On Hybrid Systems and Closed-Loop MPC Systems

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Abstract

The following five classes of hybrid systems were recently proved to be equivalent: linear complementarity (LC) systems, extended linear complementarity (ELC) systems, mixed logical dynamical (MLD) systems, piecewise affine (PWA) systems, and max-min-plus-scaling (MMPS) systems. Some of the equivalences were obtained under additional assumptions, such as boundedness of system variables. In this paper, for linear or hybrid plants in closed-loop with a model predictive control (MPC) controller based on a linear model and fulfilling linear constraints on input and state variables, we provide a simple and direct proof that the closed-loop system (cl-MPC) is a subclass of any of the former five classes of hybrid systems. This result opens the use of tools developed for hybrid systems (such as stability, robust stability, and safety analysis tools) to study closed-loop properties of MPC.

1 Introduction

Hybrid dynamical models describe systems where both analog (continuous) and logical (discrete) components are relevant and interacting [1]. Recently, hybrid systems received a lot of attention from both the computer science and the control community, but general analysis and control design methods for hybrid systems are not yet available. For this reason, several authors have focused on special subclasses of hybrid systems for which safety analysis, stability analysis, and control design techniques are currently being developed.

Some examples of such subclasses are: linear complementarity (LC) systems [2], extended linear complementarity systems (ELC) [3, 4], mixed logical dynamical (MLD) systems [5, 6], piecewise affine (PWA) systems [7], and max-min-plus-scaling (MMPS) systems [8].

Each subclass has its own advantages over the others, for instance stability criteria were proposed for PWA systems [9], control and verification techniques for MLD hybrid models [5,6,10], and conditions for existence and uniqueness of solution trajectories (well-posedness) for LC systems [2]. In particular, MLD models were proven successful for recasting hybrid dynamical optimization problems into mixed-integer linear and quadratic programs, and a language was developed in [11] for automatically generating MLD models from a high level textual description of the hybrid dynamics.

In [12] we showed that several subclasses of discrete-time hybrid systems are equivalent. Some of the equivalences were obtained under additional assumptions related to well-posedness (i.e., existence and uniqueness of solution trajectories) and boundedness of (combinations of) input, state, output and auxiliary variables. These results are extremely important, as they allow to transfer all the above analysis and synthesis tools to any of the equivalent subclasses of hybrid systems.

In this paper we show that closed-loop Model Predictive Control (cl-MPC) systems can be written as LC and MLD systems. The latter result is of extreme practical importance, as it allows to analyze cl-MPC systems by using hybrid tools (e.g., for safety/reachability and (robust) stability analysis). Related results were obtained in [13], where the authors showed that MPC control is equal to a piecewise affine control law that can be computed off-line by using multiparametric quadratic programming solvers (and, therefore, that the closed-loop system is a PWA system). Based on this result, in [14] the authors used reachability analysis for stability and performance characterization of cl-MPC. Rather than exploiting the equivalence results of [12] to convert from PWA to LC and MLD, which would require additional assumptions on the boundedness of the Lagrange multipliers associated with the MPC optimization problem, we provide a simple, direct, and constructive proof to rewrite cl-MPC systems as LC and MLD systems.

Despite the fact that MPC schemes are typically designed so that they are intrinsically stable and fulfill operating constraints, stability is guaranteed through the introduction of stability constraints, which are often removed in practical MPC schemes as they typically deteriorate performance or complicate the optimization problem. Moreover, such guarantees only hold *when the nominal model of the plant and the prediction model coincide*. An important issue is to analyze the behavior of the feedback loop when the nominal model and the plant model differ, e.g., because of the presence of nonlinearities. Robust MPC techniques [15] partially solve this issue, by taking into account a class of linear uncertain models rather than one single prediction model, although this typically requires increased computation effort and, again, leads to deterioration of performance.

The results of this paper allow to transfer the stability analysis, robust stability analysis, well-posedness, and safety analysis tools developed for hybrid systems to any combination of a linear MPC controller and a linear plant, possibly against disturbances and model

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uncertainties. The results can be easily extended to arbitrary combinations of linear MPC controllers and *hybrid* plants, such as hybrid approximations of complex nonlinear dynamic models of the process to be controlled.

