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Suboptimal Explicit MPC via Approximate Multiparametric Quadratic Programming

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Abstract

Algorithms for solving multiparametric quadratic programming (mp-QP) were recently proposed in [4, 12] for computing explicit Model Predictive Control (MPC) laws. The reason for this interest is that the solution to mp-QP is a piecewise affine function of the state vector and thus it is easily implementable on-line. The main drawback of solving mp-QP *exactly* is that whenever the number of linear constraints involved in the optimization problem increases, the number of polyhedral cells in the piecewise affine partition of the parameter space may increase exponentially. In this paper we address the problem of finding *approximate* solutions to mp-QP, where the degree of approximation is arbitrary and allows to trade off between optimality and a smaller number of cells in the piecewise affine solution.

1 Introduction

Model Predictive Control (MPC) has become the accepted standard for complex constrained multivariable control problems in the process industries [10]. Here at each sampling time, starting at the current state, an open-loop optimal control problem is solved over a finite horizon. At the next time step the computation is repeated starting from the new state and over a shifted horizon, leading to a moving horizon policy. The solution relies on a linear dynamic model, respects all input and output constraints, and optimizes a quadratic performance index.

For MPC based on linear prediction models and a quadratic performance index, in [4] the authors proposed a new approach to move off-line all the computations necessary for the implementation of MPC while preserving all its other characteristics. The approach consists of solving off-line the optimization problem associated with MPC for all the expected measurement values by using *multiparametric quadratic programming* (mp-QP) solvers. The resulting feedback controller inherits all the stability and performance properties of linear MPC, and is piecewise linear. For this reason, the on-line computation associated with *explicit* MPC controllers reduces to a function evaluation of a piecewise linear mapping.

The main drawback of explicit MPC techniques is that whenever the number of constraints involved in the optimization problem increases, the number of linear gains associated with the piecewise linear control algorithm may increase exponentially, with consequent heavy memory and CPU loads. The technique proposed in [7] attempts to reduce the complexity by reducing *a priori* the allowed combinations of active constraints, based on engineering insight on the control problem. In [5], the authors propose a method to efficiently store and compute explicit MPC solutions by exploiting properties of the value function.

In this paper we compute *suboptimal* solutions to the multiparametric quadratic problem, by relaxing the first order Karush-Kuhn-Tucker (KKT) optimality conditions (except primal feasibility, so that the computed move is feasible) by some arbitrary degree ϵ, which serves as a design knob for tuning the complexity of the controller. We show that for $\epsilon \to \infty$ the complexity of the controller is reduced to a linear control law (highly suboptimal), while for $\epsilon \to 0$ the controller converges to the explicit MPC controller [4] (fully optimal with respect to the chosen performance index). We analyze a general relaxation scheme where all KKT conditions (except primal feasibility) may be relaxed, and a particular one where only dual feasibility is relaxed. For the general perturbation scheme, we show how to compute *a posteriori* the maximum error between the optimizer and the suboptimizer. For the particular perturbation scheme, we also provide a criterion for choosing ϵ so that the distance between the exact and the approximate solution is bounded *a priori*.

2 Model Predictive Control

Consider the discrete-time linear time invariant system

$$\begin{cases} \chi(t+1) &= \mathcal{A}\chi(t) + \mathcal{B}u(t) \\ y(t) &= \mathcal{C}\chi(t) \end{cases} \tag{1}$$

where $\chi(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, and $y(t) \in \mathbb{R}^p$ are the state, input, and output vector respectively, and the pair $(\mathcal{A}, \mathcal{B})$ is stabilizable. The linear MPC problem associated with system (1) and based on a strictly convex quadratic performance index, amounts to solve at each t the quadratic program (QP)

$$\begin{aligned} \min_U \quad & \tfrac{1}{2}U'QU + \chi'(t)C'U + \tfrac{1}{2}\chi'(t)Y\chi(t) \\ \text{subj. to} \quad & AU \leq b + F\chi(t) \end{aligned} \tag{2}$$

and set $u(t) = u_t^* = I^1 U(t)$ as input to system (1), where $I^1 \triangleq [I_m\ 0\ \dots\ 0]$. See [4] for details. In (2) $Q = Q' \succ 0$, $Q \in \mathbb{R}^{s\times s}$, $C \in \mathbb{R}^{s\times n}$, $Y \in \mathbb{R}^{n\times n}$, $A \in \mathbb{R}^{q\times s}$, $b \in \mathbb{R}^q$, $F \in \mathbb{R}^{q\times n}$. As only the optimizer $U(t)$ is needed, the term involving Y is usually removed

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from (2). As (2) depends on the current state $\chi(t)$, it is referred to as a *multiparametric quadratic* program (mp-QP).

