie on a finite number of hyperplanes. This has important implications when formulating finite-time optimal control problems with equality constraints on the terminal state, as discussed later in Section 4.

3 Multiparametric Solver

Linear, quadratic, and mixed-integer linear multiparametric programming solvers rely upon the fact that the optimizer is a piecewise affine function of the parameters defined over *convex* polyhedra. On the other hand, Theorem 1 provides a characterization of the solution over a partition of *nonconvex* (in general) polyhedra. Although nonconvex polyhedra may be split into several convex components, this approach would largely increase the number of partitions. Moreover, mixed-integer solvers rely on the relaxation of integer constraints, an approach that cannot be followed in our context due to the arbitrary nonlinear dependence on the optimization variables.

A direct application of the ideas used to prove Theorem 1 would lead to fully enumerating all $2^N - 1$ possible combinations of indices $I \in C$, test for nonemptiness of R_I, and characterize the value function and the optimizer on R_I according to (2). We provide here a more efficient solution method.

Before proceeding further, for any set of indices $I = \{i_1, \dots, i_K\} \subseteq \{1, \dots, N\}$, where N is the cardinality of $\mathcal{Q}$, let $P_I \triangleq \bigcap_{i \in I} P_i$, where $P_i = \{\theta \in \Theta : \; g_1(q_i) \leq g_2'(q_i)\theta\}$. Note that $R_I \subseteq P_I$. Moreover, denote by $V_i : \mathbb{R}^m \in \mathbb{R}$ the linear function that maps θ to $V_i(\theta) = f_1(q_i) + f_2'(q_i)\theta$, $i = 1, \dots, N$.

The method we propose here is based on two simple considerations. Let $I = \{i_1 \dots i_K\} \subseteq \{1, \dots, N\}$ and j any index such that $j \in \{i_K + 1, \dots, N\}$. The first consideration relates to *feasibility*: if P_I is empty, then $P_{I \cup \{j\}}$ is certainly empty. The second relates to *optimality*: we can avoid considering a polyhedral region $P_{I \cup \{j\}}$ if $V_j(\theta) \geq V_i(\theta)$ for all $i \in I$ and for all $\theta \in P_{I \cup \{j\}}$, or if $P_{I \cup \{j\}} \subset P_{I \cup \{h\}}$ and $V_j(\theta) \geq V_h(\theta)$ for all $\theta \in P_{I \cup \{h\}}$.

Based on the above considerations, a recursive algorithm for determining the feasible parameter set Θ^*, its subpartition, the value function V^*, and the optimizer function x^*, is summarized by Algorithm 3.1.

The algorithm builds an *optimality tree* $\mathcal{T}$, as depicted in Figure 2, where each node is characterized by a sequence $I = I_0 \cup \{j\}$ and a polyhedron $W_{I_0,j} = \{\theta \in \Theta : \; g_1(q_i) \leq g_2(q_i)'\theta, \; \forall i \in I, V_j(\theta) \leq V_i(\theta), \; \forall i \in I_0\}$, where I_0 is the sequence characterizing the father node.

The root node corresponds to $I = \emptyset$, $W_\emptyset = \Theta$. The maximum depth of the tree is $N = |\mathcal{Q}|$. The maximum number of nodes is 2^N. Clearly, $\mathcal{T}$ is always unbalanced by construction: a feasible combination $\{i_1, i_2, i_3\}$ will be always child of $\{i_1, i_2\}$ rather than $\{i_2, i_3\}$; in particular $\{N\}$ will always be a leaf node.

As the number of nodes in $\mathcal{T}$ depends not only on f_1, f_2, g_1, g_2, and on the number N of elements of $\mathcal{Q}$, but also on the order of the elements of $\mathcal{Q}$, at Step 2. the elements q_j that are infeasible for all $\theta \in \Theta$ (i.e., P_j is an empty convex polyhedron) are eliminated, and the remaining ones pre-ordered by increasing values of $f_1(q_j)$. An alternative is to consider the value $f_1(q_j) + \min_\theta \{f_2'(q_j)\theta$ subject to $g_1(q_j) \leq g_2'(q_j)\theta\}$ as an ordering criterion, which can be easily computed via linear programming for each feasible element $q_j \in Q$.

