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On-line optimization via off-line parametric optimization tools

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Abstract

In this paper, model predictive control (MPC) based optimization problems with a quadratic performance criterion and linear constraints are formulated as multi-parametric quadratic programs (mp-QP), where the input and state variables, corresponding to a plant model, are treated as optimization variables and parameters, respectively. The solution of such problems is given by (i) a complete set of profiles of all the optimal inputs to the plant as a function of state variables, and (ii) the regions in the space of state variables where these functions remain optimal. It is shown that these profiles are linear and the corresponding regions are described by linear inequalities. An algorithm for obtaining these profiles and corresponding regions of optimality is also presented. The key feature of the proposed approach is that the on-line optimization problem is solved off-line via parametric programming techniques. Hence (i) no optimization solver is called on-line, and (ii) only simple function evaluations are required, to obtain the optimal inputs to the plant for the current state of the plant. © 2002 Elsevier Science Ltd. All rights reserved.

Keywords: Real-time optimization; Model predictive control; Multi-parametric quadratic programming

1. Introduction

In an optimization framework, where the objective is to minimize or maximize a performance criterion subject to a given set of constraints and where some of the parameters in the optimization problem vary between specified lower and upper bounds, parametric programming is a technique for obtaining (i) the objective function and the optimization variables as a function of these parameters and (ii) the regions in the space of the parameters where these functions are valid (Fiacco, 1983; Gal, 1995; Acevedo & Pistikopoulos, 1996, 1997b; Pertsinidis, Grossmann & McRae, 1998; Papalexandri and Dimkou, 1998; Acevedo & Pistikopoulos, 1999; Dua & Pistikopoulos, 1999). Some recent applications of this technique are,

- hybrid parametric/stochastic programming (Acevedo & Pistikopoulos, 1997a; Hené, Dua & Pistikopoulos, 2001);
- process planning under uncertainty (Pistikopoulos & Dua, 1998);
- material design under uncertainty (Dua & Pistikopoulos, 1998);
- multi-objective optimization (Pistikopoulos & Grossmann, 1988; Pertsinidis, 1992; Papalexandri & Dimkou, 1998);
- flexibility analysis (Bansal, Perkins & Pistikopoulos, 2000a); and
- computation of singular multi-variate normal probabilities (Bansal, Perkins & Pistikopoulos, 2000b).

The main advantage of using the parametric programming techniques to address such problems is that for problems pertaining to plant operations, such as for process planning (Pistikopoulos & Dua, 1998) and scheduling, one can obtain a complete map of all the optimal solutions. Moreover, as the operating conditions fluctuate, one does not have to re-optimize for the new set of conditions since the optimal solution as a function of parameters (or the new set of conditions) is already available. Mathematically, such problems can be posed as multi-parametric mixed-integer nonlinear programming problems of the following form:

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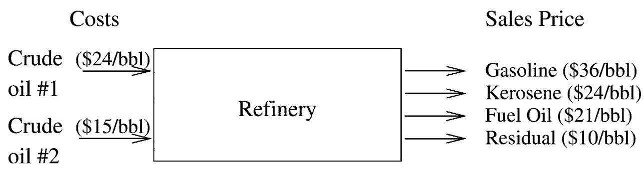

Fig. 1. Crude oil refinery.

$$z(\theta) = \min_{y,x} \; d^T y + f(x)$$
$$\text{s.t. } Ey + g(x) \le b + F\theta$$
$$\theta_{\min} \le \theta \le \theta_{\max}$$
$$x \in X \subseteq \mathcal{R}^n$$
$$y \in Y = \{0, 1\}^m$$
$$\theta \in \Theta \subseteq \mathcal{R}^s, \quad (1)$$

where y is vector of 0–1 binary variables, x a vector of continuous variables, f a scalar, continuously differentiable and convex function of x, g a vector of continuously differentiable and convex functions of x, b and d are constant vectors, E and F are constant matrices, θ is a vector of parameters, $\theta_{\min}$ and $\theta_{\max}$ are the vectors of lower and upper bounds on θ, and X and Θ are compact and convex polyhedral sets of dimensions n and s, respectively.

While the detailed theory and algorithms for solving Eq. (1) are presented in Dua and Pistikopoulos (1999, 2000), the engineering significance of solving Eq. (1) by using parametric programming techniques is highlighted in the next motivating example.

1.1. Example 1

Consider the refinery blending and production problem depicted in Fig. 1 (Edgar & Himmelblau, 1989). The objective is to maximize the profit for the operating conditions given in Table 1, where θ_1 and θ_2 are the parameters representing the additional maximum allowable production of gasoline and kerosene production, respectively. This results in a multi-parametric linear programming problem given in Table 2, where x_1 and x_2 are the flowrates of the crude oils, 1 and 2, respectively, in bbl/day and the units of profit are \$/day. This problem corresponds to a special case of (Eq. (1)) where no binary variables, y, are present and $f(x)$ and $g(x)$ are linear in x. The solution of this problem by using the algorithm of Gal and Nedoma (1972) is given in Table 3 and Fig. 2. The engineering significance of obtaining this solution is as follows.

Table 2
Refinery model

Profit	$= \max_x 8.1x_1 + 10.8x_2$
s.t.	$0.80x_1 + 0.44x_2 \le 24\,000 + \theta_1$
	$0.05x_1 + 0.10x_2 \le 2000 + \theta_2$
	$0.10x_1 + 0.36x_2 \le 6000$
	$x_1 \ge 0,\ x_2 \ge 0$
	$0 \le \theta_1 \le 6000$
	$0 \le \theta_2 \le 500$

Table 3
Solution of the refinery example

i	CR^i	Optimal solution
1	$-0.14\theta_1 + 4.21\theta_2 \le 896.55$	Profit $(\theta) = 4.66\theta_1 + 87.52\theta_2 + 286758.6$
	$0 \le \theta_1 \le 6000$	$x_1 = 1.72\theta_1 - 7.59\theta_2 + 26206.90$
	$0 \le \theta_2$	$x_2 = -0.86\theta_1 + 13.79\theta_2 + 6896.55$
2	$-0.14\theta_1 + 4.21\theta_2 \ge 896.55$	Profit $(\theta) = 7.53\theta_1 + 305409.84$
	$0 \le \theta_1 \le 6000$	$x_1 = 1.48\theta_1 + 24590.16$
	$\theta_2 \le 500$	$x_2 = -0.41\theta_1 + 9836.07$