3 Multi-Parametric Quadratic Programming

Before proceeding further, in order to follow the standard notation of the mathematical programming literature, we denote by $\theta = \chi(t)$ the vector of parameters, $\theta \in \mathbb{R}^n$, and by $x = U \in \mathbb{R}^r$ the vector of optimization variables. The mp-QP problem (2) is rewritten as

$$(QP_\theta): \quad \min \left\{ \theta' C' x + \frac{1}{2} x' Q x : Ax \leq b + F\theta \right\} \quad (3)$$

Usually, polyhedra are defined as closed subsets of $\mathbb{R}^n$. In the study of problem (QP_θ), it is convenient to adopt the following relaxed definition.

Definition 1 *A* polyhedral set *is the intersection of a finite number of closed and/or open affine subspaces. Equivalently, $X \subseteq \mathbb{R}^n$ is a polyhedral set if there exist matrices F and G and vectors f and g such that*

$$X = \{x \in \mathbb{R}^n : Fx \leq f, Gx < g\}. \quad (4)$$

We assume that $\theta \in \Theta$, where Θ is a given polyhedral set of $\mathbb{R}^n$ (i.e., a set of states, with respect to the MPC problem described in the previous section). As $Q \succ 0$, for every $\theta \in \Theta$, the corresponding quadratic program either has a unique optimal solution or is infeasible. Let $\Theta_f \subseteq \Theta$ denote the set of parameter vectors such that (QP_θ) has a feasible solution, and let $\phi^*(\theta) : \Theta_f \mapsto \mathbb{R}$ denote the *value function*, which associates with every $\theta \in \Theta_f$ the optimal value of (QP_θ).

Multiparametric quadratic programming (mp-QP) amounts to determining the optimal solution $x^*(\theta)$, and the value function $\phi^*(\theta)$ as explicit functions of θ, for all $\theta \in \Theta_f$.

For any $N \subseteq M \triangleq \{1, 2, \ldots, q\}$ we denote by A_N the submatrix of A consisting of rows indexed by N. Analogously, if $s \in \mathbb{R}^q$ then we denote by s_N the subvector of s consisting of entries indexed by N.

Definition 2 *Let $x^*(\theta)$ be the optimal solution of (QP_θ). The* optimal partition *of M associated with θ is the partition $(B(\theta), N(\theta))$ where $N(\theta)$ is the index set of active constraints at $x^*(\theta)$.*

Definition 3 *Let $(B, N) = (B(\theta_0), N(\theta_0))$ for some $\theta_0 \in \Theta_f$. We call* critical region *associated with (B, N) the set of parameters $CR_0 = \{\theta \in \Theta_f : N(\theta) = N\}$.*

The first-order KKT conditions for problem (QP_θ) are given by [1]:

$$Qx + C\theta + A'\lambda = 0, \quad (5a)$$
$$\lambda'(Ax - b - F\theta) = 0, \quad (5b)$$
$$\lambda \geq 0, \quad (5c)$$
$$Ax \leq b + F\theta, \quad (5d)$$

where $\lambda \in \mathbb{R}^m$ is the vector of Lagrange multipliers. Once an optimal partition (B, N) is fixed, the above conditions may be written as follows:

$$Qx + C\theta + A'\lambda = 0, \quad (6a)$$
$$A_B x + s_B = b_B + F_B\theta, \ \ s_B \geq 0, \quad (6b)$$
$$A_N x + s_N = b_N + F_N\theta, \ \ s_N = 0, \quad (6c)$$
$$\lambda_B = 0, \quad (6d)$$
$$\lambda_N \geq 0. \quad (6e)$$

where s_B, s_N are a partition of the vector of primal slack variables $s \in \mathbb{R}^q$. We solve (6a) for x,

$$x = -Q^{-1}(A_N'\lambda_N + C\theta) \quad (7)$$

and substitute the result into (6c), getting $-A_N Q^{-1}(A_N'\lambda_N + C\theta) - b_N - F_N\theta = 0$. Assuming that A_N is full row rank, $(A_N Q^{-1} A_N')^{-1}$ exists and therefore we obtain

$$\lambda_N = -(A_N Q^{-1} A_N')^{-1}(b_N + (F_N + A_N Q^{-1} C)\theta). \quad (8)$$

Thus λ is an affine function of θ. We can substitute λ_N from (8) into (7) to obtain

$$x = Q^{-1} A_N' (A_N Q^{-1} A_N')^{-1}(b_N + (F_N + A_N Q^{-1} C)\theta) - Q^{-1} C\theta \quad (9)$$

and note that x is also an affine function of θ. Relations (8) and (9) lead to the following result [4].

