ELSEVIER

Computers and Chemical Engineering 24 (2000) 183–188

Computers & Chemical Engineering

www.elsevier.com/locate/compchemeng

On-line optimization via off-line parametric optimization tools

Efstratios N. Pistikopoulos [a,*], Vivek Dua [a], Nikolaos A. Bozinis [a], Alberto Bemporad [b], Manfred Morari [b]

[a] *Department of Chemical Engineering, Centre for Process Systems Engineering, Imperial College of Science Technology and Medicine, Prince Consort Road, London SW7 2BY, UK*

[b] *Automatic Control Laboratory, ETH Zentrum, ETL I 29, CH-8092 Zurich, Switzerland*

Abstract

In this paper, on-line optimization problems with a quadratic performance criteria and linear constraints are formulated as multi-parametric quadratic programs, where the input and state variables, corresponding to a plant, are treated as optimization variables and parameters, respectively. The solution of such problems is given by (i) a complete set of profiles of all the optimal inputs to the plant as a function of state variables, and (ii) the regions in the space of state variables where these functions remain optimal. It is shown that these profiles are linear and the corresponding regions are described by linear inequalities. An algorithm for obtaining these profiles and corresponding regions of optimality is also presented. The key feature of the proposed approach is that the on-line optimization problem is solved off-line via parametric programming techniques, hence, at each time interval (i) no optimization solver is called on-line, (ii) simple function evaluations are required for obtaining the optimal inputs to the plant for the current state of the plant.

Keywords: Real-time optimization; Model predictive control; Multi-parametric quadratic programming

1. Introduction

The benefits of on-line optimization, from the point of view of costs and efficiency of operations, have long been recognized by process engineers. On-line optimization not only provides the maximum output from a given plant, but also takes into account various constraint violations while simultaneously considering the current state and history of the plant to predict future corrective actions. While the benefits of on-line optimization are tremendous, its application is rather restricted, considering its profit potential, primarily due to its large 'on-line' computational requirements which involve a repetitive solution of an optimization problem at regular time intervals (see Fig. 1). This limitation is in spite of the significant advances in the computational power of the modern computers and in the area of on-line optimization over the past many years. Thus, it is fair to state that an efficient implementation of on-line optimization tools relies on a quick and repetitive on-line computation of optimal control actions. In this work, we propose a parametric programming approach, which avoids this repetitive solution. By using this approach the control variables are obtained as a function of the state variables, and therefore on-line optimization breaks down to simple function evaluations, at regular time intervals, for the given state of the plant — to compute the corresponding control actions. This results in a very small computational effort in comparison to repetitively solving an optimization problem. The rest of the paper is structured as follows. In the next section, the key concepts of on-line optimization are reviewed. Then on-line optimization problem is formulated as a parametric programming problem and an algorithm for its solution is presented. The solution steps and the engineering significance of the proposed approach are reviewed and highlighted via an illustrative example.

2. On-line optimization

Model predictive control (MPC) (Morari & Lee, 1999) has been widely adopted by industry to address

* Corresponding author. Tel.: +44-20-75946620; fax: +44-20-75946606.

E-mail address: e.pistikopoulos@ic.ac.uk (E.N. Pistikopoulos)

0098-1354/00/$ - see front matter
PII: S0098-1354(00)00510-X

on-line optimization problems with input and output constraints. MPC is based on the so called receding horizon philosophy, a sequence of future control actions is chosen according to a prediction of the future evolution of the system and applied to the plant until new measurements are available. Then, a new sequence is determined which replaces the previous one. Each sequence is evaluated by means of an optimization procedure which takes into account two objectives, optimize the tracking performance; and protect the system from possible constraint violations. In a mathematical framework, MPC problems can be formulated as follows.

Consider the following state-space representation of a given process model:

$$\begin{cases} x(t+1) = Ax(t) + Bu(t) \\ \quad y(t) = Cx(t), \end{cases} \tag{1}$$

subject to the following constraints:

$$\begin{aligned} &y_{\min} \le y(t) \le y_{\max} \\ &u_{\min} \le u(t) \le u_{\max}, \end{aligned} \tag{2}$$

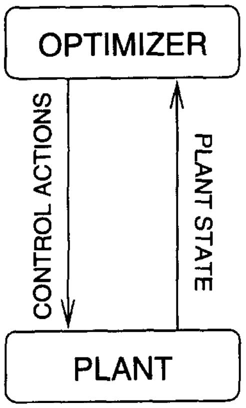

Fig. 1. On-line optimization.