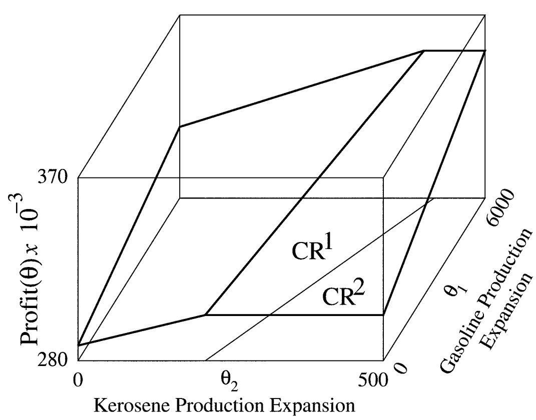

Fig. 2. Solution of refinery example.

Table 1
Refinery data

	Volume yield (%)		Maximum allowable production (bbl/day)
	Crude # 1	Crude # 2	
Gasoline	80	44	$24\,000 + \theta_1$
Kerosene	5	10	$2000 + \theta_2$
Fuel oil	10	36	6000
Residual	5	10	–
Processing cost (\$/bbl)	0.50	1.00	–

1. A complete map of all the optimal solutions, profit and crude oil flowrates as a function of θ_1 and θ_2, is available.
2. The space of θ_1 and θ_2 has been divided into two regions, CR^1 and CR^2, where the profiles of profit and flowrates of crude oils remain optimal and hence (a) one does not have to exhaustively enumerate the complete space of θ_1 and θ_2 and (b) the optimal solution can be obtained by simply substituting the value of θ_1 and θ_2 into the parametric profiles without any further optimization calculations.
3. The sensitivity of the profit to the parameters can be identified. In CR^1 the profit is more sensitive to θ_2, whereas in CR^2 it is not sensitive to θ_2 at all. Thus, for any value of θ that lies in CR^2, any expansion in kerosene production will not affect the profit.

This type of information is quite useful for solving reactive or on-line optimization problems. Such problems usually require a repetitive solution of optimization problems so as to compute the actions that must be taken at regular time intervals. This requirement comes from variations, such as demand fluctuations, during plant operation and to optimally control the plant under such dynamic behavior. In this work, we show that model predictive control (MPC) based on-line optimization problems can be reformulated as multi-parametric quadratic programming (mp-QP) problems, the solution of which is given by optimal plant input profiles as a function of variations. The on-line optimization problem thus reduces to a function evaluation problem where optimal inputs are computed by substituting the current level of variations into these profiles.

The rest of the paper is organized as follows. Section 2 shows how on-line optimization problems can be reformulated as mp-QP. An algorithm for the solution of mp-QP is also presented and is illustrated with an example in Section 3. A summary of work presented in this paper and some directions for future work are provided in Section 4.

2. On-line optimization

2.1. Introduction

The benefits of on-line optimization, from the point of view of costs and efficiency of operations, have long been recognized by process engineers. On-line optimization not only provides the maximum output from a given plant, but also takes into account various constraint violations while simultaneously considering the current state and history of the plant to predict future corrective actions. The benefits of on-line optimization are tremendous. Nevertheless, its application is rather restricted, considering its profit potential, primarily due to its large 'on-line' computational requirements, which involve a repetitive solution of an optimization problem at regular time intervals. This limitation is in spite of the significant advances in the computational power of the modern computers and in the area of on-line optimization over the past many years (Wright, 1997; Marlin & Hrymak, 1997; Engell, Kowalewski & Krogh, 1997). Thus, it is fair to state that an efficient implementation of on-line optimization tools relies on a quick and repetitive on-line computation of optimal control actions. In this work, we propose a parametric programming approach which avoids this repetitive solution. By using this approach the control variables are obtained as an explicit function of the state variables, and therefore on-line optimization breaks down to simple function evaluations, at regular time intervals, for the given state of the plant-to compute the corresponding control actions. This results in a very small computational effort in comparison to repetitively solving an optimization problem.

2.2. Model predictive control

MPC (Morari & Lee, 1999) has been widely adopted by industry to address on-line optimization problems with input and output constraints. MPC is based on the so called *receding horizon* philosophy—a sequence of future control actions is chosen according to a prediction of the future evolution of the system and applied to the plant until new measurements are available. Then, a new sequence is determined which replaces the previous one. Each sequence is evaluated by means of an optimization procedure which takes into account two objectives, optimize the tracking performance; and protect the system from possible constraint violations. In a mathematical framework, MPC problems can be formulated as follows.

Considering the state-space representation of a given process model:

$$\begin{cases} x(t+1) = Ax(t) + Bu(t) \\ \qquad y(t) = Cx(t), \end{cases} \tag{2}$$

subject to the following constraints:

$$y_{\min} \le y(t) \le y_{\max}$$
$$u_{\min} \le u(t) \le u_{\max}, \tag{3}$$

where $x(t) \in \mathcal{R}^n$, $u(t) \in \mathcal{R}^m$, and $y(t) \in \mathcal{R}^p$ are the state, input, and output vectors, respectively, subscripts min and max denote lower and upper bounds, respectively, and (A, B) is stabilizable. MPC problems for regulating to the origin can then be posed as the following optimization problem:

$$\min_U J(U, x(t))$$
$$= x'_{t+N_y|t} P x_{t+N_y|t} + \sum_{k=0}^{N_y-1} [x'_{t+k|t} Q x_{t+k|t} + u'_{t+k} R u_{t+k}]$$
$$\text{s.t.} \quad y_{\min} \leq y_{t+k|t} \leq y_{\max}, \quad k = 1, \ldots, N_c$$
$$u_{\min} \leq u_{t+k} \leq u_{\max}, \quad k = 0, 1, \ldots, N_c$$
$$x_{t|t} = x(t)$$
$$x_{t+k+1|t} = A x_{t+k|t} + B u_{t+k}, \quad k \geq 0$$
$$y_{t+k|t} = C x_{t+k|t}, \quad k \geq 0$$
$$u_{t+k} = K x_{t+k|t}, \quad N_u \leq k \leq N_y \tag{4}$$