Theorem 1 *Let Q be positive definite. Let (B, N) be an optimal partition, and let CR_0 be the corresponding critical region. Assume that the rows of A_N are linearly independent. Then, the optimal x^* and the associated vector of Lagrange multipliers λ^* are the following, uniquely defined, affine functions of θ over CR_0: $x^*(\theta) = H_x\theta + k_x$, $\lambda_N^*(\theta) = H_\lambda\theta + k_\lambda$, $\lambda_B^*(\theta) = 0$, where $H_\lambda = -(A_N Q^{-1} A_N')^{-1}(F_N + A_N Q^{-1} C)$, $k_\lambda = -(A_N Q^{-1} A_N')^{-1} b_N$, $H_x = Q^{-1} A_N' H_\lambda - Q^{-1} C$, $k_x = Q^{-1} A_N' k_\lambda$.*

Theorem 1 characterizes the solution only locally in the neighborhood of a specific θ_0, as it does not provide the construction of the set CR_0 where this characterization remains valid. On the other hand, this region can be characterized immediately. By construction, conditions (6a), (6c) and (6d) are satisfied as identities by $x^*(\theta)$ and $\lambda^*(\theta)$. By substituting in (6b) and (6e) the expressions of $x^*(\theta)$ and $\lambda^*(\theta)$ we get

$$(A_B H_x - F_B)\theta \leq b_B - A_B k_x, \quad (10a)$$
$$-H_\lambda\theta \leq k_\lambda. \quad (10b)$$

After removing the redundant inequalities from (10), we obtain a compact representation of CR_0. Obviously, CR_0 is a polyhedron in the θ-space, and represents the largest set of $\theta \in \Theta_f$ such that the combination of active constraints at the minimizer remains unchanged.

3.1 Degeneracy

So far, we have assumed that the rows of A_N are linearly independent. It can happen, however, that by solving (QP_θ) one determines a set of active constraints for which this assumption is violated. For instance, this happens when more than n constraints are active at the optimizer $x^* \in \mathbb{R}^r$, i.e., in a case of *primal degeneracy.* In this case the vector of Lagrange multipliers λ^* might not be uniquely defined, as the dual problem of (QP_θ) is not strictly convex (instead, *dual degeneracy* cannot occur because we assumed $Q \succ 0$, which implies that the minimizer is always unique). The problem of degeneracy is addressed in [4], where the authors suggest a simpler way to handle such a degenerate situation, which consists of collecting $r = rank(A_N)$ constraints arbitrarily chosen, and proceed with the new reduced set.

3.2 Continuity and Convexity Properties

The result stated below makes use of the following definition.

Definition 4 *A function $z : X \mapsto \mathbb{R}^s$, where $X \subseteq \mathbb{R}^n$ is a polyhedral set, is* piecewise affine *(resp.* piecewise quadratic*) if the following hold: (1) it is possible to partition X into finitely many convex polyhedral regions CR_i, $i = 1, \dots, p$; (2) inside CR_i, z is an affine (resp. quadratic) function, for all $i = 1, \dots, p$.*

Continuity of the value function $\phi^*(\theta)$ and the solution $x^*(\theta)$, can be shown as simple corollaries of the linearity result of Theorem 1. This fact, together with the convexity of the set of feasible parameters $\Theta_f \subseteq \Theta$, and of the value function $\phi^*(\theta)$, is proved in the next theorem [4].

Theorem 2 *Consider the multiparametric quadratic program (QP_θ) and let Q be positive definite, Θ convex. Then the set of feasible parameters $\Theta_f \subseteq \Theta$ is convex, the optimizer $x^*(\theta) : \Theta_f \mapsto \mathbb{R}^r$ is continuous and piecewise affine, and the value function $\phi^*(\theta) : \Theta_f \mapsto \mathbb{R}$ is continuous, convex and piecewise quadratic.*

4 Approximate mp-QP

Let the parameter vector $\theta_0 \in \Theta_f$ be arbitrarily chosen[1], and let (B, N) be the corresponding optimal partition. In order to obtain a suboptimal solution to (QP_θ), we relax the KKT conditions (6) as

$$\begin{aligned} -\epsilon_1 \le Qx + C\theta + A'\lambda \le \epsilon_1, && && \text{(11a)}\\ A_B x + s_B = b_B + F_B\theta, && s_B \ge 0, && \text{(11b)}\\ A_N x + s_N = b_N + F_N\theta, && 0 \le s_N \le \epsilon_2, && \text{(11c)}\\ -\epsilon_4 \le \lambda_B \le \epsilon_4, && \lambda_N \ge -\epsilon_3. && \text{(11d)} \end{aligned}$$

where $\epsilon_1 \in \mathbb{R}^r$, $\epsilon_2, \epsilon_3 \in \mathbb{R}^{|N|}$, $\epsilon_4 \in \mathbb{R}^{|B|}$ are the relaxation vectors that determine the degree of approximation, $\epsilon_k \ge 0$ (componentwise) for $k = 1, \dots, 4$. Let $\epsilon = (\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4)$.

The relaxed KKT conditions (11) define a polyhedron in the (x, θ, λ, s)-space. The *approximate critical region* CR_ϵ, is defined as the projection onto the θ-space of such a polyhedron.

