Optimal Controllers for Hybrid Systems: Stability and Piecewise Linear Explicit Form

A. Bemporad[†‡], F. Borrelli[†], M. Morari[†]

[†]Automatic Control Laboratory, ETH-Zentrum, ETL I 26, 8092 Zurich, Switzerland
[‡]Dip. Ingegneria dell'Informazione, Università di Siena, Via Roma 56, 53100 Siena, Italy
bemporad,borrelli,morari@aut.ee.ethz.ch

Abstract

In this paper we propose a procedure for synthesizing piecewise linear optimal controllers for hybrid systems and investigate conditions for closed-loop stability. Hybrid systems are modeled in discrete-time within the *mixed logical dynamical* (MLD) framework [8], or, equivalently [7], as *piecewise affine* (PWA) systems. A stabilizing controller is obtained by designing a *model predictive controller* (MPC), which is based on the minimization of a weighted $1/\infty$-norm of the tracking error and the input trajectories over a finite horizon. The control law is obtained by solving a *mixed-integer linear program* (MILP) which depends on the current state. Although efficient branch and bound algorithms exist to solve MILPs, these are known to be $\mathcal{NP}$-hard problems, which may prevent their on-line solution if the sampling-time is too small for the available computation power. Rather than solving the MILP on line, in this paper we propose a different approach where all the computation is moved off line, by solving a *multiparametric* MILP (mp-MILP). As the resulting control law is piecewise affine, on-line computation is drastically reduced to a simple linear function evaluation. An example of piecewise linear optimal control of the heat exchange system [16] shows the potential of the method.

Keywords: Hybrid systems, model predictive control, mixed-integer programming, multi-parametric programming

1 Introduction

Hybrid systems provide a unified framework for describing processes evolving according to continuous dynamics, discrete dynamics, and logic rules [2, 3, 20]. The interest in hybrid systems is mainly motivated by the large variety of practical situations, for instance real-time systems, where physical processes interact with digital controllers. Several modeling formalisms have been developed to describe hybrid systems, and among others Bemporad and Morari [8] introduced a new class of hybrid systems called *mixed logical dynamical* (MLD) systems. The MLD framework allows specifying the evolution of continuous variables through linear dynamic discrete-time equations, of discrete variables through propositional logic statements and automata, and the mutual interaction between the two. The key idea of the approach consists of embedding the logic part in the state equations by transforming Boolean variables into 0-1 integers, and by expressing the relations as mixed-integer linear inequalities. Therefore MLD systems are capable to model a broad class of systems arising in many applications: linear hybrid dynamical systems, hybrid automata, nonlinear dynamic systems where the nonlinearity can be approximated by a piecewise linear function, some classes of discrete event systems. Examples of real-world applications that can be naturally modeled within the MLD framework are reported in [8, 9, 11].

Recently, in [7] the authors proved in a constructive way that MLD systems are equivalent to *piecewise affine* (PWA) systems, confirming the result of equivalence between hybrid and PWA systems shown in [24]. PWA systems are defined by partitioning the state space into polyhedral regions, and associating with each region a different linear dynamic equation. Besides the fact that PWA systems are an important system class, the equivalence allows one to extend all the techniques developed for PWA models to the general MLD description of hybrid systems, and vice versa. This renders the PWA framework a useful companion both for investigating system theoretical properties and for designing algorithms.

MLD systems are formulated in discrete time. Despite the fact that the effects of sampling can be neglected in most applications, subtle phenomena such as Zeno behaviors cannot be captured in discrete time. On the other hand, although reformulating MLD systems in continuous time would be quite easy from a theoretical point of view, a discrete-time formulation allows developing numerically tractable schemes for solving complex analysis and synthesis problems. Several questions of interest for MLD systems can indeed be suitably formulated as mixed-integer linear/quadratic optimization problems. For feedback control, Bemporad and Morari [8] propose *Model Predictive Control* (MPC) as a general approach to control hybrid systems. MPC has been widely adopted in industry to solve control problems of systems subject to input and output constraints. MPC is based on the so called *receding horizon* philosophy: a sequence of future control actions is chosen according to a prediction of the future evolution of the system and applied to the plant until new measurements are available. Then, a new sequence is established which replaces the previous one. Each sequence is determined by means of an optimization procedure which takes into account two objectives: optimize the tracking performance and protect the system from possible constraint violations. When the model of the system is a hybrid MLD model and the performance index is quadratic, the optimization problem is a Mixed-Integer Quadratic Programming (MIQP) problem. Similarly, $1-$ and ∞-norm performance indices lead to Mixed-Integer Linear Programming (MILP) problems. The main drawback of such a control approach are its intensive on-line computation requirements. Although efficient branch and bound algorithms exist to solve MIQP/MILP [15, 17, 23], these are known to be $\mathcal{NP}$-hard problems.