2 Classes of Hybrid Dynamical Models

In this paper we consider discrete-time models of the form

$$\begin{aligned} x(k+1) &= f(x(k), u(k), w(k)) && \text{(1a)} \\ y(k) &= g(x(k), u(k), w(k)) && \text{(1b)} \\ 0 &\leq h(x(k), u(k), w(k)) && \text{(1c)} \end{aligned}$$

where the variables $u(k) \in \mathbb{R}^m$, $x(k) \in \mathbb{R}^n$ and $y(k) \in \mathbb{R}^l$ denote the input, state and output, respectively, at time k, and $w(k) \in \mathbb{R}^r$ is a vector of auxiliary variables (this notation also holds for all the hybrid models introduced later), $f : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^r \mapsto \mathbb{R}^n$, $g : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^r \mapsto \mathbb{R}^p$, $h : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^r \mapsto \mathbb{R}^q$, and the last inequality should be interpreted componentwise. Specific choices of the form of the functions f, g, h will determine different classes of hybrid systems, as we will detail in the rest of this section.

Remark 1 The general formulation (1) allows to specify that some of the state, input, output, or auxiliary variables only assume discrete values, for instance $w_i(k) \in \{0, 1\}$ can be represented by the two inequalities $\max(w_i(k) - 1, -w_i(k)) \geq 0$, $-\max(w_i(k) - 1, -w_i(k)) \geq 0$. Equivalently, it can be also represented as $w_i(k)(1 - w_i(k)) \leq 0$, $w_i(k) \geq 0$, $1 - w_i(k) \geq 0$. □

Definition 1 *Let $\Omega \subseteq \mathbb{R}^n \times \mathbb{R}^m$ be a set of input+state pairs. A hybrid system of the form (1) is called* well-posed *on Ω, if (1) is uniquely solvable in $x(k+1)$ and $y(k)$ for all pairs $(x(k), u(k)) \in \Omega$.*

Definition 2 *Let $X(0) \subseteq \mathbb{R}^n$ be a set of initial conditions, and $U \subseteq \mathbb{R}^m$ a set of inputs. A hybrid system of the form (1) is called* persistently well-posed *on $(X(0), U)$ if for all $k \geq 0$ (1) is uniquely solvable in $x(k+1)$ and $y(k)$, for all pairs $(x(k), u(k))$ such that $x(0) \in X(0)$, $u(k) \in U$.*

Definitions 1, 2 imply that $x(k+1)$, $y(k)$ are unique functions of $(x(k), u(k))$, and therefore that the components of $w(k)$ which affect $x(k+1)$, $y(k)$ through f, and g, respectively, are implicitly defined by the vector inequality (1c).

Note that while Definition 1 concerns a spatial property of f, g, and h, Definition 2 also involves temporal properties, namely the dynamics of system (1). The property of persistent well-posedness is therefore more difficult to test than simpler well-posedness on a given set Ω of state+inputs. Nevertheless, it can be addressed by using formal verification methods and reachability analysis [10].

Remark 2 As will be also clarified later for PWA and MLD systems, for well-posedness of several instances of (1) over compact sets of $\mathbb{R}^n \times \mathbb{R}^m$, the inequalities in (1) should be split into strict inequalities $h_i(x(k), u(k), w(k)) > 0$, $i \in I$, and nonstrict inequalities $h_j(x(k), u(k), w(k)) \geq 0$, $j \in J$, $I \cap J = \emptyset$, $I \cup J = \{1, \dots, q\}$. Although this would be important from a system theoretical point of view, it is not of practical interest from a numerical point of view, as "$>$" cannot be represented in numerical algorithms working in finite precision. Indeed, $h > 0$ can be only represented as $h \geq \epsilon$, and ϵ is some pre-specified tolerance, e.g., the machine precision. □

2.1 Piecewise Affine (PWA) Systems

Piecewise affine (PWA) systems [7] are described by

$$\begin{aligned} x(k+1) &= A_i x(k) + B_i u(k) + f_i \\ y(k) &= C_i x(k) + D_i u(k) + g_i \end{aligned} \quad \text{for } \begin{bmatrix} x(k) \\ u(k) \end{bmatrix} \in \Omega_i, \tag{2}$$

where $\Omega_i \triangleq \{[\begin{smallmatrix} x \\ u \end{smallmatrix}] : H_i^x x + H_i^u u \leq K_i\}$, $i = 1, \dots, s$, are convex polyhedra in the input+state space. PWA systems have been studied by several authors (see [6, 7, 9, 16, 17] and the references therein) as they form the "simplest" extension of linear systems that can still model non-linear and non-smooth processes with arbitrary accuracy and are capable of handling hybrid phenomena.