1. $\mathcal{T} \leftarrow$ {`root_node`};
2. Remove the elements of $\mathcal{Q}$ that are infeasible for all $\theta \in \Theta$ and order the remaining elements by increasing cost f_1;
3. Execute `examine`($\mathcal{T}$,`root_node`,$\emptyset$);
4. End.

5. Function `examine`($\mathcal{T}$,`node`,I_0);
   - 5.1. If $I_0 \neq \emptyset$ then let $i \leftarrow$ largest element of I_0, otherwise let $i \leftarrow 0$;
   - 5.2. For $j \in \{i+1, \dots, N\}$:
     - 5.2.1. Let $W_{I_0,j} = \{\theta \in P_{I_0} : g_1(q_j) \leq g_2(q_j)'\theta, \; V_j(\theta) \leq V_i(\theta), \; \forall i \in I_0\}$;
     - 5.2.2. If $W_{I_0,j} \neq \emptyset$ and the set $\{h : i+1 \leq h < j, W_{I_0,j} \subseteq W_{I_0,h}$, and $V_j(\theta) \geq V_h(\theta), \; \forall \theta \in W_{I_0,j}\} = \emptyset$:
       - 5.2.2.1 Append child node `node`$_j$ to `node` in $\mathcal{T}$;
       - 5.2.2.2 Execute `examine`($\mathcal{T}$,`node`$_j$,$I_0 \cup \{j\}$);
   - 5.3. End.

Algorithm 3.1: Multiparametric integer programming solver.

At step 5.2.1. the set $W_{I_0,j}$ represents the set of all vectors θ for which q_j is feasible, q_i is feasible for all $i \in I_0$, and that have a cost smaller than the cost at the father node (and, by induction, than the cost at all parent nodes). At step 5.2.2., the algorithm determines if a child node must be generated. A node is not generated if $W_{I_0,j}$ is empty or if it is included in $W_{I_0,h}$ for some "brother" node labeled by $I_0 \cup \{h\}$ already considered so far, and if everywhere on $W_{I_0,j}$ the cost $V_j(\theta)$ is larger than $V_h(\theta)$.

After the execution of Algorithm 3.1 and the construction of the tree $\mathcal{T}$, the multiparametric solution can be simplified by removing branches from $\mathcal{T}$ according to a criterion similar to the one in Step 5.2.2.: for each node `node`$_j$ characterized by $I_0 \cup \{j\}$, we can check if there exists a "brother" node `node`$_h$, $j < h \leq N$, such that $W_{I_0,j} \subseteq W_{I_0,h}$ and $V_j(\theta) \geq V_h(\theta)$, $\forall \theta \in W_{I_0,j}$. If this happens, `node`$_j$ and its whole sub-tree can be safely removed, without affecting the multiparametric solution.

Remark 3.1 Complexity and suboptimality of the multiparametric solution can be traded off with minor modifications to Algorithm 3.1. In fact, given a suboptimality tolerance $\epsilon \geq 0$, we can modify the optimality requirement in Step 5.2.1. by imposing that $V_j(\theta) \leq V_i(\theta) - \epsilon$, so that a child node is added only if the cost improves at least by ϵ, and, similarly, in Step 5.2.2. by asking that $V_j(\theta) \geq V_h(\theta) - \epsilon$. □

3.1 Evaluation of the Solution

The tree structure $\mathcal{T}$ constructed by Algorithm 3.1 can be immediately used for storing the multiparametric solution in the form (3), and for evaluating the optimal value and the optimizer for a given $\theta \in \mathbb{R}^m$, as detailed

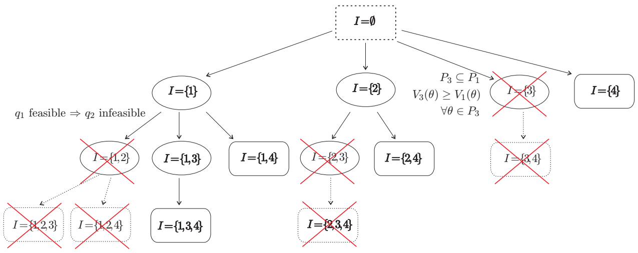


Figure 2: Optimality tree $\mathcal{T}$, related to the partition depicted in Figure 1

in the recursive Algorithm 3.2.