In this paper we propose a different approach where all the computation is moved off line, by extending the result of [10] for linear systems to the hybrid case. By formulating the MPC problem as the minimization of a weighted 1-norm of the tracking error and command input, and by treating the current state as a vector of parameters, the optimization problem can in fact be recast as a *multiparametric* MILP (mp-MILP), for which efficient solvers are available [1, 14]. After the solution of the mp-MILP has been determined, the resulting feedback control law is piecewise affine with respect to the state. Therefore on-line computation reduces to a simple linear function evaluation, instead of an expensive mixed-integer linear program. Needless to say, this makes the approach attractive for fast processes and/or low cost control hardware.

Preliminary ideas along this approach appeared in [5], where the authors included a terminal state constraint to ensure stability. In this paper, we remove such a constraint and provide conditions to choose the weights in the performance index so that the resulting MPC law is stabilizing under hard input and soft state constraints. An example illustrates the basic ideas of the synthesis technique.

2 Model Predictive Control of Hybrid Systems

Consider the *mixed logical dynamical* (MLD) system described by the relations

$$x(t+1) = \Phi x(t) + G_1 u(t) + G_2 \delta(t) + G_3 z(t) \tag{1a}$$

$$y(t) = Hx(t) + D_1 u(t) + D_2 \delta(t) + D_3 z(t) \tag{1b}$$

$$E_2 \delta(t) + E_3 z(t) \leq E_1 u(t) + E_4 x(t) + E_5 \tag{1c}$$

where $x \in \mathbb{R}^{n_c} \times \{0,1\}^{n_\ell}$ is a vector of continuous and binary states, $u \in \mathbb{R}^{m_c} \times \{0,1\}^{m_\ell}$ are the inputs, $y \in \mathbb{R}^{p_c} \times \{0,1\}^{p_\ell}$ the outputs, $\delta \in \{0,1\}^{r_\ell}$, $z \in \mathbb{R}^{r_c}$ represent auxiliary binary and continuous variables respectively, which are introduced when transforming logic relations into mixed-integer linear inequalities [8], and Φ, G_1, G_2, G_3, H, D_1, D_2, D_3, $E_1,\ldots,E_5$ are matrices of suitable dimensions. As mentioned in the introduction, the authors show in [7] that MLD systems are equivalent to the class of *piecewise affine* (PWA) discrete-time systems described by the equations

$$\begin{array}{rcl} x(t+1) & = & A_i x(t) + B_i u(t) + f_i \\ y(t) & = & C_i x(t) + g_i \end{array}, \text{ for } \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} \in \mathcal{X}_i \tag{2}$$

where $\{\mathcal{X}_i\}_{i=1}^s$ is a partition of the state+input set[1], and f_i, g_i are suitable constant vectors.

It is interesting from both a theoretical and practical point of view to ask whether or not an MLD/PWA system can be stabilized to an equilibrium state or can track a desired reference trajectory, possibly via feedback control. Despite the fact that the system is neither linear nor even smooth, we show in this section how model predictive control (MPC) provides successful tools to perform this task. As recalled above, the main idea of MPC is to use the *model* of the plant to *predict* the future evolution of the system, and based on this prediction to optimize a certain performance index under operating constraints to generate the *control* action. Only the first sample of the optimal sequence is actually applied to the plant at time t. At time $t+1$, a new sequence is evaluated to replace the previous one. This on-line "re-planning" provides the desired feedback control feature.