where $U \triangleq \{u_t, \ldots, u_t + \mathcal{N}_{u-1}\}$ $Q = Q' \succsim 0$, $R = R' \succ 0$, $P \succsim 0$, $N_y \geq N_u$, and K is some feedback gain. Problem (Eq. (4)) is solved repetitively at each time t for the current measurement $x(t)$ and the vector of predicted state variables, $x_{t+1|t}, \ldots, x_{t+k|t}$ at time $t+1, \ldots, t+k$, respectively, and corresponding control actions $u_t, \ldots, u_{t+k-1}$ is obtained. In the next section, we present a parametric programming approach where the repetitive solution of Eq. (4) at each time interval is avoided and instead an optimization problem is solved only once. For a similar treatment for constrained linear quadratic regulation problems (Chisci & Zappa, 1999; Chmielewski & Manousiouthakis, 1996; Scokaert & Rawlings, 1998; Sznaier & Damborg, 1987) see Bemporad, Morari, Dua and Pistikopoulos (1999).

2.3. Multi-parametric quadratic programming

In the following paragraphs, we present a parametric programming approach which avoids the repetitive solution of Eq. (4). By making the following substitution in Eq. (4):

$$x_{t+k|t} = A^k x(t) + \textstyle\sum_{j=0}^{k-1} A^j B u_{t+k-1-j} \tag{5}$$

results in the following QP problem:

$$\min_U \frac{1}{2} U' H U + x'(t) F U + \frac{1}{2} x'(t) Y x(t)$$
$$\text{s.t.} \quad GU \leq W + Ex(t) \tag{6}$$

where $U \triangleq [u'_t, \ldots, u'_{t+N_u-1}]' \in \mathcal{R}^s$, $s \triangleq mN_u$, is the vector of optimization variables, $H = H' \succsim 0$, and H, F, Y, G, W, E are obtained from Q, R and Eqs. (4) and (5). The QP problem in Eq. (6) can now be formulated as the following mp-QP:

$$V_z(x) = \min_z \frac{1}{2} z' H z$$
$$\text{s.t} \quad Gz \leq W + Sx(t), \tag{7}$$

where $z \triangleq U + H^{-1} F' x(t)$, $z \in \mathcal{R}^s$, represents the vector of optimization variables, $S \triangleq E + GH^{-1}F'$ and x represents the vector of parameters. Note that x in Eq. (6) is present in the objective function and on the right hand side (RHS) of the constraints, whereas it is present only on the RHS in Eq. (7). The main advantage of writing Eq. (4) in the form given in Eq. (7) is that z (and therefore U) can be obtained as an affine function of x for the complete feasible space of x. To derive these results, we first state the following theorem (see also Zafiriou, 1990).

Theorem 1. *For the problem in* Eq. (7) *let x_0 be a vector of parameter values and (z_0, λ_0) a KKT pair, where $\lambda_0 = \lambda(x_0)$ is a vector of nonnegative Lagrange multipliers, λ, and $z_0 = z(x_0)$ is feasible in* Eq. (7). *Also assume that (i) linear independence constraint qualification (LICQ) and (ii) strict complementary slackness conditions hold. Then,*

$$\begin{bmatrix} z(x) \\ \lambda(x) \end{bmatrix} = -(M_0)^{-1} N_0 (x - x_0) + \begin{bmatrix} z_0 \\ \lambda_0 \end{bmatrix} \tag{8}$$

where,

$$M_0 = \begin{pmatrix} H & G_1^T & \cdots & G_q^T \\ -\lambda_1 G_1 & -V_1 & & \\ \vdots & & \ddots & \\ -\lambda_p G_q & & & -V_q \end{pmatrix}$$

$$N_0 = (Y, \lambda_1 S_1, \ldots, \lambda_p S_p)^T$$

where G_i denotes the ith row of G, S_i denotes the ith row of S, $V_i = G_i z_0 - W_i - S_i x_0$, W_i denotes the ith row of W and Y is a null matrix of dimension $(s \times n)$.

Proof 1. See Appendix A.

The set of x where this solution, Eq. (8), remains optimal is defined as the critical region (CR^0) and can be obtained as follows. Let CR^R represent the set of inequalities obtained (i) by substituting $z(x)$ into the inactive constraints in Eq. (7) and (ii) from the positivity of the Lagrange multipliers corresponding to the active constraints, as follows:

$$CR^R = \{\breve{G} z(x) \leq \breve{W} + \breve{S} x(t),\ \tilde{\lambda}(x) \geq 0\}, \tag{9}$$

where $\breve{}$ and $\tilde{}$ correspond to inactive and active constraints respectively as discussed in Appendix A and then CR^0 is obtained by removing the redundant constraints from CR^R as follows:

$$CR^0 = \Delta\{CR^R\}, \tag{10}$$

where Δ is an operator which removes the redundant constraints—for a procedure to identify the redundant constraints, see Gal (1995). Note that for simplicity in presentation, we use the notation CR to denote the set of points in X that lie in CR as well as to denote the set of inequalities which define CR'. Since for a given space of state-variables, X, so far we have characterized only a subset of X i.e. $CR^0 \subseteq X$, in the next step the rest of the region CR^{rest}, is obtained as follows:

$$CR^{rest} = X - CR^0, \tag{11}$$

by using the procedure described by Dua and Pistikopoulos (2000) (see Appendix B for details). The above steps, (Eqs. (8)–(11)) are repeated and a set of $z(x)$, $\lambda(x)$ and corresponding CR^0s is obtained. The solution procedure terminates when no more regions can be obtained, i.e. when $CR^{rest} = \emptyset$. For the regions which have the same solution and can be unified to give a convex region, such a unification is performed and a compact representation is obtained. The continuity and convexity properties of the optimal solution are summarized in the next theorem.