During the execution of Algorithm 3.2, children nodes must be visited in lexicographic order, namely if $j < h$, the node corresponding to the sequence $I = \{i_1, \ldots, i_k, j\}$ must be visited before the node corresponding to the sequence $I = \{i_1, \ldots, i_k, h\}$. This ordering comes naturally by the way Algorithm 3.1 constructs tree $\mathcal{T}$. At Step 2.2., one can avoid evaluating the whole inclusion $\theta \in P_I$. Indeed, only checking $\theta \in P_{i_M}$, where $i_M \triangleq \max(I)$, is enough, as the remaining conditions $\theta \in P_i$, by recursion, were already checked for all $i \in I \setminus \{i_M\}$. Moreover, only the inequalities of P_{i_M} which are not redundant on $P_{I\setminus\{i_M\}}$ need to be evaluated, which allows one to save memory space and computation time.

The solution can be expressed as a multi-level conditional expression (i.e., as a tree of nested conditionals), similarly to what is done in [13] for multiparametric integer linear programming[6]. In fact, the multiparametric solution can be written as:

$$
\begin{array}{l}
\text{if } \theta \in \Theta \text{ then} \\
\quad \text{if } H_1\theta \le K_1 \text{ then} \\
\quad\quad \vdots \\
\quad\quad \text{if } H_i\theta \le K_i \text{ then} \\
\quad\quad\quad x^*(\theta) = q_i \\
\quad\quad \vdots \\
\quad \text{elseif } H_k\theta \le K_k \text{ then} \\
\quad\quad \vdots \\
\quad \text{else} \\
\quad\quad \text{problem is infeasible} \\
\quad \text{end} \\
\text{else} \\
\quad \text{solution is undefined } (\theta \notin \Theta) \\
\text{end}
\end{array}
\tag{3}
$$

where $H_{()}$, $K_{()}$, are (possibly empty) matrices/vectors of suitable dimensions.

4 Explicit Quantized Optimal Control

Consider the following linear discrete time invariant system

$$x(t+1) = Ax(t) + Bu(t) \tag{4}$$

where $x \in \mathbb{R}^{n_x}$, $u \in \mathcal{U} \triangleq \{\bar{u}_1, \bar{u}_2, \ldots, \bar{u}_L\}$, $\bar{u}_i \in \mathbb{R}^{n_u}$ are the levels of quantization, and (A, B) is a stabilizable pair. Starting from the initial state $x(0)$, we wish to control the final state $x(T)$ to a target set Ω while satisfying the constraints

$$\bar{A}x(t) + \bar{B}u(t) \le \bar{C},\ t = 0, \ldots, T-1. \tag{5}$$

Constraints (5) are generic linear constraints on input and state variables. A typical instance are box constraints of the form $x_{\min} \le x_k \le x_{\max}$ (constraints of the form $u_{\min} \le u_k \le u_{\max}$ can be immediately taken into account by simply excluding from $\mathcal{U}$ those values $\bar{u}_i$ outside the bounds). We assume that the set Ω is a full-dimensional polyhedral set[7].

We want to show how the multiparametric integer solver developed earlier can be used to derive explicit optimal control laws. To this end, consider the following optimal control problem:

$$
\begin{array}{ll}
\min_U & \left\{ J(U,\theta) = x_T' P x_T + \sum_{k=0}^{T-1} (x_k' Q x_k + u_k' R u_k) \right\} \\
\text{s.t.} & \left\{ \begin{array}{l}
x_0 = \theta \\
x_{k+1} = Ax_k + Bu_k,\ k = 0, \ldots, T-1, \\
\bar{A}x_k + \bar{B}u_k \le \bar{C},\ k = 0, \ldots, T-1, \\
x_T \in \Omega \\
u_k \in \mathcal{U} \triangleq \{\bar{u}_1, \ldots, \bar{u}_L\},
\end{array} \right.
\end{array}
\tag{6}
$$

where $R = R' > 0$, $Q = Q' \ge 0$, $P \ge 0$ are matrices of suitable dimensions, θ represents a generic initial condition, $U \triangleq [u_0'\ u_1'\ \ldots\ u_{T-1}']' \in \mathbb{R}^{mT}$ is the set of free control moves, $U \in \mathcal{Q}$, where $\mathcal{Q} \triangleq \mathcal{U}^T = \mathcal{U} \times \ldots \times \mathcal{U}$, and $U^*(\theta) \triangleq [u_0^{*'}\ u_1^{*'}\ \ldots\ u_{T-1}^{*'}](\theta)'$ is the minimizer (or, in case of multiple optima, the lexicographic minimum of the set of optimizers).