Suppose for simplicity of notation that we want to regulate the state of system (2) to the origin, and that the origin is an equilibrium state for $u = 0$[2]. Then,

$$\forall i = 1, \ldots, s: \ 0 \in \mathcal{X}_i \Rightarrow f_i = 0 \tag{3}$$

Let t be the current time, and $x(t)$ the current state. Consider the following optimal control problem

$$\min_{\{\mathcal{U},\ \sigma\}} J(\mathcal{U}, \sigma, x(t)) \triangleq \sum_{k=0}^{T-1} \|Rv(k|t)\|_\infty + \|Qx(k|t)\|_\infty + \|Px(T|t)\|_\infty + \rho\sigma \tag{4}$$

$$\text{subj. to} \begin{cases} x(k+1|t) & = & A_i x(k|t) + B_i v(k) + f_i \\ y(k|t) & = & C_i x(k|t) + g_i, \quad \text{for } \begin{bmatrix} x(k|t) \\ v(k) \end{bmatrix} \in \mathcal{X}_i \\ u_{\min} & \leq & v(t+k) \leq u_{\max},\ k = 0, \ldots, T-1 \\ x_{\min} - \sigma & \leq & x(t+k|t) \leq x_{\max} + \sigma,\ k = 1, \ldots, N_c \\ 0 & \leq & \sigma \leq \sigma(t-1) \end{cases} \tag{5}$$

where $\mathcal{U} = \{v(0), \ldots, v(T-1)\}$ is the sequence of future control moves (i.e., the optimization vector), T and $N_c \leq T$ are the prediction and state constraint horizons, respectively, P, Q, and R are full rank (not necessarily square) matrices, $x(k|t)$ is the state predicted

[1]In general, when the PWA system (2) is obtained by translating an MLD system, the sets $\mathcal{X}_i$ are a partition of a bounded polyhedral set. Here we assume that the partition covers the whole set $\mathbb{R}^{n+m}$, in order to avoid problems of non-wellposedness of (2).

[2]The results of this paper do not change if the equilibrium state and/or its corresponding equilibrium input are nonzero.

at time $t+k$ by applying the input $u(t+k) = v(k)$ to (2) from $x(0|t) = x(t)$, $u_{\min}$, $u_{\max}$ and $x_{\min}$, $x_{\max}$ are hard bounds on the inputs and soft bounds on the states, respectively (more in general, we can deal with hard/soft constraints of the form $Sv(t+k) + Tx(t+k|t) \leq W$). The variable σ was introduced to soften the constraints on the state, as typically these bounds are not as critical as actuator limitations. The decreasing condition $\sigma \leq \sigma(t-1)$, where $\sigma(t-1)$ is the optimal slack variable computed at time $t-1$, is necessary for the stability of the control law, as we will detail later. Such a condition, which in the linear case can be easily avoided by using an infinite prediction horizon [25], may be restrictive in certain situations, for instance in case of disturbance rejection, where one should reset the bound on σ to some nonzero value whenever the arrival of a disturbance step is detected.

From an optimization point of view, soft state constraints enlarge the set of feasible input sequences for the MPC optimization (5). The weight ρ is the trade-off between performance and constraint violation.

According to the *receding horizon* philosophy mentioned above, we set

$$u(t) = v_t^*(0), \tag{6}$$

disregard the subsequent optimal inputs $v_t^*(1), \ldots, v_t^*(T-1)$, and repeat the whole optimization procedure at time $t+1$. In the next section we will show how to formulate the problem (5) as a mixed integer linear program (MILP).

2.1 Stability

We remark that an infinite horizon formulation [19, 22, 6] would be inappropriate in the present hybrid context for both practical and theoretical reasons. In fact, approximating the infinite horizon with a large T is computationally prohibitive, as the number of possible switches (i.e., the combinations of 0-1 variables involved in the MILP, as will be shown later) depends exponentially on T. Moreover, from a theoretical point of view, for a PWA system it is not clear in general how to reformulate an infinite dimensional optimization problem into a finite dimensional one, which instead can be always done for linear systems through Lyapunov or Riccati algebraic equations.

In order to synthesize MPC controllers with stability guarantees, in [5] we adopted the standard stability constraint on the final state $x(T|t) = 0$. On the other hand, such a constraint typically deteriorates the overall performance, especially for short prediction horizons. In order to avoid such a constraint, we can either compute an invariant set for the hybrid system (2) and force the final state $x(T|t)$ to belong to such a set, or use the formulation (4)-(5). While the computation of invariant sets for hybrid systems is still an open problem, the following theorem shows that, by appropriately choosing the terminal weight P, the control law (4)–(6) stabilizes system (2) asymptotically

Theorem 1 *Let the origin be an equilibrium for system (2), and assume that condition (3) is satisfied. If there exist vectors u_n, $u_{\min} \leq u_n \leq u_{\max}$, such that*

$$-\|Px\|_\infty + \|P(A_i x + B_i u_i)\|_\infty + \|Qx\|_\infty + \|Ru_n\|_\infty \leq 0 \tag{7}$$

is satisfied for all $(x, u_i) \in \mathcal{X}_i$, $\forall i = 1, \ldots, s$, the MPC law (4)–(6) stabilizes system (2), in that $\lim_{t\to\infty} x(t) = 0$, $\lim_{t\to\infty} u(t) = 0$, *while fulfilling the input constraints* $u_{\min} \leq u(t) \leq u_{\max}$.