Theorem 2. *For the mp-QP problem*, (Eq. (7)), *the set of feasible parameters $X_f \subseteq X$ is convex*, *the optimal solution*, $z(x)$: $X_f \rightarrow \mathcal{R}^s$ *is continuous and piecewise affine*, *and the optimal objective function* $V_z(x)$: $X_f \rightarrow \mathcal{R}$ *is continuous*, *convex and piecewise quadratic*.

Table 4
mp-QP algorithm

Step 1	For a given space of x solve Eq. (7) by treating x as a free variable and obtain $[x_0]$
Step 2	In Eq. (7) fix $x = x_0$ and solve Eq. (7) to obtain $[z_0, \lambda_0]$
Step 3	Obtain $[z(x), \lambda(x)]$ from Eq. (8)
Step 4	Define CR^R as given in Eq. (9)
Step 5	From CR^R remove redundant inequalities and define the region of optimality CR^0 as given in Eq. (10)
Step 6	Define the rest of the region, CR^{rest}, as given in Eq. (11)
Step 7	If no more regions to explore, go to the next step, otherwise go to Step 1
Step 8	Collect all the solutions and unify a convex combination of the regions having the same solution to obtain a compact representation

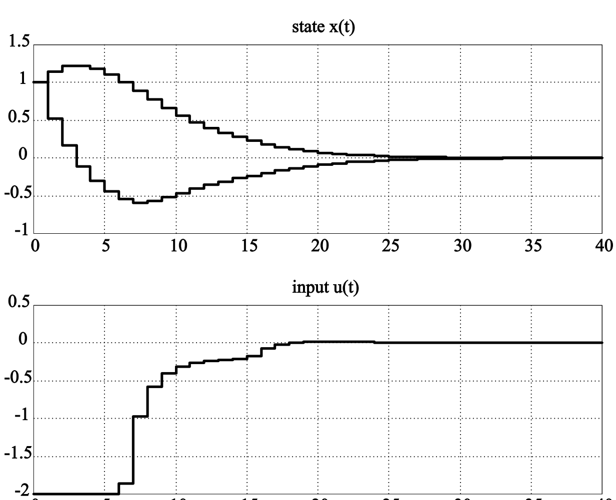

Fig. 3. Closed loop response.

Proof 2. See Appendix C.

Based upon the above theoretical developments, an algorithm for the solution of an mp-QP of the form given in Eq. (7) to calculate U as an affine function of x and characterize X by a set of polyhedral regions, CRs, has been developed which is summarized in Table 4.

This approach provides a significant advancement in the solution and on-line implementation of MPC problems, since its application results in a complete set of control actions as a function of state-variables (from Eq. (8)) and the corresponding regions of validity (from Eq. (10)), which are computed off-line, i.e. the explicit control law. Therefore during on-line optimization, no optimizer call is required and instead for the current state of the plant, the region, CR^0, where the value of the state variables is valid, can be identified by substituting the value of these state variables into the inequalities which define the regions. Then, the corresponding control actions can be computed by using a function evaluation of the corresponding affine function. In the next section, we present an example, to illustrate these concepts, and the worst case computational complexity of the mp-QP algorithm.

3. Numerical example and computational complexity

3.1. Example

Consider the following state-space representation:

$$\begin{cases} x(t+1) = \begin{bmatrix} 0.7326 & -0.0861 \\ 0.1722 & 0.9909 \end{bmatrix} x(t) + \begin{bmatrix} 0.0609 \\ 0.0064 \end{bmatrix} u(t) \\ y(t) = [0 \quad 1.4142] x(t) \end{cases}. \tag{12}$$

The constraints on input are as follows:

$$-2 \leq u(t) \leq 2 \tag{13}$$

The corresponding optimization problem of the form Eq. (4) for regulating to the origin is given as follows:

$$\min_{u_t, u_{t+1}} x'_{t+2|t} P x_{t+2|t} + \sum_{k=0}^{1} [x'_{t+k|t} x_{t+k|t} + .01 u^2_{t+k}]$$

$$\text{s.t.} \quad -2 \leq u_{t+k} \leq 2, \quad k = 0, 1$$

$$x_{t|t} = x(t) \tag{14}$$

where P solves the Lyapunov equation $P = A'PA + Q$,

$$Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \ R = 0.01, \ N_u = N_y = N_c = 2.$$

The closed-loop response from the initial condition $x(0) = [1 \ 1']$ is shown in Fig. 3. The same problem is

Table 5
Parametric solution of the numerical example

Region #	Region	U
1	$\begin{bmatrix} -5.9220 & -6.8883 \\ 5.9220 & 6.8883 \\ -1.5379 & 6.8291 \\ 1.5379 & -6.8291 \end{bmatrix} x \leq \begin{bmatrix} 2.0000 \\ 2.0000 \\ 2.0000 \\ 2.0000 \end{bmatrix}$	$[-5.9220 \quad -6.8883]x$
2, 4	$\begin{bmatrix} -3.4155 & 4.6452 \\ 0.1044 & 0.1215 \\ 0.1259 & 0.0922 \end{bmatrix} x \leq \begin{bmatrix} 2.6341 \\ -0.0353 \\ -0.0267 \end{bmatrix}$	2.0000
3	$\begin{bmatrix} 0.0679 & -0.0924 \\ 0.1259 & 0.0922 \end{bmatrix} x \leq \begin{bmatrix} -0.0524 \\ -0.0519 \end{bmatrix}$	2.0000
5	$\begin{bmatrix} -0.1259 & -0.0922 \\ -0.0679 & 0.0924 \end{bmatrix} x \leq \begin{bmatrix} -0.0519 \\ -0.0524 \end{bmatrix}$	2.0000
6	$\begin{bmatrix} -6.4159 & -4.6953 \\ -0.0275 & 0.1220 \\ 6.4159 & 4.6953 \end{bmatrix} x \leq \begin{bmatrix} 1.3577 \\ -0.0357 \\ 2.6423 \end{bmatrix}$	$[-6.4159 \quad -4.6953]x+0.6423$
7, 8	$\begin{bmatrix} 3.4155 & -4.6452 \\ -0.1044 & -0.1215 \\ -0.1259 & -0.0922 \end{bmatrix} x \leq \begin{bmatrix} 2.6341 \\ -0.0353 \\ -0.0267 \end{bmatrix}$	−2.0000
9	$\begin{bmatrix} 6.4159 & 4.6953 \\ 0.0275 & -0.1220 \\ -6.4159 & -4.6953 \end{bmatrix} x \leq \begin{bmatrix} 1.3577 \\ -0.0357 \\ 2.6423 \end{bmatrix}$	$[-6.4159 \quad -4.6953]x-0.6423$

now solved by using the parametric programming approach. The corresponding mp-QP problem of the form Eq. (7) has the following constant vectors and matrices.