System (2) belongs to the general class (1) by letting f, g be PWA functions defined over $\breve{\Omega} \triangleq \cup_{i=1}^s \Omega_i$, and $r = q = 0$ (i.e., $w(k)$ and $h(x(k), u(k), w(k))$ are not required). A necessary and sufficient condition for the PWA system (2) to be well-posed over $\breve{\Omega}$ is therefore that f, g are single-valued PWA functions. Therefore, typically the sets Ω_i have mutually disjoint interiors, and are often defined as the partition of a convex polyhedral set $\breve{\Omega}$. In case of discontinuities of f, g over overlapping boundaries of the regions Ω_i, to ensure well-posedness we should write some of the inequalities in the form $(H_i^x)^j x + (H_i^u)^j u < K_i^j$ (see Remark 2). In the following, for the sake of compactness of notation, we shall neglect this issue.

2.2 Mixed Logical Dynamical (MLD) Systems

In [5] the authors have introduced a class of hybrid systems in which logic, dynamics and constraints are integrated. This leads to a description of the form

$$\begin{aligned} x(k+1) &= Ax(k) + B_1 u(k) + B_2 \delta(k) + B_3 z(k) && \text{(3a)} \\ y(k) &= Cx(k) + D_1 u(k) + D_2 \delta(k) + D_3 z(k) && \text{(3b)} \\ E_1 x(k) + E_2 u(k) &+ E_3 \delta(k) + E_4 z(k) \leqslant g_5, && \text{(3c)} \end{aligned}$$

where $x(k) = [\ x_{\rm r}'(k)\ \ x_{\rm b}'(k)\ ]'$, $x_{\rm r}(k) \in \mathbb{R}^{n_{\rm r}}$ and $x_{\rm b}(k) \in \{0, 1\}^{n_{\rm b}}$ ($y(k)$ and $u(k)$ have a similar structure), and where $z(k) \in \mathbb{R}^{r_{\rm r}}$ and $\delta(k) \in \{0, 1\}^{r_{\rm b}}$ are auxiliary variables. The inequalities (3c) have to be interpreted componentwise. Systems that can be described by model (3) are called Mixed Logical Dynamical (MLD) systems. By letting $w(k) \triangleq [z'(k)\ \delta'(k)]'$, clearly (3) together with the integrality conditions over δ, $x_{\rm b}$, $y_{\rm b}$, and $u_{\rm b}$ (expressed as inequalities, see Remark 1), is a subclass of (1).

The MLD formalism allows specifying the evolution of continuous variables through linear dynamic equations, of discrete variables through propositional logic statements and automata, and the mutual interaction between the two. The key idea of the approach consists of embedding the logic part in the state equations by transforming Boolean variables into 0-1 integers, and by expressing the relations as mixed-integer linear inequalities (see [5] and references therein). MLD systems are therefore capable of modeling a broad class of systems, and several tools were introduced for control [5], state estimation and fault detection [18], verification and safety analysis [10]. Moreover, the language HYSDEL (HYbrid Systems DEscription Language) was developed in [11] to obtain MLD models from of a high level textual description of the hybrid dynamics.

2.3 Linear Complementarity (LC) Systems

Linear complementarity (LC) systems are given in discrete-time by the equations

$$
\begin{aligned}
x(k+1) &= Ax(k) + B_1u(k) + B_2w(k) && \text{(4a)}\\
y(k) &= Cx(k) + D_1u(k) + D_2w(k) && \text{(4b)}\\
v(k) &= E_1x(k) + E_2u(k) + E_3w(k) + g_4 && \text{(4c)}\\
0 \le v(k) &\perp w(k) \ge 0 && \text{(4d)}
\end{aligned}
$$

with $v(k), w(k) \in \mathbb{R}^s$ and where $\perp$ denotes the orthogonality of vectors (i.e. $v(k) \perp w(k)$ means that $v'(k)w(k) = 0$). We call $v(k)$ and $w(k)$ the complementarity variables. Clearly, (4) is a subclass of (1).