It is immediate to cast problem (6) as an integer quadratic program (IQP). Indeed, by substituting $x_k = A^k x(t) + \sum_{j=0}^{k-1} A^j B u_{k-1-j}$, Eq. (6) can be rewritten

1. $[V^*(\theta), x^*(\theta)] \leftarrow$eval($\mathcal{T}$,`root_node`,$\theta$);

2. Function $[V^*, x^*] \leftarrow$eval($\mathcal{T}$,`node`,θ);
   - 2.1. Let $V^* \leftarrow +\infty$, $x^* \leftarrow \emptyset$;
   - 2.2. If $\theta \in P_I$:
     - 2.2.1. Let $I \leftarrow$ combination associated with `node`;
     - 2.2.2. Let $c \leftarrow$ number of children of `node`; Let $i \leftarrow 0$;
     - 2.2.3. While $i < c$ and $V^* = +\infty$:
       - 2.2.3.1 $i \leftarrow i + 1$;
       - 2.2.3.2 Let $\texttt{node}_i \leftarrow i$-th child of `node`;
       - 2.2.3.3 $[V^*, x^*] \leftarrow$eval($\mathcal{T}$,$\texttt{node}_i$,θ);
     - 2.2.4. If $V^* = +\infty$ and $I \ne \emptyset$:
       - 2.2.4.1 Let $i^* \leftarrow$ largest element of I;
       - 2.2.4.2 Let $x^* \leftarrow q_{i^*}$, $V^* \leftarrow f_1(q_{i^*}) + f_2'(q_{i^*})\theta$;
   - 2.3. Return $[V^*, x^*]$;
   - 2.4. End.

Algorithm 3.2: Evaluation of the optimal value $V^*(\theta)$ and of the lexicographic minimum $x^*(\theta)$

[6] In [13] the authors denominate a multi-level conditional expression a *quast*.

[7] In case of non full-dimensional sets Ω, the set Θ of initial states $x(0)$ for which (5) are feasible may be lower-dimensional, for instance if $\Omega = \{0\}$, corresponding to the constraint $x(T) = 0$, Θ would be a lattice, as remarked earlier.

as

$$\begin{array}{ll} \min\limits_{U} & \left\{\frac{1}{2}U'HU + U'F'\theta + \frac{1}{2}\theta' Y\theta\right\} \\ \text{subj. to} & GU \le W + E\theta \\ & U \in \mathcal{Q}, \end{array} \quad (7)$$

where the column vector $U \triangleq [u'_0, \ldots, u'_{T-1}]' \in \mathbb{R}^{mT}$ is the optimization vector, $H = H' > 0$, and H, F, Y, G, W, E are easily obtained from Q, R, and (6).

The optimization problem (6) is an IQP which depends on the initial state θ. The multiparametric nonlinear integer programming algorithm developed earlier can be conveniently used to compute the piecewise constant solution $U^*(x_0)$ to the optimal control problem (6). In fact, after taking apart the quadratic term $\frac{1}{2}\theta' Y\theta$ that does not affect the optimizer $U^*(\theta)$, problem (7) can be recast in the form (1) by setting $f_1(U) = \frac{1}{2}U'HU$, $f_2(U) = FU$, $g_1(U) = GU - W$, $g_2(U) = E'$.

From Theorem 1, it follows that the set $\Theta^* \subseteq \mathbb{R}^{n_x}$ of initial states x_0 for which a solution to (6) exists is the union of at most L^T convex polyhedra, that the value function $V^* : \mathbb{R}^{n_x} \in \mathbb{R}$ is a piecewise quadratic function of x_0 (more exactly, the sum of a convex quadratic and a piecewise affine function) over a partition of Θ^* in at most $2^{L^T} - 1$ (possibly nonconvex) polyhedra, and that the optimizer function $U^* : \mathbb{R}^{n_x} \in \mathcal{U}^T$ is a piecewise constant function of x_0 defined over the same partition of Θ^*. Moreover, in the absence of inequality constraints $\bar{A}x_k + \bar{B}u_k \le \bar{C}$, $k = 0, \ldots, T-1$, and $x_T \in \Omega$, V^* is the sum of a convex quadratic and a piecewise affine concave function of x_0.