Assume for the moment that CR_ϵ has been computed (this issue will be discussed in Section 4.1). Then, the rest of the space $R^{\text{rest}} = \Theta \backslash CR_\epsilon$ has to be explored and new critical regions generated. An effective approach for partitioning the rest of the space by polyhedral sets, in the sense of Definition 1, was proposed in [4], and is based on the following theorem:

Theorem 3 *Let $Y \subseteq \mathbb{R}^p$ be a polyhedron, and let $CR_\epsilon = \{\theta \in Y : W\theta \le r\}$ be a nonempty polyhedral subset of Y, where $W \in \mathbb{R}^{q \times p}$. Also let*

$$R_i = \left\{ \theta \in Y : \begin{array}{l} W_{\{i\}}\theta > r_{\{i\}} \\ W_{\{j\}}\theta \le r_{\{j\}}, \forall j < i \end{array} \right\} i = 1, \dots, q$$

where $W_{\{i\}}$ denotes the ith row of W and $r_{\{i\}}$ denotes the ith entry of r. Finally, let $R^{\text{rest}} = \cup_{i=1}^q R_i$. Then (i) $R^{\text{rest}} \cup CR_\epsilon = Y$; (ii) $CR_\epsilon \cap R_i = \emptyset$ for all i and $R_i \cap R_j = \emptyset$ for all $i \ne j$, i.e., $\{CR_\epsilon, R_1, \dots, R_m\}$ is a partition of Y.

After partitioning the rest of the space, we may proceed recursively: we choose for each region R_i a new vector θ_0, compute the approximate critical region CR_ϵ, compute the rest of the space $R_i \setminus CR_\epsilon$, and so on.

4.1 Orthogonal Projections

Before proceeding further, it is useful to rewrite the approximate KKT conditions (11a) in the form

$$Qx + C\theta + A'_N\lambda_N + A'_B\lambda_B + \nu = 0, \quad -\epsilon_1 \le \nu \le \epsilon_1. \tag{12}$$

where $\nu \in \mathbb{R}^r$ represents the violation of the first KKT condition (5a). From (12) we obtain $x = -Q^{-1}(A'_N\lambda_N + A'_B\lambda_B + C\theta + \nu)$ and thus, assuming that $A_N Q^{-1} A'_N$ is invertible,

$$\lambda_N = E_\nu \nu + E_s s_N + E_\lambda \lambda_B + H_\lambda \theta + k_\lambda, \tag{13}$$

where $E_s \triangleq (A_N Q^{-1} A'_N)^{-1}$, $E_\nu \triangleq E_s A_N Q^{-1}$ and $E_\lambda \triangleq -E_\nu A'_B$.

The approximated critical region CR_ϵ is now the projection onto the θ-space of the polyhedron in the $(\nu, s_N, \lambda_B, \theta)$-space described by the inequalities

$$\begin{aligned} & -\epsilon_1 \le \nu \le \epsilon_1, && \text{(14a)}\\ & -A_B Q^{-1}(A'_N(E_\nu\nu + E_s s_N + E_\lambda\lambda_B + H_\lambda\theta + k_\lambda) \\ & \qquad + A'_B\lambda_B + C\theta + \nu) \le b_B + F_B\theta, && \text{(14b)}\\ & 0 \le s_N \le \epsilon_2, && \text{(14c)}\\ & -\epsilon_4 \le \lambda_B \le \epsilon_4,\ E_\nu\nu + E_s s_N + E_\lambda\lambda_B + \\ & \qquad + H_\lambda\theta + k_\lambda \ge -\epsilon_3. && \text{(14d)} \end{aligned}$$

Rather than projecting with respect to the whole set of variables ν, s_N, λ_B, we can restrict the amount of relaxations, and accordingly distinguish among the following three cases:

A. Case $\epsilon_2 = 0$. This special case implies $s_N = 0$, and therefore amounts to fix the index set N of

[1] A vector $\theta_0 \in \Theta_f$ can be computed by finding a pair (x, θ) satisfying $Ax - F\theta \le b$, $\theta \in \Theta$, e.g., via linear programming.,

constraints which are active at the optimizer of (QP_θ). The projection is performed only with respect to ν, λ_B.

B. Case $\epsilon_2 = 0$, $\epsilon_4 = 0$. This special case implies $\lambda_B = 0$, $s_N = 0$, and corresponds to avoid the relaxation of the second KKT condition (5b). Equivalently, it implies that the given optimal partition (B, N) is maintained. The simplification of the projection procedure is obvious: we only need to project with respect to ν.