Proof: The proof follows from standard Lyapunov arguments. Let $\mathcal{U}_t^*$ be the optimal control sequence $\{v_t^*(0), \ldots, v_t^*(T-1)\}$, let

$$V(t) \triangleq J(\mathcal{U}_t^*, x(t))$$

be the corresponding value attained by the performance index, i such that $(x^*(T|t), u_i) \in \mathcal{X}_i$, and let $\mathcal{U}_1 \triangleq \{v_t^*(1), \ldots, v_t^*(T-2), v_t^*(T-1), u_i\}$. Then, $\mathcal{U}_1$ is feasible at time $t+1$, and hence $V(t+1) - V(t) = -\|Qx(t)\|_\infty - \|Ru(t)\|_\infty - \|Px^*(T|t)\|_\infty + \|Px^*(T+1|t)\|_\infty + \|Qx^*(T|t)\|_\infty + \|Ru(T|t)\|_\infty + \rho(\sigma(t+1) - \sigma(t)) \leq -\|Qx(t)\|_\infty - \|Ru(t)\|_\infty - \|Px^*(T|t)\|_\infty + \|Px^*(T+1|t)\|_\infty + \|Qx^*(T|t)\|_\infty + \|Ru(T|t)\|_\infty$ As the condition (7) is satisfied for $x = x^*(T|t)$, $V(t)$ is a decreasing sequence. Since $V(t)$ is lower-bounded by 0, there exists $V_\infty = \lim_{t\to\infty} V(t)$, which implies $V(t+1) - V(t) \to 0$. Therefore, each term of the sum

$$\|Qx(t)\|_\infty + \|Ru(t)\|_\infty \tag{8}$$

converges to zero as well, which proves the theorem as Q and R are nonsingular. □

Remark 1 Condition (7) amounts to finding a common polyhedral Lyapunov function for the hybrid system (2). Of course, the existence of such a function is not guaranteed in general. An alternative is to replace the final weight $\|Px(T|t)\|_\infty$ by a more general piecewise linear function of $x(T|t)$, which would require one to find a piecewise linear Lyapunov function [18, 21] for the hybrid system (2). Note that, although the piecewise linear weight can still be tackled by mixed-integer linear programming, the complexity of the optimization problem increases.

Remark 2 Given matrices P, Q, R, and vectors $\{u_n\}$ *checking* if condition (7) is satisfied can be performed through mixed-integer linear programming. On the other hand, it is not clear how to formulate an algorithm for *synthesizing* P and $\{u_n\}$ given Q and R (which provides the solution to a sort of "equivalent" ∞-/norm-based Lyapunov equation for PWA systems), although such an algorithm is reported in [4] for linear systems.

Example 2.1

We slightly modify the example in [5] so that we can exploit the result of [4] for computing matrix P satisfying (7). Consider the system

$$\left\{\begin{array}{rcl} x(t+1) & = & 0.7\begin{bmatrix} \cos\alpha(t) & -\sin\alpha(t) \\ \sin\alpha(t) & \cos\alpha(t) \end{bmatrix} x(t)+ \\ & & \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \\ y(t) & = & [1\ 0]x(t) \\ \alpha(t) & = & \left\{\begin{array}{ll} \frac{\pi}{3} & \text{if} \quad [1\ 0]x(t) \geq 0 \\ -\frac{\pi}{3} & \text{if} \quad [1\ 0]x(t) < 0 \end{array}\right. \\ x(t) & \in & [-5,5]\times[-5,5] \\ u(t) & \in & [-1,1] \end{array}\right. \tag{9}$$

It is easy to verify that the matrix

$$P = \begin{bmatrix} -15.6 & 27.02 \\ 27.02 & 15.6 \end{bmatrix} \tag{10}$$

satisfies condition (7). Matrix P has been determined by applying the result of [4] to each of the two linear subsystems. Unfortunately, this procedure for determining P cannot be generalized. □

3 Piecewise Linear Solution of MPC

In the previous section we have defined an optimal receding horizon control law for PWA systems. By exploiting the equivalence between MLD and PWA systems, we will use the MLD form from now on, as it is more attractive from a computational point of view.