In [19,20] (linear) complementarity systems in *continuous* time have been studied. Applications include constrained mechanical systems, electrical networks with ideal diodes or other dynamical systems with piecewise linear relations, variable structure systems, constrained optimal control problems, projected dynamical systems and so on [19, Ch. 2]. Issues related to modeling, well-posedness (existence and uniqueness of solution trajectories) [19,20], simulation and discretization [19] have been of particular interest.

For the definition of extended linear complementarity (ELC) systems and min-max-plus-scaling (MMPS) systems, the reader is referred to [12], where we also proved the following result:

Theorem 1 *PWA systems, MLD systems, LC systems, ELC systems, and MMPS systems are equivalent (possibly under some assumptions on the boundedness of input, state, and auxiliary variables), and are a subset of the general class of hybrid systems (1).*

3 Closed-Loop Model Predictive Control (cl-MPC) Systems and Hybrid Systems

Model Predictive Control (MPC) has become the accepted standard for complex constrained multivariable control problems in the process industries. Here at each sampling time, starting at the current state, an open-loop optimal control problem is solved over a finite horizon. Only the first computed control value in the sequence is implemented. At the next time step the computation is repeated starting from the new state and over a shifted horizon, leading to a moving horizon policy [21].

For the discrete-time linear time invariant system

$$
\left\{
\begin{aligned}
x(k+1) &= Ax(k) + Bu(k)\\
y(k) &= Cx(k)
\end{aligned}
\right. \tag{5}
$$

where $x(k) \in \mathbb{R}^n$, $u(k) \in \mathbb{R}^m$, and $y(k) \in \mathbb{R}^p$ are the state, input, and output vector respectively, and the pair (A, B) is stabilizable, consider the problem of tracking the output reference signal $r(k) \in \mathbb{R}^p$ while fulfilling the constraints

$$
D_1x(k) + D_2u(k) + D_3\Delta u(k) \le d_4 \tag{6}
$$

at all time instants $k \ge 0$, where $\Delta u(k) \triangleq u(k) - u(k-1)$ are the increments of the input.

Assume that a full measurement of the state $x(k)$ is available at the current time k. Then, the optimization problem

$$
\begin{aligned}
\min_U &\left(\sum_{t=1}^{N_y} \epsilon'_{k+t|k} Q \epsilon_{k+t|k} + \sum_{t=1}^{N_y-1} \Delta u'_{k+t} R \Delta u_{k+t}\right)\\
\text{subj. to } & D_1x_{k+t|k} + D_2u_{k+t} + D_3\Delta u_{k+t} \le d_4,\\
& \qquad t = 0, 1, \dots, N_c\\
& x_{k+t+1|k} = Ax_{k+t|k} + Bu_{k+t},\ t \ge 0\\
& y_{k+t|k} = Cx_{k+t|k},\ t \ge 1\\
& u_{k+t} = u_{k+t-1} + \Delta u_{k+t},\ t \ge 1\\
& \Delta u_{k+t} = 0,\ N_u \le t < N_y\\
& x_{k|k} = x(k),\ u_k = \Delta u_k + u(k-1)
\end{aligned} \tag{7}
$$

is solved with respect to the column vector $U \triangleq [\Delta u'_k, \dots, \Delta u'_{k+N_u-1}]' \in \mathbb{R}^s$, $s \triangleq mN_u$, at each time k, where $x_{k+t|k}$ denotes the predicted state vector at time $k+t$, obtained by applying the input sequence $u_k, \dots, u_{k+t-1}$ to model (5) starting from the state $x(k)$, and $\epsilon_{k+t|k} \triangleq y_{k+t|k} - r(k)$ is the predicted tracking error[1]. In (7), we assume that $Q = Q' \succeq 0$, $R = R' \succ 0$ ("$\succ$" denotes matrix positive definiteness), $(Q^{\frac{1}{2}}, A)$ detectable (for instance $Q = C'C$ with (C, A) detectable), N_y, N_u, N_c are the output, input, and constraint horizons, respectively, with $N_u \le N_y$ and $N_c \le N_y - 1$.