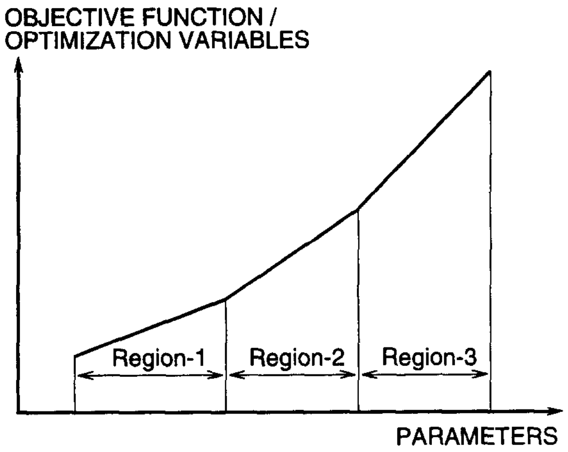

Fig. 2. Parametric programming.

where $x(t)\in\Re^n$, $u(t)\in\Re^m$, and $y(t)\in\Re^p$ are the state, input, and output vectors, respectively, subscripts min and max denote lower and upper bounds, respectively, and (A, B) is stabilizable. MPC problems for regulating to the origin can then be posed as the following optimization problems:

$$\begin{aligned} &\min_U J(U,x(t)) \\ &= x'_{t+N_y|t}Px_{t+N_y|t} + \sum_{k=0}^{N_y-1} x'_{t+k|t}Qx_{t+k|t} + u'_{t+k}Ru_{t+k} \\ &\text{s.t. } y_{\min} \le y_{t+k|t} \le y_{\max}, \quad k = 1, \ldots, N_c \\ &\quad u_{\min} \le u_{t+k} \le u_{\max}, \quad k = 0,1, \ldots, N_c \\ &\quad x_{t|t} = x(t) \\ &\quad x_{t+k+1|t} = Ax_{t+k|t} + Bu_{t+k}, \quad k \ge 0 \\ &\quad y_{t+k|t} = Cx_{t+k|t}, \quad k \ge 0 \\ &\quad u_{t+k} = Kx_{t+k|t}, \quad N_u \le k \le N_y \end{aligned} \tag{3}$$

where $U \triangleq \{ut, \ldots, u_{t+N_u-1}\}$, $Q = Q' \ge 0$, $R = R' > 0$, $P \ge 0$, $N_y \ge N_u$, and K is some feedback gain. The problem (Eq. (3)) is solved repetitively at each time t for the current measurement $x(t)$ and the vector of predicted state variables, $x_{t+1|t}, \ldots, x_{t+k|t}$ at time $t+1$, $\ldots$, $t+k$, respectively, and corresponding control actions $u_t, \ldots, u_{t+k-1}$ is obtained. In the next section, we present a parametric programming approach where the repetitive solution of Eq. (3) at each time interval is avoided and instead an optimization problem is solved only once. For a similar treatment for constrained linear quadratic regulation problems (Sznaier & Damborg, 1987; Chmielewski & Manousiouthakis, 1996; Scokaert & Rawlings, 1998; Chisci & Zappa, 1999) see Bemporad, Morari, Dua and Pistikopoulos (1999).

3. Multi-parametric quadratic programming

In an optimization framework, where the objective is to minimize or maximize a performance criterion subject to a given set of constraints and where some of the parameters in the optimization problem are uncertain, parametric programming is a technique for obtaining (i) the objective function and the optimization variables as a function of these parameters, and (ii) the regions in the space of the parameters where these functions are valid (Fiacco, 1983; Gal, 1995; Acevedo & Pistikopoulos, 1997; Papalexandri & Dimkou, 1998; Pertsinidis, Grossmann & McRae, 1998; Dua & Pistikopoulos, 1999) (see Fig. 2). The main advantage of using the parametric programming techniques to address such problems is that for problems pertaining to plant operations, such as for process planning (Pistikopoulos & Dua, 1998) and scheduling, one obtains a complete map of all the optimal solutions and as the operating conditions fluctuate, one does not have to re-optimize for the new set of conditions since the optimal solution as a function of parameters (or the new set of conditions) is already available.