By considering again the MLD system (1), problem (4) can be rewritten as

$$\min_{\{v_0^{T-1},\,\sigma\}} J(v_0^{T-1},\sigma,x(t)) \triangleq \sum_{k=0}^{T-1} \|Rv(k|t)\|_\infty + \|Qx(k|t)\|_\infty + \\ +\|Px(T|t)\|_\infty + \rho\sigma \tag{11}$$

$$\text{subj. to} \left\{\begin{array}{rcl} x(k+1|t) & = & \Phi x(k|t) + G_1v(k) + G_2\delta(k|t) + G_3z(k|t) \\ y(k|t) & = & Hx(k|t) + D_1v(k) + D_2\delta(k|t) + D_3z(k|t) \\ E_2\delta(k|t) & + & E_3z(k|t) \leq E_1v(k) + E_4x(k|t) + E_5 \\ u_{\min} & \leq & v(t+k) \leq u_{\max},\ k=0,1,\ldots,T-1 \\ x_{\min}-\sigma & \leq & x(t+k|t) \leq x_{\max}+\sigma,\ k=1,\ldots,N_c \\ 0 & \leq & \sigma \leq \sigma(t-1) \end{array}\right. \tag{12}$$

The MPC formulation (11)-(12) can be rewritten as a mixed-integer linear program by using the following standard approach. The sum of the components of any vector $\{\varepsilon_0^u,\ldots,\varepsilon_{T-1}^u,\varepsilon_0^x,\ldots,\varepsilon_{T-1}^x,\varepsilon_T^x,\rho\sigma\}$ that satisfies

$$\begin{array}{l} -\mathbf{1_m}\varepsilon_k^u \leq Ru(k|t)\ k=0,1,\ldots,T-1 \\ -\mathbf{1_m}\varepsilon_k^u \leq -Ru(k|t)\ k=0,1,\ldots,T-1 \\ -\mathbf{1_n}\varepsilon_k^x \leq Qx(k|t)\ k=0,1,\ldots,T-1 \\ -\mathbf{1_n}\varepsilon_k^x \leq -Qx(k|t)\ k=0,1,\ldots,T-1 \\ -\mathbf{1_n}\varepsilon_k^x \leq Px(T|t) \\ -\mathbf{1_n}\varepsilon_k^x \leq -Px(T|t) \end{array} \tag{13}$$

represents an upper bound on $J(v_0^{T-1},\sigma,x(t))$, where $-\mathbf{1_k}$ is a column vector of ones of length k, and $x(k|t) = A^kx(t) + \sum_{j=0}^{k-1} A^j(B_1u_{k-1-j} + B_2\delta_{k-1-j} + B_3z_{k-1-j})$. Similarly to what was shown in [13], it is easy to prove that the vector $p \triangleq \{\varepsilon_0^u,\ldots,\varepsilon_{T-1}^u,\varepsilon_0^x,\ldots,\varepsilon_{T-1}^x,\varepsilon_T^x,u_t,\ldots,u_{t+T-1},\delta_t,\ldots,\delta_{t+T-1},z_t\ldots,z_{t+T-1},\sigma\}$ that satisfies equations (13) and simultaneously minimizes

$$J(p) = \varepsilon_0^u + \ldots + \varepsilon_{T-1}^u + \varepsilon_o^x + \ldots + \varepsilon_{T-1}^x + \varepsilon_T^x \tag{14}$$

also solves the original problem, i.e. the same optimum $J^*(v_0^{T-1},x(t))$ is achieved. Therefore, problem (11)-(12) can be reformulated as the following MILP problem