The MPC control law is based on the following idea: At time k compute the optimal solution $U(k) = \{\Delta u^*_k, \dots, \Delta u^*_{k+N_u-1}\}$ to problem (7), apply

$$
u(k) = x_u(k) + \Delta u^*_k \tag{8}
$$

as input to system (5), where $x_u(k) \triangleq u(k-1)$ is an additional state required to store the input from the previous step, and repeat the optimization (7) at the next time step $k+1$, based on the new measured (or estimated) state $x(k+1)$. By substituting $x_{k+t|k} = A^t x(k) + \sum_{j=0}^{t-1} A^j B u_{k+t-1-j}$ in (7), this can be written as

$$
\begin{aligned}
\min_U \quad & \tfrac{1}{2}U'HU + \xi'(k)FU + \tfrac{1}{2}\xi'(k)Y\xi(k)\\
\text{subj. to} \quad & GU \le W + S\xi(k)
\end{aligned} \tag{9}
$$

where $\xi(k) \triangleq [x'(k)\ x'_u(k)\ r'(k)]'$, $H = H' \succ 0$, and H, F, Y, G, W, S are easily obtained from (7) (as only

[1] If the reference is known in advance, in (7) one can replace $r(k)$ with $r(k+t)$, with a consequent anticipative action of the resulting MPC controller. Otherwise, we assume that $r(k+t) = r(k)$ for $t \ge 0$.

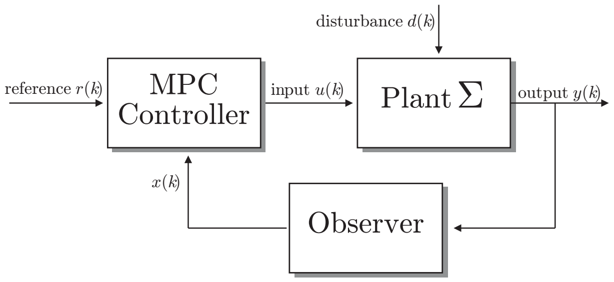

Figure 1: Closed-loop model predictive control system

the optimizer $U(k)$ is needed, the term involving Y is usually removed from (9)).

The optimization problem (9) is a quadratic program (QP), which depends on the current state $x(k)$, past input $x_u(k) = u(k-1)$, and reference $r(k)$. The optimal input $u(k)$ is obtained by defining the new input increment $\Delta u(k)$ to be used in (8) as the first m components of the optimizer $U(k)$ of (9),

$$\Delta u(k) = I_1 U(k), \tag{10}$$

where $I_1 \triangleq [I_m\ 0\ \dots\ 0]$.

Consider the closed-loop model predictive control system depicted in Fig. 1. The plant Σ is described by the difference equations

$$\Sigma : \begin{cases} \chi(k+1) &= \mathcal{A}\chi(k) + \mathcal{B}u(k) + \mathcal{H}d(k) \\ y(k) &= \mathcal{C}\chi(k) + \mathcal{D}d(k), \end{cases} \tag{11}$$

where $\chi(k) \in \mathbb{R}^{\bar{n}}$ is the state vector, and $d(k) \in \mathbb{R}^{\ell}$ is a vector of unmeasured disturbances. We distinguish between model Σ in (11), which is the model generating the data, and model (5), which is the linear model used for designing the MPC controller. Typically (5) is an approximation of (11), for instance a low-order approximation where only the relevant dynamics are kept. As the MPC optimization problem (7) is based on model (5), it requires a state $x(k)$ which is coherent with the same model (5). A common solution consists of generating $x(k)$ via the state observer

$$x(k+1) = Ax(k) + Bu(k) + K_e(y(k) - Cx(k)). \tag{12}$$

In [13], by exploiting the fact that the coefficients of the linear term in the cost function and the right hand side of the constraints in (9) depend linearly on a vector $\xi(k)$ of parameters, the authors tackle the quadratic program (9) as a *multi-parametric quadratic program* (mp-QP), analyze the properties of mp-QP, develop an efficient algorithm to solve it, and show that the optimal solution is a piecewise affine function of the state. As a result, the authors show that the MPC controller admits the explicit continuous PWA form $u(k) = F_i\xi(k) + g_i$ if $\xi(k) \in \Omega_i^{\xi}$, $i = 1, \dots, N$, where $\Omega_i^{\xi} \triangleq \{\xi :\ H_i^{\xi}\xi(k) \leq K_i^{\xi}\}$, and $\{\Omega_i\}_{i=1}^{N}$ is a partition of a given state+input+reference set Ξ. This allows to immediately state the following result:

Theorem 2 *Every cl-MPC system (7), (10)–(12) can be written as a continuous PWA system.*

By applying the results 4 and 1 of Theorem 1, one can also show that cl-MPC systems can be equivalently rewritten as LC systems. However, this requires boundedness assumptions over some of the variables, as the transformation through MLD is involved, plus a large number of complementarity pairs. Here below we prove directly that cl-MPC systems are a subclass of LC systems, which does not require such assumptions and limits the number of required complementarity pairs.