$$H=\begin{bmatrix} 0.0196 & 0.0063 \\ 0.0063 & 0.0199 \end{bmatrix}, \quad F=\begin{bmatrix} 0.1259 & 0.0679 \\ 0.0922 & -0.0924 \end{bmatrix},$$

$$G=\begin{bmatrix} 1 & 0 \\ -1 & 0 \\ 0 & 1 \\ 0 & -1 \end{bmatrix}, \quad W=\begin{bmatrix} 2 \\ 2 \\ 2 \\ 2 \end{bmatrix}, \quad E=\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

The solution of the mp-QP problem as computed by using the algorithm given in Table 4 is provided in Table 5 and is depicted in Fig. 4. Note that the CRs 2, 4 and 7, 8 in Table 5 are combined together and a compact convex representation is obtained. To illustrate how on-line optimization reduces to a function evaluation problem, consider the starting point $x(0)=[1 \ 1']$. This point is substituted into the constraints defining the CRs in Table 5 and it satisfies only the constraints of $CR^{7,8}$ (see also Fig. 4). The control action corresponding to $CR^{7,8}$ from Table 5 is $u^{7,8}=-2$, which is obtained without any further optimization calculations and it is same as the one obtained from the closed loop response depicted in Fig. 3.

The same example is repeated with the additional constraint on the state

$$x_{t+k|t} \geq x_{\min}, \quad x_{\min} \triangleq \begin{bmatrix} -0.5 \\ -0.5 \end{bmatrix}, \quad k=1.$$

The closed-loop behavior from the initial condition $x(0)=[1 \ 1']$ is depicted in Fig. 5a. The MPC controller is given in Table 6. The polyhedral partition of the

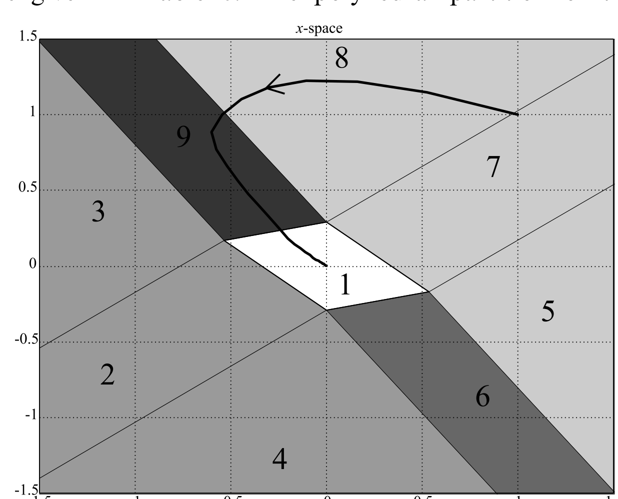

Fig. 4. Polyhedral partition of the state-space.

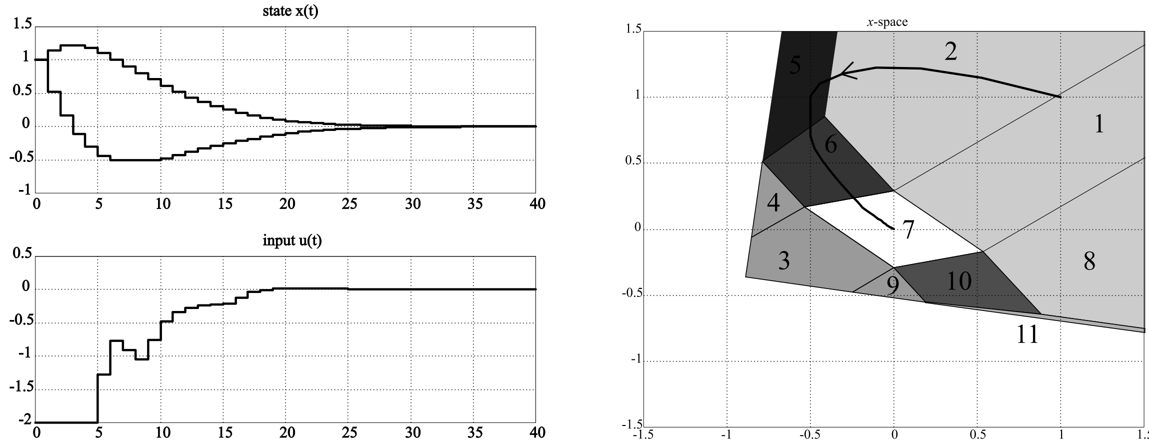

Fig. 5. Example with the additional constraint $x_{t+k|t} \geq -0.5$. (a) Closed loop MPC. (b) Polyhedral partition of the state-space and closed-loop MPC trajectories.

state-space corresponding to the modified MPC controller is depicted in Fig. 5b. The partition consists now of 11 regions. Note that there are feasible states smaller than $x_{\min}$, and vice versa, infeasible states $x \geq x_{\min}$. This is not surprising. For instance, the initial state $x(0) = [-0.6,\ 0]'$ is feasible for the MPC controller (which checks state constraints at time $t+k$, $k=1$), because there exists a feasible input such that $x(1)$ is within the limits. On the contrary, for $x(0) = [-0.47,\ -0.47]$ no feasible input is able to produce a feasible $x(1)$. Moreover, the union of the regions depicted in Fig. 5b should not be confused with the region of attraction of the MPC closed-loop. For instance, by starting at $x(0) = [46.0829,\ -7.0175]'$ (for which a feasible solution exists), the MPC controller runs into infeasibility after $t = 9$ time steps.