$$\begin{array}{lrcl} \min_p & J(p) & = & \varepsilon_0^u + \ldots + \varepsilon_{T-1}^u + \varepsilon_o^x + \ldots + \varepsilon_{T-1}^x + \varepsilon_T^x \\ \text{subj. to} & -\mathbf{1_m}\varepsilon_k^u & \leq & \pm R(u(k|t))\ k=0,1,\ldots,T \\ & -\mathbf{1_n}\varepsilon_k^x & \leq & \pm Q(A^kx(0|t)+ \\ & & & \sum_{j=0}^{k-1} A^j(B_1u_{k-1-j|t} + B_2\delta_{k-1-j|t} \\ & & & +B_3u_{k-1-j|t}))\ k=1,\ldots,T \\ & x_{\min} & \leq & A^kx(t) + \sum_{j=0}^{k-1} A^j(B_1u_{k-1-j|t}+ \\ & & & +B_2\delta_{k-1-j|t} + B_3u_{k-1-j|t}))+ \\ & & & D_1u(k) + D_2\delta(k|t) + D_3z(k|t) \leq x_{\max}, \\ & & & k=1,\ldots,T \\ & u_{\min} & \leq & u_{k|t} \leq u_{\max},\ k=0,1,\ldots,T \\ & E_2\delta(k|t) & + & E_3z(k|t) \leq E_1v(k)+ \\ & & & E_4\sum_{j=0}^{k-1} A^j(B_1u_{k-1-j|t} + B_2\delta_{k-1-j|t} \\ & & & +B_3u_{k-1-j|t}) + E_5,\ k\geq 0 \end{array} \tag{15}$$

or, in the more compact form,

$$\begin{array}{ll} \min_p & l(p_c,p_d,\xi(t)) = f_c^Tp_c + f_d^Tp_d \\ \text{subj. to} & G_cp_c + G_cp_d \leq S + F\xi(t) \end{array} \tag{16}$$

where $\xi(t) = [x'(t)\ \sigma(t-1)]'$, the matrices G, S, F can be defined from (15), and p_c, p_d represent continuous and discrete variables, respectively.

The MILP problem (16) depends on the current value of $\xi(t)$, and needs to be solved in order to compute the command input. Rather than solving the MILP on line, we follow the ideas of [10, 4], and propose an approach where all the computation is moved off line. In fact, by treating $\xi(t)$ as a vector of parameters, the MILP becomes a *multiparametric* MILP (mp-MILP), and its solution for all admissible initial states $\xi(t)$ will be the explicit MPC controller law for PWA systems. We will also show that such a control law is piecewise affine with respect to the state vector.

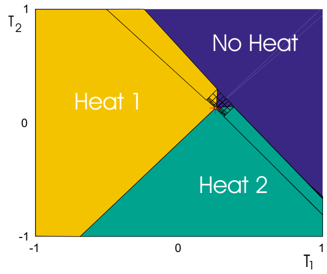
As we will describe in the next section, we use the algorithm developed in [14] for solving the mp-MILP formulated above. Once the multi-parametric problem (15) has been solved off line, i.e. the solution $p_t^* = f(\xi(t))$ of (16) has been found, the model predictive controller (4)-(5) is available explicitly, as the optimal input $u(t)$ consists simply of m components of p_t^*

$$u(t) = [0\ \ldots 0\ I\ 0\ \ldots\ 0]f(\xi(t)). \tag{17}$$

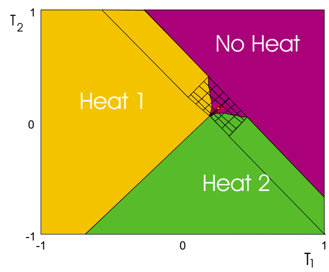
As the solution p^* of the mp-MILP problem is piecewise affine with respect to the state $x(t)$, the same property is inherited by the controller because of (17).

4 Multiparametric-MILP Solvers

Two main approaches have been proposed for solving mp-MILP problems. In [1], the authors develop an algorithm based on branch and bound (B&B) methods. At each node of the B&B tree an mp-LP is solved. The solution at the root node represent a valid lower bound, while the solution at a node where all the integer variables have been fixed represents a valid upper bound. As in standard B&B methods, the complete enumeration of combinations of 0-1 integer variables is avoided by comparing the multiparametric solutions, and by fathoming the nodes where there is no improvement of the value function. In [14] an alternative algorithm was proposed, which instead of solving mp-LP problems with integer variables relaxed in the interval $[0, 1]$, only solves mp-LPs where the integer variables are fixed at the optimal value determined by an MILP. More in detail, problem (16) is alternatively decomposed into an mp-LP and an MILP subproblem. When the values of the binary variable are fixed, an mp-LP is solved, and its solution provides a parametric upper bound. On the other hand, when the parameters in $\xi(t)$ are treated as free variables, an MILP is solved, which provides a new integer vector (see [14] for more details). The algorithmic implementation of the mp-MILP algorithm adopted in this paper relies on [12] for solving mp-LP problems, and on [17] for solving MILP's.