Theorem 3 *Every cl-MPC system (7), (10)–(12) can be written as an LC system.*

Proof: The proof simply follows from the first-order Karush Kuhn Tucker (KKT) conditions for QP (9) [22, Ch. 10.6]

$$HU(k) + F'\xi(k) + G'\lambda(k) = 0,\ \lambda(k) \in \mathbb{R}^q \tag{13a}$$
$$\lambda'(k)(GU(k) - W - S\xi(k)) = 0 \tag{13b}$$
$$\lambda(k) \geq 0 \tag{13c}$$
$$W + S\xi(k) - GU \geq 0 \tag{13d}$$

From (13a), it follows that

$$\begin{aligned} U(k) &= -H^{-1}F'\xi(k) - H^{-1}G'\lambda(k) \\ &\triangleq Tx(k) + Vx_u(k) + Zr(k) + \Lambda\lambda(k). \end{aligned} \tag{14}$$

By letting $Mx(k) + Nx_u(k) + Lr(k) \triangleq S\xi(k)$, $v(k) \triangleq W + Mx(k) + Nx_u(k) + Lr(k) - GU(k)$, $w(k) \triangleq \lambda(k)$, and recalling (10) we can rewrite the closed-loop MPC system in the LC form

$$\begin{aligned}
\begin{bmatrix} \chi(k+1) \\ x(k+1) \\ x_u(k+1) \end{bmatrix} &= \begin{bmatrix} \mathcal{A} & \mathcal{B}I_1T & \mathcal{B}(I_m + I_1V) \\ K_e\mathcal{C} & A - K_eC + BI_1T & B(I_m + I_1V) \\ 0 & I_1T & (I_m + I_1V) \end{bmatrix} \cdot \\
&\begin{bmatrix} \chi(k) \\ x(k) \\ x_u(k) \end{bmatrix} + \begin{bmatrix} \mathcal{B}I_1Z & \mathcal{H} \\ BI_1Z & K_e\mathcal{D} \\ I_1Z & 0 \end{bmatrix} \begin{bmatrix} r(k) \\ d(k) \end{bmatrix} + \begin{bmatrix} \mathcal{B}I_1\Lambda \\ BI_1\Lambda \\ \Lambda \end{bmatrix} w(k) \\
y(k) &= \mathcal{C}\chi(k) + \mathcal{D}d(k) \\
v(k) &= \begin{bmatrix} 0 & M - GT & N - GV \end{bmatrix} \begin{bmatrix} \chi(k) \\ x(k) \\ x_u(k) \end{bmatrix} \\
&\quad + (L - GZ)r(k) - G\Lambda w(k) + W \\
&0 \leq v(k) \perp w(k) \geq 0
\end{aligned} \tag{15}$$

where $\left[\begin{smallmatrix} \chi \\ x \\ x_u \end{smallmatrix}\right]$, $\left[\begin{smallmatrix} r \\ d \end{smallmatrix}\right]$ are the state and input vectors, respectively, of the LC system. ■

Note that the result of Theorem 3 also holds when model (11) is replaced by any of the hybrid models described in the previous sections. Consequently, stability, feasibility/safety, and performance properties of cl-MPC where a simple linear model is used in the *synthesis* of the controller, and a more accurate hybrid model approximating the plant dynamics is used for *analysis*, can be tested using tools developed for hybrid systems.

Remark 3 For each weight matrix $R \succ 0$, the cl-MPC system (7), (11) is well-posed on the set of $x(k)$, $x_u(k)$,

$r(k)$ where (9) is feasible. In fact, the Hessian matrix $H \succ 0$ in (9), and therefore $\Delta u(k)$ is uniquely determined once $x(k)$, $r(k)$, $x_u(k)$ are assigned. Consequently, the equivalent LC form (15) is well-posed, despite the fact that $w(k)$ might not be uniquely defined by the KKT conditions (e.g., in case of primal degeneracy of the QP problem (9)). $\square$

In order to show directly that cl-MPC systems are also a subclass of MLD systems, we prove the following Lemma:

Lemma 1 *Let $\xi \triangleq [x' \ x_u' \ r']'$ belong to a bounded set Ξ. Then, there exists an upper-bound $\lambda^+ \geq 0$ such that at least one vector of Lagrange multipliers λ is optimal for (9) and satisfies $0 \leq \lambda \leq \lambda^+$.*

Proof: Consider the combination $I \subseteq \{1, \ldots, k\}$ of active constraints $G_I U = W_I + S_I \xi$ at the optimum, where $_I$ denotes the submatrix obtained by collecting the rows indicized by the elements of I, and assume that G_I is full row rank. From the KKT conditions (13), $U = -H^{-1}(F'\xi + G_I' \lambda_I(\xi))$, where $\lambda_I(\xi) \geq 0$ is a vector collecting the subset of Lagrange multipliers relative to the active constraints (the remaining multipliers are zero). Substituting U, we obtain $\lambda_I(\xi) = -(G_I H^{-1} G_I')^{-1}[W_I + S_I \xi + G_I H^{-1} F' \xi]$, which admits an upper-bound $\lambda_I^+ \triangleq \max_{\xi \in \Xi} \lambda_i(\xi) \geq 0$. Take $\lambda^+ \triangleq \max \lambda_I^+$ over all combinations I of linearly independent active constraints. If for some ξ a linearly dependent combination of constraints is active at the optimum, (i.e., the QP is primal degenerate, and λ is not unique), then a subset of linearly independent constraints and a vector $\lambda(\xi) \leq \lambda^+$ can be chosen which provides the same solution U (cf. [19, Lemma 4.4.5] and [23, Theorem 2.6.12]).
An alternative proof follows by considering the KKT conditions (13) in LCP form: $v = (GH^{-1}G')\lambda + [W + (GH^{-1}F' + S)\xi]$, $0 \leq v \perp \lambda \geq 0$, and directly applying [19, Lemma 7.6.14], showing that, for all $\xi \in \Xi$ such that the QP (9) is feasible, there exists a unique least-norm solution $\lambda(\xi)$ satisfying, for some scalar $\alpha \in \mathbb{R}$, $\|\lambda(\xi)\| \leq \alpha \|W + (GH^{-1}F' + S)\xi\| \leq \alpha(\|W\| + \|GH^{-1}F' + S)\| \cdot \max_{\xi \in \Xi} \|\xi\|)$. $\blacksquare$

Remark 4 A more efficient way of computing λ^+ than enumerating all possible combinations of linearly independent active constraints (as proposed in the first part of the proof of Lemma 1) consists of computing the solution to the mp-QP problem (9) by applying the algorithm of [13], which provides all and only the combinations of linearly independent active constraints which are optimal for some $\xi \in \Xi$ (Ξ is partitioned into polyhedral cells, each one characterized by a different combination). This is illustrated in the example reported in [24]. $\square$

Remark 5 In case $GH^{-1}G \succ 0$ [25, Theorem 2] also provides a recursive method to write the optimal solution $U(k)$ as an explicit piecewise affine function of $x(k)$, $x_u(k)$ and $r(k)$. Unfortunately, in most QP problems arising from MPC, only $GH^{-1}G \succeq 0$ holds, and therefore such a technique cannot be applied. $\square$

Using arguments similar to those used to prove the second point of Theorem 1, we obtain the following result:

Proposition 1 *Every linear MPC closed-loop system can be written as an MLD system, provided that bounds on the states, inputs, and references, are specified.*

Proof: Introduce a vector of binary variables $\delta(k) \in \{0,1\}^q$. The idea is to represent $v_i(k) = 0$, $w_i(k) \geq 0$ with $\delta_i(k) = 1$, and $v_i(k) \geq 0$, $w_i(k) = 0$ with $\delta_i(k) = 0$. This can be achieved by introducing the constraints $w(k) \leq M_w \delta(k)$, $v(k) \leq M_v(e - \delta(k))$, $w(k) \geq 0$, $v(k) \geq 0$, where M_w and M_v are diagonal matrices containing upper bounds on $w(k)$, and $v(k)$ (provided by Lemma 1), respectively, and e denotes the vector for which all entries are equal to one. By setting $z(k) = w(k)$ and replacing $v(k)$ as in (15) it easy to rewrite the MPC closed-loop system in the MLD form