3.2. Computational complexity

The algorithm given in Table 4 solves an mp-QP by partitioning X in N_r convex polyhedral regions. This number N_r depends on the dimension n of the state, the product $s = mN_u$ of the number N_u of control moves and the dimension m of the input vector, and the number of constraints q in the optimization problem (Eq. (7)).

In an LP the optimum is reached at a vertex, and therefore s constraints must be active. In a QP the optimizer can lie everywhere in the admissible set. As the number of combinations of ℓ constraints out of a set of q is

$$\binom{q}{\ell} = q!/(q-\ell)!\ell!,$$

the number of possible combinations of active constraints at the solution of a QP is at most

$$\sum_{\ell=0}^{q} \binom{q}{\ell} = 2^q. \tag{15}$$

This number represents an upper-bound on the number of different linear feedback gains which describe the controller. In practice, far fewer combinations are usually generated as x spans X. Furthermore, the gains for the future input moves $u_{t+1}, \ldots, u_{t+N_u-1}$ are not relevant for the control law. Thus, several different combinations of active constraints may lead to the same first m components $u_t^*(x)$ of the solution. On the other hand, the number N_r of regions of the piecewise affine solution is in general larger than the number of feedback gains, because the regions have to be convex sets.

A worst case estimate of N_r can be computed from the way the algorithm in Table 4 generates critical regions CR to explore the set of parameters X. The following analysis does not take into account (i) the reduction of redundant constraints, and (ii) possible empty sets are not further partitioned. The first critical region CR_0 is defined by the constraints $\lambda(x) \geq 0$ (q constraints) and $Gz(x) \leq W + Sx$ (q constraints). If the strict complementary slackness condition holds, only q constraints can be active, and hence CR is defined by q constraints. From Appendix B, CR^{rest} consists of q convex polyhedra CR_i, defined by at most q inequalities. For each CR_i, a new CR is determined which consists of $2q$ inequalities (the additional q inequalities come from the condition $CR \subseteq CR_i$), and therefore the corresponding CR^{rest} partition includes $2q$ sets defined by $2q$ inequalities. As mentioned above, this way of generating regions can be associated with a search tree. By induction, it is easy to prove that at the tree level $k+1$ there are $k!m^k$ regions defined by $(k+1)q$ constraints. As observed earlier, each CR is the largest set corresponding to a certain combination of active con-

Table 6
Parametric solution of the numerical example for $x_{t+k/t} \geq -0.5$

Region #	Region	u
1, 2	$\begin{bmatrix} 3.4155 & -4.6452 \\ -0.1044 & -0.1215 \\ -0.7326 & 0.0861 \\ -0.1259 & -0.0922 \end{bmatrix} x \leq \begin{bmatrix} 2.6341 \\ -0.0353 \\ 0.3782 \\ -0.0267 \end{bmatrix}$	-2.0000
3, 9	$\begin{bmatrix} -3.4155 & 4.6452 \\ -0.7326 & 0.0861 \\ 0.1044 & 0.1215 \\ 0.1259 & 0.0922 \\ -0.1722 & -0.9909 \end{bmatrix} x \leq \begin{bmatrix} 2.6341 \\ 0.6218 \\ -0.0353 \\ -0.0267 \\ 0.5128 \end{bmatrix}$	2.0000
4	$\begin{bmatrix} -0.7326 & 0.0861 \\ 0.1259 & 0.0922 \\ 0.0679 & -0.0924 \end{bmatrix} x \leq \begin{bmatrix} 0.6218 \\ -0.0519 \\ -0.0524 \end{bmatrix}$	2.0000
5	$\begin{bmatrix} -12.0326 & 1.4142 \\ 12.0326 & -1.4142 \\ 1.8109 & -1.9698 \end{bmatrix} x \leq \begin{bmatrix} 10.2120 \\ -6.2120 \\ -2.4406 \end{bmatrix}$	$[-12.0326\ \ 1.4142]\ x - 8.2120$
6	$\begin{bmatrix} -6.4159 & -4.6953 \\ -0.3420 & 0.3720 \\ 6.4159 & 4.6953 \\ 0.0275 & -0.1220 \end{bmatrix} x \leq \begin{bmatrix} 2.6423 \\ 0.4609 \\ 1.3577 \\ -0.0357 \end{bmatrix}$	$[-6.4159\ \ -4.6953]x - 0.6423$
7	$\begin{bmatrix} -1.5379 & 6.8291 \\ -5.9220 & -6.8883 \\ 1.5379 & -6.8291 \\ 5.9220 & 6.8883 \end{bmatrix} x \leq \begin{bmatrix} 2.0000 \\ 2.0000 \\ 2.0000 \\ 2.0000 \end{bmatrix}$	$[-5.9220\ \ -6.8883]x$
8	$\begin{bmatrix} -0.1722 & -0.9909 \\ -0.1259 & -0.0922 \\ -0.0679 & 0.0924 \end{bmatrix} x \leq \begin{bmatrix} 0.4872 \\ -0.0519 \\ -0.0524 \end{bmatrix}$	-2.0000
10	$\begin{bmatrix} -0.1311 & -0.9609 \\ 6.4159 & 4.6953 \\ -0.0275 & 0.1220 \\ -6.4159 & -4.6953 \end{bmatrix} x \leq \begin{bmatrix} 0.5041 \\ 2.6423 \\ -0.0357 \\ 1.3577 \end{bmatrix}$	$[-6.4159\ \ -4.6953]x + 0.6423$
11	$\begin{bmatrix} -26.8936 & -154.7504 \\ 26.8936 & 154.7504 \\ 62.7762 & 460.0064 \end{bmatrix} x \leq \begin{bmatrix} 80.0823 \\ -76.0823 \\ -241.3367 \end{bmatrix}$	$[-26.8936\ \ -154.7504]x - 78.0823$

straints. Therefore, the search tree has a maximum depth of 2^q, as at each level there is one admissible combination less. In conclusion, the number of regions is $N_r \leq \Sigma_{k=0}^{2^q-1} k!q^k$, each one defined by at most $q2^q$ linear inequalities.