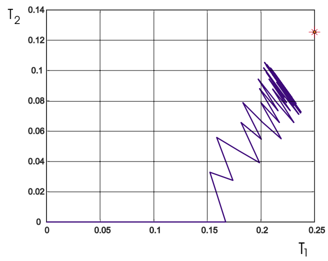


(a) Explicit optimal controller for $u_0 = 0.4$

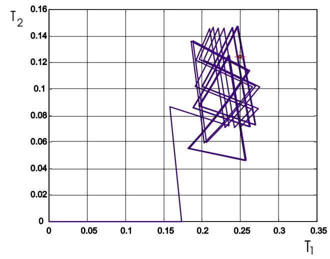


(b) Explicit optimal controller for $u_0 = 0.8$

Figure 1: Polyhedral partition of the state-space associated with the optimal control law



(a) Closed-loop MPC control of system (18) for $u_0 = 0.4$



(b) Closed-loop MPC control of system (18) for $u_0 = 0.8$

Figure 2: Closed-loop MPC control of system (9)

5 An Example

Consider the following hybrid control problem for the heat exchange example proposed by Hedlund and Ranzter [16]. The temperature of two furnaces should be controlled to a given set-point by alternate heating. Only three modes of operation are allowed: heat only the first furnace, heat only the second one, do not heat. The amount of power u_0 to be fed to the furnaces at each time instant is fixed. The system is described by the following equations:

$$\left\{\begin{array}{lcl} \dot{T} & = & \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} T + B_i u_0 \\ B_i & = & \left\{\begin{array}{lll} \begin{bmatrix} 1 \\ 0 \end{bmatrix} & \text{if} & \text{heating the first furnace} \\ \begin{bmatrix} 0 \\ 1 \end{bmatrix} & \text{if} & \text{heating the second furnace} \\ \begin{bmatrix} 0 \\ 0 \end{bmatrix} & \text{if} & \text{no heating} \end{array}\right. \end{array}\right. \tag{18}$$

System (18) is discretized with a sampling time $T_s = 0.08s$, and its equivalent MLD form (1) is computed as described in [8] by introducing an auxiliary vector $z(t) \in \mathbb{R}^9$.

In order to optimally control the two temperatures to the desired values $T_e^1 = 1/4$ and $T_e^2 = 1/8$, the following performance index is minimized:

$$\min_{\{v_0^2\}} J(v_0^2, x(t)) \triangleq \sum_{k=0}^{2} \|R(v(k+1) - v(k))\|_\infty + \|Q(T(k|t) - T_e)\|_\infty \tag{19}$$

subject to the MLD system dynamics, along with the weights $Q = 1$, $R = 700$. The cost function weights the tracking error and trades it off with the number of input switches occurring along the prediction horizon. By solving the mp-MILP associated with the MPC problem we obtain the explicit controller for the range $T \in [-1, 1] \times [-1, 1]$, $u_0 \in [0, 1]$. In Fig. 1 two slices of the three-dimensional state-space partition for different constant power levels u_0 are depicted. Around the equilibrium, the solution appears more finely partitioned, in order to perform an optimal control action. The resulting optimal trajectories are shown in Fig. 2. For a low power $u_0 = 0.4$ the set-point is never reached.

6 Conclusions

We have proposed a procedure for synthesizing piecewise linear stabilizing controllers for hybrid systems modeled in discrete-time. The controllers are optimal with respect to a weighted $1/\infty$-norm of the tracking error and the input trajectories.

7 Acknowledgments

This research has been supported by the Swiss National Science Foundation.

References

[1] J. Acevedo and E. N. Pistikopoulos. A multiparametric programming approach for linear process engineering problems under uncertainty. *Ind. Eng. Chem. Res.*, 36:717–728, 1997.

[2] R. Alur, C. Courcoubetis, T.A. Henzinger, and P.-H. Ho. Hybrid automata: An algorithmic approach to the specification and verification of hybrid systems. In A.P. Ravn R.L. Grossman, A. Nerode and H. Rischel, editors, *Hybrid Systems*, volume 736 of *Lecture Notes in Computer Science*, pages 209–229. Springer Verlag, 1993.

[3] A. Asarin, O. Maler, and A. Pnueli. On the analysis of dynamical systems having piecewise-constant derivatives. *Theoretical Computer Science*, 138:35–65, 1995.