In the following paragraphs, we present a parametric programming approach, which avoids a repetitive solution of Eq. (3). First, we do some algebraic manipulations to recast (Eq. (3)) in a form suitable for using and developing some new parametric programming concepts. By making the following substitution in Eq. (3):

$$x_{t+k|t} = A^k x(t) + \sum_{j=0}^{k-1} A^j B u_{t+k-1-j} \tag{4}$$

the objective $J(U, x(t))$ can be formulated as the following quadratic programming (QP) problem:

$$\begin{aligned}&\min_U \frac{1}{2} U'HU + x'(t)FU + \frac{1}{2} x'(t)Yx(t)\\ &\text{s.t. } GU \le W + Ex(t)\end{aligned} \tag{5}$$

where $U \triangleq [u'_t, \ldots, u'_{t+N_u-1}]' \in \Re^s$, $s \triangleq mN_u$, is the vector of optimization variables, $H = H' \succ 0$, and H, F, Y, G, W, E are obtained from Q, R and Eqs. (3) and (4). The QP problem (Eq. (5)) can now be formulated as the following multi-parametric quadratic program (mp-QP):

$$\begin{aligned}\mu(x) = &\min_z \frac{1}{2} z'Hz\\ &\text{s.t. } Gz \le W + Sx(t),\end{aligned} \tag{6}$$

where $z \triangleq U + H^{-1}F'x(t)$, $z \in \Re^s$ represents the vector of optimization variables, $S \triangleq E + GH^{-1}F'$ and x represents the vector of parameters. The main advantage of writing Eq. (3) in the form given in Eq. (6) is that z (and therefore U) can be obtained as an affine function of x for the complete feasible space of x. To derive these results, we first state the following theorem (see also Zafiriou (1990)).

Theorem 1. *For the problem in* Eq. (6) *let x_0 be a vector of parameter values and (z_0, λ_0) a KKT pair, where $\lambda_0 = \lambda(x_0)$ is a vector of non-negative Lagrange multipliers, λ, and $z_0 = z(x_0)$ is feasible in* Eq. (6). *Also assume that (i) linear independence constraint qualification and (ii) strict complementary slackness conditions hold. Then,*

$$\begin{bmatrix} z(x) \\ \lambda(x) \end{bmatrix} = -(M_0)^{-1} N_0 (x - x_0) + \begin{bmatrix} z_0 \\ \lambda_0 \end{bmatrix} \tag{7}$$

where,

$$M_0 = \begin{pmatrix} H & G_1^T & \cdots & G_q^T \\ -\lambda_1 G_1 & -V_1 & & \\ \vdots & & \ddots & \\ -\lambda_p G_q & & & -V_q \end{pmatrix}$$

$$N_0 = (Y, \lambda_1 S_1, \ldots, \lambda_p S_p)^T$$

where G_i denotes the ith row of G, S_i denotes the ith row of S, $V_i = G_i z_0 - W_i - S_i x_0$, W_i denotes the ith row of W and Y is a null matrix of dimension $(s \times n)$.

The set of x where this solution, (Eq. (7)), remains optimal is defined as the critical region (CR^0) and can be obtained as follows. Let CR^R represent the set of inequalities obtained (i) by substituting $z(x)$ into the inequalities in Eq. (6), and (ii) from the positivity of the Lagrange multipliers, as follows:

$$CR^R = \{Gz(x) \le W + Sx(t), \lambda(x) \ge 0\}, \tag{8}$$

then CR^0 is obtained by removing the redundant constraints from CR^R as follows:

$$CR^0 = \Delta\{CR^R\}, \tag{9}$$

where Δ is an operator which removes the redundant constraints — for a procedure to identify the redundant constraints, see Gal (1995). Since for a given space of state variables, X, so far we have characterized only a subset of X i.e. $CR^0 \subseteq X$, in the next step the rest of the region CR^{rest}, is obtained as follows (Dua & Pistikopoulos, 1999):

$$CR^{rest} = X - CR^0. \tag{10}$$

The above steps, (Eqs. (7)–(10)) are repeated and a set of $z(x)$, $\lambda(x)$ and corresponding CR^0s are obtained. The solution procedure terminates when no more regions can be obtained, i.e. when $CR^{rest} = \emptyset$. For the regions, which have the same solution and can be unified to give a convex region, such unification is performed and a compact representation is obtained. The continuity and convexity properties of the optimal solution are summarized in the next theorem.