The case of optimal control problems based on infinity-norms, leading to a multiparametric linear integer programming problem, has been dealt with in [16]. As also detailed in [16], problems involving logic constraints over Boolean variables can be also dealt with by transforming Boolean formulas into linear integer inequalities [17].

4.1 Explicit Quantized Receding Horizon Control

A useful way for transforming the $U^*(\theta)$ into a closed-loop control law is to adopt the so called *receding horizon* philosophy. The receding horizon controller is defined as

$$u(t) = u_0^*(x(t)), \quad (8)$$

where $u_0^*(x(t))$ is the first element of the minimizer $U^*(x(t))$ of the finite-time quantized optimal control problem, initialized at the current state $\theta = x(t)$.

An immediate corollary of Theorem 1 is that the control law (8) is a piecewise constant law defined over a polyhedral partition. Criteria for selecting the terminal set Ω in order to guarantee practical stability properties of the quantized control law (8) were analyzed in [2].

Remark 4.1 As only the first part $u_0^*(x(t))$ of the minimizer $U^*(x(t))$ is of interest, after the execution of Algorithm 3.1 the multiparametric solution can be simplified by removing subtrees of $\mathcal{T}$ where the first optimal move u_0^* is the same in all nodes (in depth search of such subtrees would just serve to determine $u_1^*, \ldots, u_{N-1}^*$). □

5 An Example

Example 5.1 Consider an extremely simplified version of the problem of landing a spacecraft on a planet, where we consider only the vertical motion described by the equations $m\frac{dv}{dt} = -\beta v + u$, $\frac{dh}{dt} = v$, where h is the height from ground, v the vertical velocity, and the overall force u acting on the spacecraft is given by

$$u = \begin{cases} -mg & \text{thruster off} \\ 0 & \text{thruster on (gravity compensation)} \\ mg & \text{double thruster on.} \end{cases} \quad (9)$$

By choosing the parameters $\beta = 1$, $m = 1$, $g = 1$ (units are omitted here, as the parameters have no particular meaning in this example), and by discretizing the dynamics with a sampling time $T_s = 1$, we obtain the discrete-time linear model $x(t+1) = \left[\begin{smallmatrix} 1 & 0.6321 \\ 0 & 0.3679 \end{smallmatrix}\right] x(t) + \left[\begin{smallmatrix} 0.7358 \\ 1.2642 \end{smallmatrix}\right] u(t)$, where $u(t) \in \mathcal{U} \triangleq \{-1, 0, 1\}$, and $x = \left[\begin{smallmatrix} h \\ v \end{smallmatrix}\right]$. We wish to design a controller that brings the height of the spacecraft and its velocity to zero while satisfying the constraints

$$h \ge 0, \quad v \ge -\bar{v}, \quad (10)$$

where $\bar{v} = 1.5$. To this end, we consider the finite-time optimal control problem $\min_{u_0,u_1} x_2' P x_2 + \sum_{k=0}^{1} (x_k' Q x_k + u_k' R u_k)$ s.t. $x_1 \ge \left[\begin{smallmatrix} 0 \\ -v \end{smallmatrix}\right]$, $u_0, u_1 \in \{-1, 0, 1\}$, where $R = 10$, $Q = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right]$, and $P \approx \left[\begin{smallmatrix} 3.1240 & 1.5677 \\ 1.5677 & 2.3241 \end{smallmatrix}\right]$ solves the Riccati equation associated with A, B, Q, R.

The associated mp-IQP problem has the form (7) with $\theta = x_0$ and

$$H = \begin{bmatrix} 0.7675 & 0.2924 \\ 0.2924 & 0.6323 \end{bmatrix}, \; F = \begin{bmatrix} 0.2160 & 0.2132 \\ 0.1477 & 0.1468 \end{bmatrix}$$

$$G = \begin{bmatrix} -0.7358 & 0 \\ -1.2642 & 0 \\ 1 & 0 \\ -1 & 0 \\ 0 & 1 \\ 0 & -1 \end{bmatrix}, \; W = \begin{bmatrix} 0 \\ 1.5 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \; E = \begin{bmatrix} 1 & 0.6321 \\ 0 & 0.3679 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix},$$

where we have neglected the constant term $\frac{1}{2}\theta' Y\theta$. By running Algorithm 3.1 on $\Theta = \{\theta : \|\theta\|_\infty \le 15\}$, the multiparametric solution is computed in 0.85 s on a Pentium III 800 Mhz running Matlab 5.3, and the associated tree $\mathcal{T}$ consists of 24 nodes and has a depth of 5 levels, as depicted in Figure 3. The number of inequalities associated with each node varies between one and four[8].