C. Case $\epsilon_1 = 0$, $\epsilon_2 = 0$, $\epsilon_4 = 0$. In this final special case, we only relax the nonnegativity condition on the Lagrange multipliers corresponding to nonactive constraints of (QP_θ). Hence we need no projection, as similarly to (10) for the exact case, the approximated critical region reduces to

$$(A_B H_x - F_B)\theta \le b_B - A_B k_x, \tag{15a}$$
$$- H_\lambda \theta \le \epsilon_3 + k_\lambda. \tag{15b}$$

4.2 Asymptotic Properties

Since the feasibility of the optimizer is never relaxed, the approximate critical region is always contained in Θ_f. It is however of interest its asymptotic behavior. For convenience, in the following result we use the extended notation $CR(\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4)$ to denote the approximated critical region CR_ϵ, and denote by CR_0 the exact critical region, represented by (10).

Lemma 1 (see [2]) *The following statements hold: (i)* $\lim_{\delta\to 0} CR(\delta, \delta, \delta, \delta) = CR_0$*; (ii) if* $\epsilon_k \le \epsilon'_k$*, for all* $k = 1, \dots, 4$*, then* $CR(\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4) \subseteq CR(\epsilon'_1, \epsilon'_2, \epsilon'_3, \epsilon'_4)$*; (iii)* $\lim_{\delta\to+\infty} CR(\delta, \delta, \epsilon_3, \epsilon_4) = \Theta_f$ *for all* $\epsilon_3, \epsilon_4 \ge 0$*; (iv)* $\lim_{\delta\to+\infty} CR(\delta, 0, \epsilon_3, \epsilon_4) = \Theta_N$ *for all* $\epsilon_3, \epsilon_4 \ge 0$*, where* Θ_N *is the projection onto the* θ*-space of* $\{(x, \theta) : A_B x - F_B\theta \le b_B, A_N x - F_N\theta = b_N\}$.

Note that point (iv) above applies to the special cases A and B. In the special case C, if $\epsilon_3 \to +\infty$ then the limit approximate critical region is simply obtained from conditions (15) by removing (15b). The obtained set may be strictly contained in Θ_N.

4.3 Approximate Optimizer

So far, we have described a suboptimal method for partitioning the parameter set Θ, but contrarily to the exact case described in Theorem 1, we have not specified yet an approximate optimizer, which will be denoted by $\widehat{x}(\theta)$. Similarly to the exact case, we wish to have $\widehat{x}(\theta)$ to be a piecewise affine function of θ (defined over the partition into approximate critical regions given by the recursive method mentioned above), and such that $\widehat{x}(\theta)$ is primal feasible for all $\theta \in CR_\epsilon$, for each approximate critical region CR_ϵ. Moreover, we wish that $\widehat{x}(\theta)$ is as close as possible to the exact solution $x^*(\theta)$.

For case C, it turns out that $\widehat{x}(\theta) = x^*(\theta)$ is a good choice, as it is primal feasible for all $\theta \in CR_\epsilon$, and optimal for $\theta \in CR_0 \subseteq CR_\epsilon$.

For cases A and B, primal feasibility should be instead explicitly enforced. To this end, the following lemma can be easily proved by convexity.

Lemma 2 *Let* $V = \{V_1, \dots, V_\ell\}$ *be a set of vectors of* $\mathbb{R}^n$ *such that* $CR_\epsilon \subseteq \operatorname{conv}(V)$*. Let* $\widehat{x}(\theta)$ *be an affine function of* θ*. Then,* $A\widehat{x}(V_i) \le b + FV_i$ *for all* $V_i \in V$ *implies* $A\widehat{x}(\theta) \le b + F\theta$ *for all* $\theta \in CR_\epsilon$.

A natural choice for V is the set of vertices of CR_ϵ. Although good packages exist for determining the set of vertices of CR_ϵ (see [6]), for high dimensional θ-spaces this might be computationally too expensive. In alternative, the set V can be obtained by determining a union of hyper-rectangles which outer approximates CR_ϵ [3]. After a set V fulfilling Lemma 2 is chosen, we compute the affine suboptimal solution $\widehat{x}(\theta) = \widehat{H}\theta + \widehat{k}$, where $\widehat{H}$ and $\widehat{k}$ are obtained by solving the following constrained quadratic least squares problem

$$\min_{H\in\mathbb{R}^{n\times m},\ k\in\mathbb{R}^n} \sum_{i=1}^{\ell} \left\| \mathcal{W}[x^*(V_i) - (H\theta^i + k)] \right\|^2 \tag{16a}$$
$$\text{subj. to } A(HV_i + k) \le b + FV_i, \quad i = 1, 2, \dots, \ell, \tag{16b}$$

which provides the best fit to the optimal solutions $x^*(V_i)$ under the constraint of primal feasibility over $\operatorname{conv}(V) \supseteq CR_\epsilon$, where $\mathcal{W}$ is a weighting matrix. When the approximate mp-QP is used to solve an MPC problem, a sensible choice for $\mathcal{W}$ is $\mathcal{W} = \left[\begin{smallmatrix} I_m & 0 \\ 0 & 0 \end{smallmatrix}\right]$, as only the first m components of the solution are used to build the suboptimal explicit MPC law. Moreover, for the approximate region which contains the origin, in (16) we impose $k = 0$, so that asymptotic convergence to the origin is allowed.