Theorem 2. *For the mp-QP problem,* Eq. (6), *the set of feasible parameters $X_f \subseteq X$ is convex, the optimal solution, $z(x)$: $X_f \mapsto \Re^s$ is continuous and piecewise affine, and the optimal objective function $\mu(x)$: $X_f \mapsto \Re$ is continuous, convex and piecewise quadratic.*

Based upon the above theoretical developments, an algorithm for the solution of a mp-QP of the form given in Eq. (6) to calculate U as an affine function of x and characterize X by a set of polyhedral regions, CRs, has been developed which is summarized in Table 1.

This approach provides a significant advancement in the solution and on-line implementation of MPC problems. Since its application results in a complete set of control actions as a function of state variables (from

Table 1
Solution steps of the algorithm

Step 1	For a given space of x solve Eq. (6) by treating x as a free variable and obtain $[x_0]$
Step 2	In Eq. (6) fix $x = x_0$ and solve Eq. (6) to obtain $[z_0, \lambda_0]$
Step 3	Obtain $[z(x), \lambda(x)]$from Eq. (7)
Step 4	Define CR^R as given in Eq. (8)
Step 5	From CR^R remove redundant inequalities and define the region of optimality CR^0 as given in Eq. (9)
Step 6	Define the rest of the region, CR^{rest}, as given in Eq. (10)
Step 7	If no more regions to explore, go to the next step, otherwise go to step 1
Step 8	Collect all the solutions and unify a convex combination of the regions having the same solution to obtain a compact representation

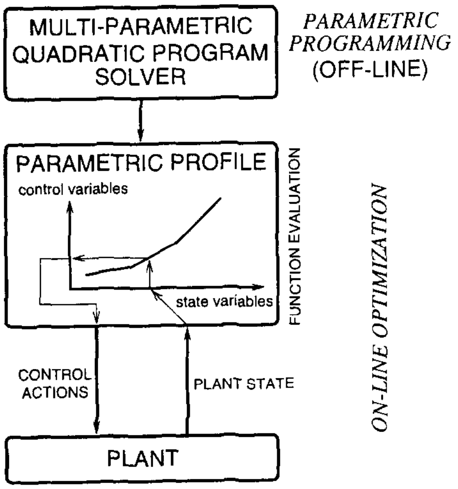

Fig. 3. On-line optimization via parametric programming.

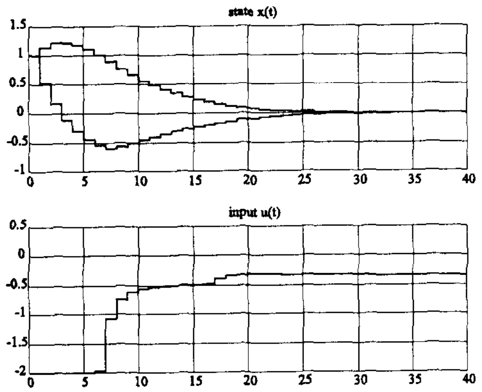

Fig. 4. Closed-loop response.

Eq. (7)) and the corresponding regions of validity (from Eq. (9)), which are computed off-line. Therefore, during on-line optimization, no optimizer needs to be called and instead for the current state of the plant, the region, CR^0, where the value of the state variables is valid, can be identified by substituting the value of these state variables into the inequalities, which define the regions. Then, the corresponding control actions can be computed by using a function evaluation of the corresponding affine function (see Fig. 3). In the next section, we present an example to illustrate these concepts.

4. Numerical example

Consider the following state-space model representation:

$$\begin{cases} x(t+1) = \begin{bmatrix} 0.7326 & -0.0861 \\ 0.1722 & 0.9909 \end{bmatrix} x(t) + \begin{bmatrix} 0.0609 \\ 0.0064 \end{bmatrix} u(t) \\ y(t) = [0 \quad 1.4142] x(t) \end{cases} \tag{11}$$

together with the following constraints:

$$-2 \leq u(t) \leq 2 \tag{12}$$

The corresponding optimization problem of the form (Eq. (3)) for regulating to the origin is given as follows:

$$\min_{u_t, u_{t+1}} \; x'_{t+2|t} P x_{t+2|t} + \sum_{k=0}^{1} x'_{t+k|t} x_{t+k|t} + 0.01 u^2_{t+k} \tag{13}$$

$$\text{s.t.} \quad -2 \leq u_{t+k} \leq 2, \quad k = 0,1$$

$$x_{t|t} = x(t)$$

where P solves the Lyapunov equation $P = A'PA + Q$, $Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $R = 0.01$, $N_u = N_y = N_c = 2$. The closed-loop response from the initial condition $x(0) = [1 \quad 1]'$ is shown in Fig. 4. The same problem is now solved by using the parametric programming approach. The corresponding mp-QP problem of the form (Eq. (6)) has the following constant vectors and matrices.