We compare now the solution $U^*(\theta)$ of the integer quadratic problem with the quantization $\hat{U}(\theta)$ to the nearest (in Euclidean norm) feasible point in $\mathcal{Q}$ of

[8] An evaluation of the value function V^* takes an average of 1.36 ms (this value is obtained by averaging over a grid of 4225 samples of Θ), against about 6.01 ms needed to compute V^* by enumeration. Even from this simple problem where the number of elements of $\mathcal{Q}$ is only $N = 9$, it is clear the advantage of having an explicit representation of V^*.

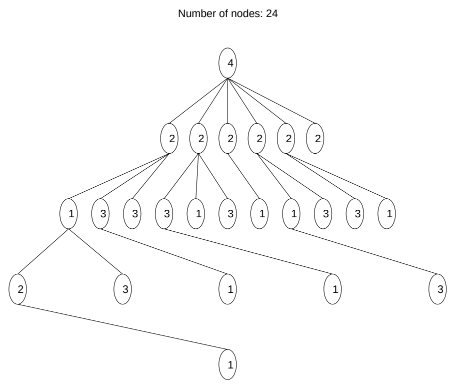
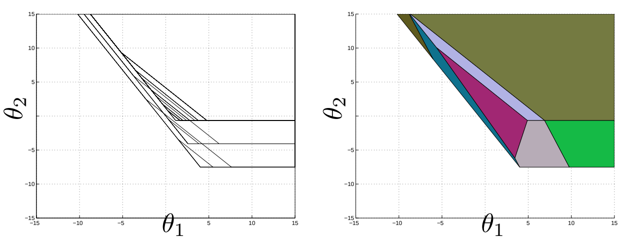


Figure 3: Optimality tree associated with the optimal control law. For each node is reported the number of linear inequalities that must be checked at that node during the on-line evaluation of the solution for a given x_0



(a) Partition associated with the solution $U^*(\theta)$ of the mp-IQP

(b) Partition associated with the solution $U^*_{\mathrm{QP}}(\theta)$ of the mp-QP

Figure 4: Comparison between the solutions of the continuous and of the integer multiparametric quadratic program

the solution $U^*_{\mathrm{QP}}(\theta)$ of the continuous quadratic program $\min_U \in \mathbb{R}^{mT}\{\frac{1}{2}U'HU + \theta'F'U$ subject to $GU \leq W + E\theta\}$. The partition associated with $U^*_{\mathrm{QP}}(\theta)$, obtained in 0.22 s using the algorithm reported in [18], is depicted in Figure 4(b), while the partition associated with $U^*(\theta)$ is depicted in Figure 4(a). In Figure 5, we report the difference $\hat{V}(\theta) - V^*(\theta)$, where $\hat{V}(\theta) = \frac{1}{2}\hat{U}'(\theta)H\hat{U}(\theta) + \theta'F'\hat{U}(\theta)$, and $V^*(\theta)$ is the optimal value function for the integer quadratic program; clearly $V^*(\theta) \leq \hat{V}(\theta)$, for all $\theta \in \Theta^*$. $\square$

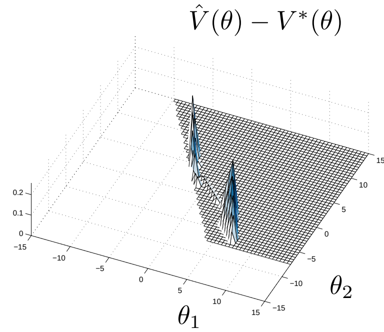


Figure 5: Difference $\hat{V}(\theta) - V^*(\theta)$, where $\hat{V}(\theta) = \frac{1}{2}\hat{U}'(\theta)H\hat{U}(\theta) + \theta'F'\hat{U}(\theta)$, and $\hat{U}(\theta)$ is obtained by quantizing the solution of the continuous quadratic program to the nearest feasible point in $\mathcal{Q}$

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