Remark 1 Apart from the special case of MPC problems with input constraints and soft output constraints, unless some other particular hypotheses on A, b, F are assumed, problem (16) may in general be infeasible, especially for large ϵ. In this case, a possibility is to iteratively reduce (e.g., halve) the entries of ϵ until a feasible solution to (16) is found. □

We finally remark that, contrarily to the exact case, the overall piecewise affine function may not be continuous.

4.4 Approximate Value Function

Because of the property of primal feasibility given by (16b) (Case A,B) or (15a) (Case C), the following proposition follows immediately.

Proposition 1 *Let* $\widehat{\phi}(\theta) = \theta' C' \widehat{x}(\theta) + \frac{1}{2}\widehat{x}(\theta)' Q \widehat{x}(\theta)$ *be the approximate value function, and* $\phi^*(\theta)$ *the (exact) value function of problem* (QP_θ)*. Then,* $\widehat{\phi}(\theta) \ge \phi(\theta)$ *for all* $\theta \in \Theta_f$*, i.e.,* $\widehat{\phi}(\theta)$ *is an upper-bound for* $\phi^*(\theta)$.

In Lemma 1 we will give a bound on the gap between $\widehat{\phi}(\theta)$ and $\phi^*(\theta)$, valid for Case C.

4.5 Suboptimality Figures

Once the suboptimal solution to the mp-QP problem has been determined, it is interesting to compute (*a posteriori*) the degree of suboptimality of the resulting approximate explicit MPC controller with respect to the original MPC problem. In other words, the difference between the first m components of $\widehat{x}(\theta)$ and $x^*(\theta)$. To this end, we define the *absolute error* $e_{\text{abs}} \triangleq$

$\max_{\theta\in\Theta_f} \|I^1(\widehat{x}(\theta) - x^*(\theta))\|_\infty$ and the *relative error* $e_{\rm rel} \triangleq \max_{\theta\in\Theta_f} \left(\|I^1(\widehat{x}(\theta) - x^*(\theta))\|_\infty / \|\theta\|_1\right)$. In [2], we prove constructively how to numerically compute such errors via Mixed Integer Linear Programming (MILP), and parametric-MILP.

4.6 A Priori Error Bounds

Analytic forms for expressing the error between the optimizer and a feasible vector can be found in [8], [9] for linear complementarity problems. Although in principle these results may be applied to our mp-QP context, they rely on the existence of constants whose determination is not constructively given. Therefore, in this paper we follow a different route and develop a direct approach to analyze the error between the optimal and the suboptimal solution.

Consider the special case $\epsilon_1 = 0$, $\epsilon_2 = 0$, and $\epsilon_4 = 0$. Our goal is to impose *a priori* a bound ρ on the absolute error $e_{\rm abs} \triangleq \max_{\theta\in\Theta_f} \|I^1(\widehat{x}(\theta) - x^*(\theta))\|_\infty \le \rho$. For convenience, we define $\Delta x(\theta) \triangleq \widehat{x}(\theta) - x^*(\theta)$ and denote by CR_{ϵ_3} the approximate critical region defined by (15).

Lemma 1 (see [2]) *Let $\epsilon_1 = 0$, $\epsilon_2 = 0$, and $\epsilon_4 = 0$. Then, for all $\theta \in CR_{\epsilon_3}$,*

$$\widehat{\phi}(\theta) - \phi^*(\theta) \le \frac{1}{2}\epsilon_3' A_N Q^{-1} A_N' \epsilon_3. \tag{17}$$

Lemma 2 *Let $\epsilon_1 = 0$, $\epsilon_2 = 0$, and $\epsilon_4 = 0$. Then, for all $\theta \in CR_{\epsilon_3}$,*

$$\Delta x'(\theta) Q \Delta x(\theta) \le \epsilon_3' A_N Q^{-1} A_N' \epsilon_3. \tag{18}$$

Proof: We have $\widehat{\phi}(\theta) - \phi^*(\theta) = \frac{1}{2}\widehat{x}'(\theta)Q\widehat{x}(\theta) + \theta' C'\widehat{x}(\theta) - \frac{1}{2}x'^*(\theta)Qx^*(\theta) + \theta'C'x^*(\theta)$, and so

$$\hat{\phi}(\theta) - \phi^*(\theta) = -\frac{1}{2}\Delta x'(\theta) Q \Delta x(\theta) + \Delta x'(\theta)(Q\widehat{x}(\theta) + C\theta) \tag{19}$$