3.3. Computational time

In Tables 7 and 8, we report the computational time and the number of regions obtained by solving a few test MPC problems. In the comparison, we vary the

Table 7
Computational time to solve the mp-QP problem (seconds)

Free moves	States			
	2	3	4	5
2	3.02	4.12	5.05	5.33
3	10.44	26.75	31.7	70.19
4	25.27	60.20	53.93	58.61

Table 8
Number of regions N_r in the MPC solution

Free moves	States			
	2	3	4	5
2	7	7	7	7
3	17	47	29	43
4	29	99	121	127

number of free moves N_u and the number of poles of the open-loop system (and consequently the number of states x). Computational times have been evaluated by running the algorithm in Table 4 in Matlab 5.3 on a Pentium II-300 MHz machine. No attempts were made to optimize the efficiency of the algorithm and its implementation.

The theory and algorithm presented in this work are quite general and seem to have great potential for large scale, industrial applications. While the framework presented in this work may still require significant computational effort, most computations are executed off-line, while on-line implementation basically reduces to simple function evaluations. The suitability and applicability of the proposed parametric optimization based approach to large scale applications is a topic of current investigation.

4. Concluding remarks

In this work, we have presented a parametric programming approach for the solution of MPC based optimization problems. Parametric programming provides a complete map of the optimal solution as a function of the parameters, by partitioning the space of parameters into characteristic regions. In the context of on-line optimization, optimal control actions are computed off-line as a function of state variables, and the space of state variables is sub-divided into characteristic regions. On-line optimization is then carried-out by taking measurements from the plant, identifying the characteristic region corresponding to these measurements, and then calculating the control actions by simply substituting the values of the measurements into the expression for the control profile corresponding to the identified characteristic region. The on-line optimization problem thus reduces to a simple map-reading and function evaluation problem. The corresponding computational effort required by this kind of implementation is very small, as no optimizer is ever called on-line. Current research efforts focus on addressing hybrid control problems (Bemporad & Morari, 1999a) and robust MPC problems (Bemporad & Morari, 1999b) via multi-parametric mixed-integer programming techniques (Dua & Pistikopoulos, 1999, 2000).

Appendix A

A.1. Proof of Theorem 1

Theorem 3. (Fiacco, 1976) *for the problem in* Eq. (7) *and under the assumptions of* Theorem 1, *in neighborhood of* x_0, *there exists a unique, once continuously differentiable function* $[z(x), \lambda(x)]$ *where* $z(x)$ *is a unique isolated minimizer for* (Eq. (7)), *and*

$$\begin{pmatrix} dz(x_0)/dx \\ d\lambda(x_0)/dx \end{pmatrix} = -(M_0)^{-1}N_0, \tag{A.1}$$

where,

$$M_0 = \begin{bmatrix} H & G_1^T & \cdots & G_q^T \\ -\lambda_1 G_1 & -V_1 & & \\ \vdots & & \ddots & \\ -\lambda_q G_q & & & -V_q \end{bmatrix}$$

$$N_0 = (Y, \lambda_1 S_1, \ldots, \lambda_q S_q)^T$$

where G_i *denotes the ith row of* G, S_i *denotes the ith row of* S, $V_i = G_i z_0 - W_i - S_i x_0$, W_i *denotes the ith row of* W *and* Y *is a null matrix of dimension* $(s \times n)$.

Theorem 4. *For the problem in* Eq. (7) *and under the assumptions of* Theorem 1, z *and* λ *are affine functions of* x.

Proof . The first-order KKT conditions for the mp-QP are given by

$$Hz + G'\lambda = 0, \tag{A.2}$$

$$\lambda_i(G_i z - W_i - S_i x) = 0, \qquad i = 1,\ldots,q \tag{A.3}$$

$$\lambda \geq 0. \tag{A.4}$$

From Eq. (A.2)

$$z = -H^{-1}G'\lambda. \tag{A.5}$$

Let $\breve{\lambda}$ and $\tilde{\lambda}$ denote the Lagrange multipliers corresponding to inactive and active constraints, respectively. For inactive constraints, $\breve{\lambda} = 0$. For active constraints,

$$\tilde{G}z - \tilde{W} - \tilde{S}x = 0, \tag{A.6}$$

where $\tilde{G}, \tilde{W}, \tilde{S}$ correspond to the set of active constraints. From Eqs. (A.5) and (A.6),

$$\tilde{\lambda} = -(\tilde{G}H^{-1}\tilde{G}')^{-1}(\tilde{W} + \tilde{S}x) \tag{A.7}$$

Note that $(\tilde{G}H^{-1}\tilde{G}')^{-1}$ exists because of the LICQ assumption. Thus λ is an affine function of x. We can substitute $\tilde{\lambda}$ from Eq. (A.7) into Eq. (A.5) to obtain

$$z = H^{-1}\tilde{G}'(\tilde{G}H^{-1}\tilde{G}')^{-1}(\tilde{W} + \tilde{S}x) \tag{A.8}$$

and note that z is also an affine function of x.

Corollary 1. *From* Theorem 3 and Theorem 4

$$\begin{bmatrix} z(x) \\ \lambda(x) \end{bmatrix} = -(M_0)^{-1} N_0 (x - x_0) + \begin{bmatrix} z_0 \\ \lambda_0 \end{bmatrix}. \tag{A.9}$$

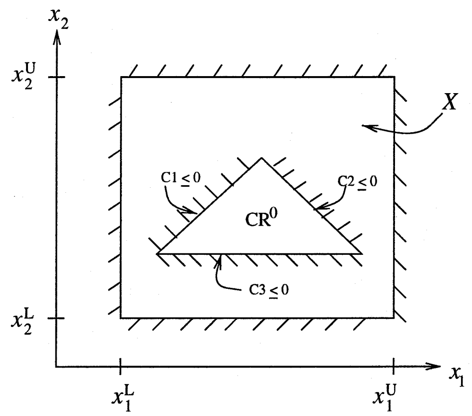

Fig. 6. Critical regions, X and CR^0.