$$\begin{aligned}
\begin{bmatrix} \chi(k+1) \\ x(k+1) \\ x_u(k+1) \end{bmatrix} &= \begin{bmatrix} \mathcal{A} & \mathcal{B}I_1T & \mathcal{B}(I_m + I_1V) \\ K_e\mathcal{C} & A - K_eC + BI_1T & B(I_m + I_1V) \\ 0 & I_1T & (I_m + I_1V) \end{bmatrix} \cdot \\
&\begin{bmatrix} \chi(k) \\ x(k) \\ x_u(k) \end{bmatrix} + \begin{bmatrix} \mathcal{B}I_1Z & \mathcal{H} \\ BI_1Z & K_e\mathcal{D} \\ I_1Z & 0 \end{bmatrix} \begin{bmatrix} r(k) \\ d(k) \end{bmatrix} + \begin{bmatrix} \mathcal{B}I_1\Lambda \\ BI_1\Lambda \\ \Lambda \end{bmatrix} z(k) \\
&y(k) = \mathcal{C}\chi(k) + \mathcal{D}d(k)
\end{aligned} \tag{16a}$$

$$\begin{aligned}
&\begin{bmatrix} 0 & 0 & 0 \\ 0 & M - GT & N - GV \\ 0 & 0 & 0 \\ 0 & -M + GT & -N + GV \end{bmatrix} \begin{bmatrix} \chi(k) \\ x(k) \\ x_u(k) \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ L - GZ & 0 \\ 0 & 0 \\ -L + GZ & 0 \end{bmatrix} \\
&\begin{bmatrix} r(k) \\ d(k) \end{bmatrix} + \begin{bmatrix} -M_w \\ M_v \\ 0 \\ 0 \end{bmatrix} \delta(k) + \begin{bmatrix} I \\ -G\Lambda \\ -I \\ G\Lambda \end{bmatrix} w(k) \leq \begin{bmatrix} 0 \\ M_v e - W \\ 0 \\ W \end{bmatrix}
\end{aligned} \tag{16b}$$

$\blacksquare$

Note that the number q of integer variables equals the number of constraints of the MPC optimization problem (9). Hence, if the MLD system were translated into PWA form as in [6], the resulting PWA system would have at most 2^q regions. This confirms the result of Theorem 2 and [13], where the explicit PWA form of the MPC controller (obtained by using multiparametric programming) is defined over a polyhedral partition of the state space composed by at most 2^q regions (note that 2^q equals the number of all possible combinations of active constraints). Since many of such combinations are infeasible, in general the resulting number of regions is much lower than 2^q.

3.1 Extensions

More generally, model (11) can be replaced by a hybrid model of the form (1). The results shown above can be all repeated also for this more general setup.

Corollary 1 *The cl-MPC system formed by an LC (MLD, PWA, ELC, MMPS) system in feedback with a linear MPC controller is an LC (MLD, PWA, ELC, MMPS) system.*

Such a result is important for studying well-posedness, stability, and constraint fulfillment properties of MPC closed-loop systems constituted by an MPC controller based on a linear model of the process (a common choice for obtain an easily implementable controller) and a plant modeled as a hybrid system. This can be for instance a PWA system obtained by linearizing a nonlinear process model at different operating points, an LC system obtained by a mechanical model, or an MLD system obtained by using the description language HYSDEL [11].

4 Conclusions

In this paper we showed that closed-loop MPC systems can be treated and analyzed as hybrid systems, in particular as linear complementarity (LC) systems, mixed logical dynamical (MLD) systems, piecewise affine (PWA) systems, and indirectly, by exploiting the equivalences of [12], also as extended linear complementarity (ELC) systems, max-min-plus-scaling (MMPS) systems. For an example of application of the results of this paper, not reported here for the lack of space, the reader is referred to [24].

The result is of paramount importance for applying the tools developed for hybrid systems, such as stability and robust stability analysis, and safety/reachability analysis, to study closed-loop properties of model predictive controllers. MPC schemes are typically designed so that they are stable and fulfill operating constraints *when the nominal model of the plant and the prediction model coincide*. The results of this paper, instead, allow to investigate stability and safety properties of any combination of a linear MPC controller, a linear observer, and a linear plant, and can be easily extended to arbitrary combinations of linear MPC controllers and *hybrid* plants, such as hybrid approximations of complex nonlinear dynamic models of the process under closed-loop control.

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