Define the function $f(t) \triangleq \frac{1}{2}(\widehat{x}(\theta) - t\Delta x(\theta))'Q(\widehat{x}(\theta) - t\Delta x(\theta)) + \theta'C'(\widehat{x}(\theta) - t\Delta x(\theta))$. Note that $f(t)$ is the objective value of (QP_θ) associated with $\widehat{x}(\theta) - t\Delta x(\theta)$, which is a feasible solution for all $t \in [0, 1]$, as $\widehat{x}(\theta)$ and $x^*(\theta)$ are both feasible. Since $f(1) = \phi^*(\theta)$, function $f(t)$ must be decreasing on a left neighbor of $t = 1$. Hence, $f'(t) = \Delta x'(\theta)Q\Delta x(\theta)t - \Delta x'(\theta)(Q\widehat{x}(\theta) + C\theta) \le 0$ if $t = 1$, and so

$$\Delta x'(\theta)(Q\widehat{x}(\theta) + C\theta) \ge \Delta x'(\theta)Q\Delta x(\theta). \tag{20}$$

From (19) we then obtain

$$\widehat{\phi}(\theta) - \phi^*(\theta) \ge \frac{1}{2}\Delta x'(\theta) Q \Delta x(\theta) \tag{21}$$

which, in addition to (17), implies the thesis. □

Lemma 3 (see [2]) *Let $z \in \mathbb{R}^s$, and consider the following optimization problem*

$$V^* \triangleq \begin{array}{ll} \max_z & \|I^1 z\|_\infty \\ \text{s.t.} & z'Qz \le \alpha \end{array} \tag{22}$$

Then, $V^* = \max\left\{\sqrt{\alpha[Q^{-1}]_{ii}} : i = 1\dots m\right\}$, where $[\cdot]_{ij}$ denotes the (i, j)th entry of $[\cdot]$.

Theorem 4 (see [2]) *Let $\epsilon_1 = 0$, $\epsilon_2 = 0$, $\epsilon_4 = 0$, and assume that for each optimal partition (B, N) the corresponding approximated critical region CR_{ϵ_3} is generated by setting $\epsilon_3 = \epsilon(N)\underline{1}$, where $\underline{1} \triangleq [1\ 1\ \dots\ 1]'$, and*

$$\epsilon(N) = \frac{\rho}{\sqrt{\underline{1}' A_N Q^{-1} A_N' \underline{1}}} \cdot \min_{i=1\dots m} \frac{1}{\sqrt{[Q^{-1}]_{ii}}} \tag{23}$$

Then $e_{\rm abs} \triangleq \max_{\theta\in\Theta_f} \|I^1(\widehat{x}(\theta) - x^*(\theta))\|_\infty \le \rho$.

5 Feasibility and Stability Issues

The two main issues regarding MPC policies are the feasibility of the optimization problem (2) at each time step $t \ge 0$, and the stability of the resulting closed-loop system.

As stressed in the previous section, primal feasibility is maintained in the approximate mp-QP solution. When the MPC setup of Section 2 is augmented by additional constraints aimed at guaranteeing feasibility at each time step t [10], such constraints will be also fulfilled by the suboptimal MPC solution.

Concerning stability, the suboptimal controller proposed in this paper does not enjoy directly intrinsic nominal stability properties. Nevertheless, the closed-loop system, composed by a linear plant in feedback with the suboptimal explicit MPC controller, is a piecewise affine system, and therefore stability criteria based for example on piecewise (or common) quadratic Lyapunov functions [11] can be applied in the present context.

6 An Example

Consider the second order non-minimum phase system $y(t) = 2(s-1)/(s^2 + 2s + 5)u(t)$, sample the dynamics with $T = 0.1$ s, and obtain the state-space representation $x(t+1) = \left[\begin{smallmatrix} 0.7969 & -0.2247 \\ 0.1798 & 0.9767 \end{smallmatrix}\right] x(t) + \left[\begin{smallmatrix} 0.1271 \\ 0.0132 \end{smallmatrix}\right] u(t)$, $y(t) = \left[\begin{smallmatrix} 1.4142 & -0.7071 \end{smallmatrix}\right] x(t)$. To regulate the system to the origin while fulfilling the input constraint $-1 \le u(t) \le 1$, we design an MPC controller based on the optimization problem: $\min_{\{u_t,\dots,u_{t+5}\}} x'_{t+6|t} P x_{t+6|t} + \sum_{k=0}^{5}[x'_{t+k|t} x_{t+k|t} + 0.1u^2_{t+k}]$ subject to $-1 \le u_{t+k} \le 1$, $k = 0, \dots, 5$, $x_{t|t} = x(t)$, where P solves the Lyapunov equation $P = \mathcal{A}'P\mathcal{A} + \mathcal{Q}$, $\mathcal{Q} = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right]$. The mp-QP problem associated with the MPC law has the form (QP_θ), where Q, C, A, b, F are reported in [2]. The exact explicit MPC controller is depicted in Figure 6.