$$H = \begin{bmatrix} 0.0196 & 0.0063 \\ 0.0063 & 0.0199 \end{bmatrix}, \quad F = \begin{bmatrix} 0.1259 & 0.0679 \\ 0.0922 & -0.0924 \end{bmatrix},$$

$$G = \begin{bmatrix} 1 & 0 \\ -1 & 0 \\ 0 & 1 \\ 0 & -1 \end{bmatrix}, \quad W = \begin{bmatrix} 2 \\ 2 \\ 2 \\ 2 \end{bmatrix},$$

$$E = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

The solution of the mp-QP problem as computed by using the algorithm given in Table 1 is provided in Table 2 and is depicted in Fig. 5. Note that the CRs 2, 4 and 7, 8 in Table 2 are combined together and a compact convex representation is obtained. To illustrate how on-line optimization reduces to a function evaluation problem, consider the starting point $x(0) = [1 \quad 1]'$. This point is substituted into the constraints defining the CRs in Table 2 and it satisfies only the constraints of $CR^{7,\,8}$ (see also Fig. 5). The control action corresponding to $CR^{\,7,\,8}$ from Table 2 is $u^{7,\,8} = -2$, which is obtained without any further optimization calculations and it is same as the one obtained from the closed-loop response depicted in Fig. 4.

Table 2
Parametric solution of the numerical example

Region #	Parametric solutions
1	$CR^1: \begin{bmatrix} -5.9220 & -6.8883 \\ 5.9220 & 6.8883 \\ -1.5379 & 6.8291 \\ 1.5379 & -6.8291 \end{bmatrix} x \le \begin{bmatrix} 2.0000 \\ 2.0000 \\ 2.0000 \\ 2.0000 \end{bmatrix}$ $u^1 = [-5.9220 \quad 6.8883]x$
2, 4	$CR^{2,\,4}: \begin{bmatrix} -3.4155 & 4.6452 \\ 0.1044 & 0.1215 \\ 0.1259 & 0.0922 \end{bmatrix} x \le \begin{bmatrix} 2.6341 \\ -0.0353 \\ -0.0267 \end{bmatrix}$ $u^{2,\,4} = -2.0000$
3	$CR^3: \begin{bmatrix} 0.0679 & -0.0924 \\ 0.1259 & 0.0922 \end{bmatrix} x \le \begin{bmatrix} -0.0524 \\ 0.0519 \end{bmatrix}$ $u^3 = -2.0000$
5	$CR^5: \begin{bmatrix} -0.1259 & -0.0922 \\ -0.0679 & 0.0924 \end{bmatrix} x \le \begin{bmatrix} -0.0519 \\ -0.0524 \end{bmatrix}$ $u^5 = -2.0000$
6	$CR^6: \begin{bmatrix} -6.4159 & -4.6953 \\ -0.0275 & 0.1220 \\ 6.4159 & 4.6953 \end{bmatrix} x \le \begin{bmatrix} 1.3577 \\ -0.0357 \\ 2.6423 \end{bmatrix}$ $u^6 = [-6.4159 \quad -4.6953]x + 0.6423$
7, 8	$CR^{7,\,8}: \begin{bmatrix} 3.4155 & -4.6452 \\ -0.1044 & -0.1215 \\ -0.1259 & -0.0922 \end{bmatrix} x \le \begin{bmatrix} 2.6341 \\ -0.0353 \\ -0.0267 \end{bmatrix}$ $u^{7,\,8} = -2.0000$
9	$CR^9: \begin{bmatrix} 6.4159 & 4.6953 \\ 0.0275 & -0.1220 \\ -6.4159 & -4.6953 \end{bmatrix} x \le \begin{bmatrix} 1.3577 \\ -0.0357 \\ 2.6423 \end{bmatrix}$ $u^9 = [-6.4159 \quad -4.6953]x - 0.6423$

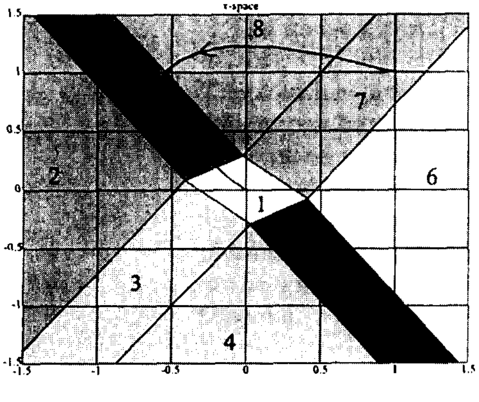

Fig. 5. Polyhedral partition of the state-space.