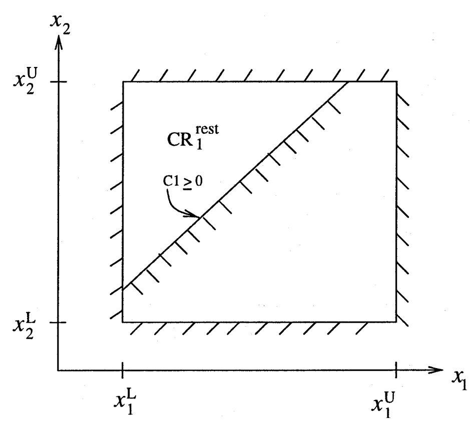

Fig. 7. Step 1.

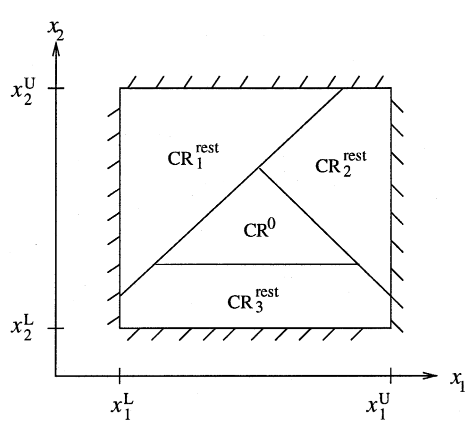

Fig. 8. Rest of the regions.

Appendix B

B.1. Definition of CR^{rest}

Given an initial region, X and a region of optimality, CR^0 such that $CR^0 \subseteq X$, a procedure is described in this section to define the rest of the region, $CR^{rest} = X - CR^0$. For the sake of simplifying the explanation of the procedure, consider the case when only two state-variables x_1 and x_2, are present (see Fig. 6), where X is defined by the inequalities: $\{x_1^L \le x_1 \le x_1^U,\ x_2^L \le x_2 \le x_2^U\}$ and CR^0 is defined by the inequalities: $\{C1 \le 0,\ C2 \le 0,\ C3 \le 0\}$ where $C1$, $C2$ and $C3$ are linear in x. The procedure consists of considering one-by-one the inequalities which define CR^0. Considering, for example, the inequality $C1 \le 0$, the rest of the region is given by, CR_1^{rest}: $\{C1 \ge 0,\ x_1^L \le x_1,\ x_2 \le x_2^U\}$, which is obtained by reversing the sign of inequality $C1 \le 0$ and removing redundant constraints in X (see Fig. 7). Thus, by considering the rest of the inequalities, the complete rest of the region is given by: $CR^{rest} = \{CR_1^{rest} \bigcup CR_2^{rest} \bigcup CR_3^{rest}\}$, where CR_1^{rest}, CR_2^{rest} and CR_3^{rest} are given in Table B.1 and are graphically depicted in Fig. 8. Note that for the case when X is unbounded, simply suppress the inequalities involving X in Table B.1.

Table B.1. Definition of rest of the regions

Region	Inequalities
CR_1^{rest}	$C1 \ge 0,\ x_1^L \le x_1,\ x_2 \le x_2^U$
CR_2^{rest}	$C1 \le 0,\ C2 \ge 0,\ x_1 \le x_1^U,\ x_2 \le x_2^U$
CR_3^{rest}	$C1 \le 0,\ C2 \le 0,\ C3 \ge 0,\ x_1^L \le x_1 \le x_1^U,\ x_2^L \le x_2$

Appendix C

C.1. Proof of Theorem 2

We first prove convexity of X_f and $V_z(x)$. Take generic x_1, $x_2 \in X_f$, and let $V_z(x_1)$, $V_z(x_2)$ and z_1, z_2 the corresponding optimal values and minimizers. Let $\alpha \in [0, 1]$, and define $z_\alpha \triangleq \alpha z_1 + (1-\alpha) z_2$, $x_\alpha \triangleq \alpha x_1 + (1-\alpha) x_2$. By feasibility, z_1, z_2 satisfy the constraints $Gz_1 \le W + Sx_1$, $Gz_2 \le W + Sx_2$. These inequalities can be linearly combined to obtain $Gz_\alpha \le W + Sx_\alpha$, and therefore z_α is feasible for the optimization problem (Eq. (7)) where $x(t) = x_\alpha$. Since a feasible solution, $z(x_\alpha)$, exists at x_α, an optimal solution exists at x_α and hence X_f is convex. The optimal solution at x_α will be less than or equal to the feasible solution, i.e. $V_z(x_\alpha) \le 1/2\ z'_\alpha H z_\alpha$, and hence

$$V_z(x_\alpha) - \frac{1}{2}[\alpha z_1' H z_1 + (1-\alpha) z_2' H z_2]$$

$$\leq \frac{1}{2} z_\alpha' H z_\alpha - \frac{1}{2}[\alpha z_1' H z_1 + (1-\alpha) z_2' H z_2]$$

$$= \frac{1}{2}[\alpha^2 z_1' H z_1 + (1-\alpha)^2 z_2' H z_2 + 2\alpha(1-\alpha) z_2' H z_1$$

$$- \alpha z_1' H z_1 - (1-\alpha) z_2' H z_2]$$

$$= -\frac{1}{2}\alpha(1-\alpha)(z_1 - z_2)' H (z_1 - z_2) \leq 0, \quad \text{(C.1)}$$

i.e. $V_z(\alpha x_1 + (1-\alpha)x_2) \leq \alpha V_z(x_1) + (1-\alpha) V_z(x_2)$, $\forall x_1$, $x_2 \in X$, $\forall \alpha \in [0, 1]$, which proves the convexity of $V_z(x)$ on X_f. Within the closed polyhedral regions CR^0 in X_f the solution $z(x)$ is affine (Eq. (A.8)). The boundary between two regions belongs to both closed regions. Because the optimum is unique the solution must be continuous across the boundary. The fact that $V_z(x)$ is continuous and piecewise quadratic follows trivially.

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