In order to reduce the number of regions, we apply the approximate mp-QP algorithm. By setting $\epsilon_1 = 0$, $\epsilon_2 = 0$, $\epsilon_4 = 0$, and choosing a constant ϵ_3, we get the solutions shown in Figure 2 (for simplicity, from now on we let all the components of ϵ_k to be equal, and denote by ϵ_k the single component). Each approximate mp-QP solution was computed in less than 15 s of cpu on a Pentium III 650 MHz running Matlab 5.3. Note that despite the relaxation of dual feasibility ($\epsilon_3 > 0$), the region containing the origin does not change with respect to the exact solution. This is justified by the fact that, being $N = \emptyset$, the constraints defining the critical region are all of the form $Ax^*(\theta) \le b + F\theta$, and therefore

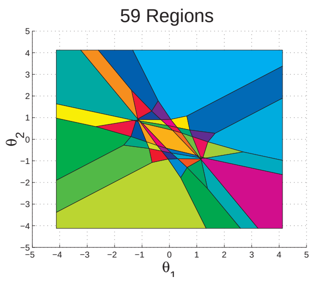
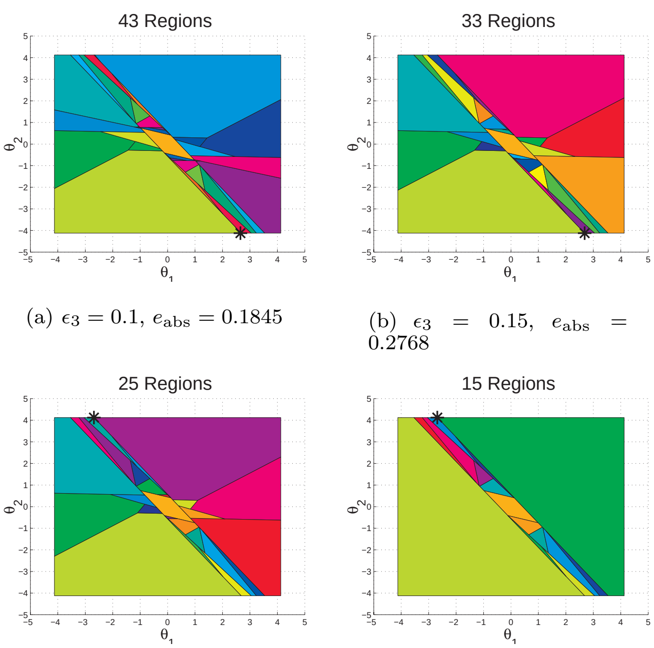


Figure 1: Explicit MPC controller: exact polyhedral partition of the state-space [4]



(a) $\epsilon_3 = 0.1$, $e_{\text{abs}} = 0.1845$

(b) $\epsilon_3 = 0.15$, $e_{\text{abs}} = 0.2768$

(c) $\epsilon_3 = 0.25$, $e_{\text{abs}} = 0.2796$

(d) $\epsilon_3 = 0.4$, $e_{\text{abs}} = 0.2796$

Figure 2: Approximate mp-QP solutions for different values of ϵ_3, and $\epsilon_1 = \epsilon_2 = \epsilon_4 = 0$

are not affected by the relaxation. For all the suboptimal MPC laws, the closed-loop system is quadratically stable, as it admits the common quadratic Lyapunov function $x'Lx$ [11], where L are reported in [2]. The maximum absolute errors e_{abs} are also reported in [2], and the state where such an error is achieved is marked by an asterisk in Figure 2.

By choosing ϵ_3 adaptively in accordance with Theorem 4, we obtain: $e_{\text{abs}} = 0.0514$ and 59 regions for $\rho = 0.1$, $e_{\text{abs}} = 0.1219$ and 43 regions for $\rho = 0.25$, $e_{\text{abs}} = 0.1951$ and 27 regions for $\rho = 0.5$, $e_{\text{abs}} = 0.2796$ and 15 regions for $\rho = 1$, $e_{\text{abs}} = 0.2796$ and 15 regions for $\rho = 5$. It is apparent that the a posteriori error bound e_{abs} is always smaller than the prespecified a priori error bound ρ. This is not surprising, as the choice for ϵ_3 suggested by Theorem 4 is based on the conservative over-estimate (18). Moreover, for $\theta \in CR_\epsilon$, the piecewise affine function $\widehat{x}(\theta) - x^*(\theta)$ does not span the whole ellipsoidal set described by the constraint in (22), so that further conservativeness is introduced. The fact that the intrinsic polyhedral structure of the partition may not allow to reach the a priori error bound ρ is further testified by the fact that as ρ increases, e_{abs} saturates at 0.2796. Results obtained by varying all ϵ_i are reported in [2].

7 Conclusions

In this paper we addressed the problem of reducing the number of polyhedral cells associated with explicit solutions to MPC problems, by relaxing the KKT conditions for optimality (except primal feasibility). The degree of approximation is arbitrary and allows to trade off between optimality and a comparatively small number of cells in the piecewise affine solution. We thank Domenico Mignone for providing us the LMI-based routine for computing common quadratic and piecewise-quadratic Lyapunov functions.

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