5. Concluding remarks

In this work, we have presented a parametric programming approach for the solution of on-line optimization problems. Parametric programming provides a complete map of the optimal solution as a function of the parameters, by partitioning the space of parameters into characteristic regions. In the context of on-line optimization, optimal control actions are computed off-line as a function of state variables, and the space of state variables is sub-divided into characteristic regions. On-line optimization is then carried-out by taking measurements from the plant, identifying the characteristic region corresponding to these measurements, and then calculating the control actions by simply substituting the values of the measurements into the expression for the control profile corresponding to the identified characteristic region. The on-line optimization problem thus reduces to a simple map-reading and function evaluation problem. The computational effort required by this kind of implementation is very small, as no optimizer is ever called on-line. Current research efforts focus on addressing hybrid control problems (Bemporad & Morari, 1999) via multi-parametric mixed-integer programming techniques (Dua & Pistikopoulos, 1999).

References

Acevedo, J., & Pistikopoulos, E. N. (1997). A multi-parametric programming approach for linear process engineering problems under uncertainty. *Industrial Engineering & Chemistry Research*, *36*, 717–728.

Bemporad, A., & Morari, M. (1999). Control of systems integrating logic, dynamics, and constraints. *Automatica*, *35*(3), 407–427.

Bemporad, A., Morari, M., Dua, V., & Pistikopoulos, E. N. (1999). *The explicit linear quadratic regulator for constrained systems.* Technical Report AUT99-16, Automatic Control Lab, ETH Zurich, http://control.ethz.ch.

Chisci, L., & Zappa, G. (1999). Fast algorithm for a constrained infinite horizon LQ problem. *International Journal on Control*, *72*(11), 1020–1026.

Chmielewski, D., & Manousiouthakis, V. (1996). On constrained infinite-time linear quadratic optimal control. *Systems & Control Letters*, *29*(3), 121–130.

Dua, V., & Pistikopoulos, E. N. (1999). Algorithms for the solution of multiparametric mixed-integer nonlinear optimization problems. *Industrial Engineering & Chemistry Research*, *38*, 3976–3987.

Fiacco, A. V. (1983). *Introduction to sensitivity and stability analysis in nonlinear programming*. New York, USA: Academic Press.

Gal, T. (1995). *Postoptimal analyses, parametric programming, and related topics*. New York: De Gruyter.

Morari, M., & Lee, J. H. (1999). Model predictive control: past, present and future. *Computers & Chemical Engineering*, *23*, 667–682.

Papalexandri, K. P., & Dimkou, T. I. (1998). A parametric mixed-integer optimization algorithm for multiobjective engineering problems involving discrete decisions. *Industrial Engineering & Chemistry Research*, *37*(5), 1866–1882.

Pertsinidis, A., Grossmann, I. E., & McRae, G. J. (1998). Parametric optimization of MILP programs and a framework for the parametric optimization of MINLPs. *Computers & Chemical Engineering*, *22*, S205.

Pistikopoulos, E. N., & Dua, V. (1998). Planning under uncertainty: a parametric optimization approach. In J. F. Pekny, & G. E. Blau, *Proceedings of the third international conference on foundations of computer-aided process operations* (pp. 164–169).

Scokaert, P. O. M., & Rawlings, J. B. (1998). Constrained linear quadratic regulation. *IEEE Transactions on Automatic Control*, *43*(8), 1163–1169.

Sznaier, M., & Damborg, M. J. (1987). Sub-optimal control of linear systems with state and control inequality constraints. In *Proceedings of the 26th IEEE conference on decision and control*, *vol. 1* (pp. 761–762).

Zafiriou, E. (1990). Robust model predictive control of processes with hard constraints. *Computers & Chemical Engineering*, *14*(4/5), 359–371.