[4] A. Bemporad, F. Borrelli, and M. Morari. Explicit solution of LP-based model predictive control. In *CDC00*, 2000.

[5] A. Bemporad, F. Borrelli, and M. Morari. Piecewise linear optimal controllers for hybrid systems. In *Proc. American Contr. Conf.*, 2000.

[6] A. Bemporad, L. Chisci, and E. Mosca. On the stabilizing property of the zero terminal state receding horizon regulation. *Automatica*, 30(12):2013–2015, 1994.

[7] A. Bemporad, G. Ferrari-Trecate, and M. Morari. Observability and controllability of piecewise affine and hybrid systems. *IEEE Trans. Automatic Control*, to appear.

[8] A. Bemporad and M. Morari. Control of systems integrating logic, dynamics, and constraints. *Automatica*, 35(3):407–427, March 1999.

[9] A. Bemporad and M. Morari. Verification of hybrid systems via mathematical programming. In F.W. Vaandrager and J.H. van Schuppen, editors, *Hybrid Systems: Computation and Control*, volume 1569 of *Lecture Notes in Computer Science*, pages 31–45. Springer Verlag, 1999.

[10] A. Bemporad, M. Morari, V. Dua, and E. N. Pistikopoulos. The explicit linear quadratic regulator for constrained systems. In *Proc. American Contr. Conf.*, 2000.

[11] A. Bemporad, F.D. Torrisi, and M. Morari. Verification of mixed logical dynamical models — the batch evaporator process benchmark. Technical Report AUT00-04, Automatic Control Laboratory, ETH Zurich, Switzerland, February 2000. `http://control.ethz.ch/`.

[12] F. Borrelli, A. Bemporad, and M. Morari. A geometric algorithm for multi-parametric linear programming. Technical Report AUT00-06, Automatic Control Laboratory, ETH Zurich, Switzerland, February 2000.

[13] P.J. Campo and M. Morari. Robust model predictive control. In *Proc. American Contr. Conf.*, volume 2, pages 1021–1026, 1987.

[14] V. Dua and E. N. Pistikopoulos. An algorithm for the solution of multiparametric mixed integer linear programming problems. *Annals of Operations Research*, to appear 1999.

[15] R. Fletcher and S. Leyffer. Numerical experience with lower bounds for MIQP branch-and-bound. *SIAM J. Optim.*, 8(2):604–616, May 1998.

[16] S. Hedlund and A. Rantzer. Optimal control of hybrid systems. In *Proc. 38th IEEE Conf. on Decision and Control*, pages 3972–3976, Phoenix, AZ, December 1999.

[17] ILOG, Inc., Gentilly Cedex, France. *CPLEX 6.5 Reference Manual*, 1999.

[18] P. Julian, J. Guivant, and A. Desages. A parametrization of piecewise linear Lyapunov functions via linear programming. *Int. J. Control*, 72(7-8):702–715, May 1999.

[19] S.S. Keerthi and E.G. Gilbert. Optimal infinite-horizon feedback control laws for a general class of constrained discrete-time systems: stability and moving-horizon approximations. *J. Opt. Theory and Applications*, 57:265–293, 1988.

[20] J. Lygeros, C. Tomlin, and S. Sastry. Controllers for reachability specifications for hybrid systems. *Automatica*, 35(3):349–370, 1999.

[21] Y. Ohta, H. Imanishi, L. Gong, and H. Haneda. Computer generated Lyapunov functions for a class of nonlinear systems. *IEEE Trans. Circ. Syst. -I*, 40(5):343–354, May 1993.

[22] J.B. Rawlings and K.R. Muske. The stability of constrained receding-horizon control. *IEEE Trans. Automatic Control*, 38:1512–1516, 1993.

[23] N. V. Sahinidis. BARON — Branch And Reduce Optimization Navigator. Technical report, University of Illinois at Urbana-Champaign, Dept. of Chemical Engineering, Urbana, IL, USA, 2000.

[24] E.D. Sontag. Interconnected automata and linear systems: A theoretical framework in discrete-time. In R. Alur, T.A. Henzinger, and E.D. Sontag, editors, *Hybrid Systems III - Verification and Control*, number 1066 in Lecture Notes in Computer Science, pages 436–448. Springer-Verlag, 1996.

[25] A. Zheng and M. Morari. Stability of model predictive control with mixed constraints. *IEEE Trans. Automatic Control*, 40:1818–1823, 1995.