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Brief paper

Squaring the circle: An algorithm for generating polyhedral invariant sets from ellipsoidal ones☆

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Abstract

This paper presents a new (geometrical) approach to the computation of *polyhedral* (robustly) positively invariant (PI) sets for general (possibly discontinuous) nonlinear discrete-time systems possibly affected by disturbances. Given a β-contractive ellipsoidal set $\mathscr{E}$, the key idea is to construct a polyhedral set that lies between the ellipsoidal sets $\beta\mathscr{E}$ and $\mathscr{E}$. A proof that the resulting polyhedral set is contractive and thus, PI, is given, and a new algorithm is developed to construct the desired polyhedral set. The problem of computing polyhedral invariant sets is formulated as a number of quadratic programming (QP) problems. The number of QP problems is guaranteed to be finite and therefore, the algorithm has finite termination. An important application of the proposed algorithm is the computation of polyhedral terminal constraint sets for model predictive control based on quadratic costs.

Keywords: Positively invariant sets; Contractive sets; Model predictive control; Stability; Robust stability

1. Introduction

Positively invariant (PI) sets and contractive sets have been used in many control theoretic problems, such as the synthesis of stabilizing controllers and the computation of domains of attraction, e.g., see Kolmanovsky and Gilbert (1998) and Blanchini (1999) for comprehensive overviews. In particular, PI sets play a very important role in the design of stabilizing model predictive controllers (MPC). For example, the terminal cost and constraint set approach in MPC (Mayne, Rawlings, Rao, & Scokaert, 2000) requires that the terminal set is PI under some appropriate local feedback law. The most utilized types of invariant sets are the *ellipsoidal* ones, which have a simple representation, but can be less flexible than *polyhedral* invariant sets, which can be arbitrarily complex. Polyhedral invariant sets are preferred in various cases due to the fact that they are often derived from physical constraints on state and control variables, which makes them a better approximation of domains of attraction. Moreover, a polyhedral set is often more suitable for usage in an optimization problem. For instance, in case of *MPC based on quadratic costs*, to guarantee recursive feasibility of the MPC optimization problem and stability one often constrains the terminal state to a *terminal set*. This set can be naturally chosen as an ellipsoidal sublevel set of a constructed (local) quadratic Lyapunov function, which is needed as terminal cost for the MPC algorithm. However, if an ellipsoidal set is used as the terminal set, then the MPC optimization problem becomes a *quadratically constrained* quadratic programming (QCQP) problem in case linear prediction models are used (or a mixed integer QCQP problem, if piecewise affine prediction models are used), which is usually not

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tackled by standard solvers. Note that QCQP problems cannot be solved by QP solvers, but rather require semi-definite programming solvers (Cannon, Kouvaritakis, & Rossiter, 2001), which are computationally much more demanding. In Lobo, Vandenberghe, Boyd, and Lebret (1998), the authors show how to reduce a QCQP problem into a second order cone program, which can be solved via a primal-dual interior-point method (Nesterov & Nemirovsky, 1994).

If a *polyhedral* invariant set is employed instead, then the MPC optimization problem is a standard QP (or mixed integer QP) problem. Since most MPC algorithms with an a priori stability guarantee are based on quadratic costs, e.g., see the survey (Mayne et al., 2000) for an overview, a lot of effort has been put in developing new approaches for computing *polyhedral* PI sets, see, for example, Raković, Mayne, Kerrigan, and Kouramas (2005), Pluymers, Rossiter, Suykens, and De Moor (2005), and Lazar, Heemels, Weiland, and Bemporad (2006).

In this paper we consider the problem of constructing a polyhedral PI set for discrete-time systems when an ellipsoidal one is already available, which is the case for MPC based on quadratic costs, as mentioned before. Given a β-contractive[1] ellipsoidal set $\mathscr{E}$, the key idea is to construct a polyhedral set that lies between the ellipsoidal sets $\beta\mathscr{E}$ and $\mathscr{E}$. We prove that the resulting polyhedral set is contractive and thus, PI. Next, the problem of fitting a polyhedral set between two ellipsoidal sets is solved by treating the ellipsoidal sets as sublevel sets of quadratic functions and constructing a piecewise affine (PWA) function that approximates the "outer" quadratic function well enough. A solution to the original problem is then obtained by retrieving a suitable sublevel set of the resulting PWA function. One of the advantages of the proposed algorithm is that it requires the solution of a finite number of QP problems, which guarantees finite termination. Also, due to its unique geometrical approach, which is *independent of the system dynamics*, the method is applicable to a wide class of systems, including linear systems affected by disturbances or subject to input saturation, switched linear systems under arbitrary switching and piecewise linear systems defined on conical regions in the state-space.

The rest of the paper is organized as follows. After introducing some basic notions in Section 1.1, the problem statement and the proposed solution are presented in Section 2. The algorithm for constructing the desired polyhedral set is given in Section 3 together with computational complexity aspects. Two examples are presented in Section 4 to illustrate the potential and the wide applicability of the method. Conclusions are summarized in Section 5.

1.1. Notation and basic definitions

Let $\mathbb{R}$, $\mathbb{R}_+$, $\mathbb{Z}$ and $\mathbb{Z}_+$ denote the field of real numbers, the set of nonnegative reals, the set of integers and the set of nonnegative integers, respectively. For a set $\mathscr{S} \subseteq \mathbb{R}^n$, we denote by $\partial\mathscr{S}$ the boundary, by $\mathrm{int}(\mathscr{S})$ the interior and by $\mathrm{cl}(\mathscr{S})$ the closure. For any real $\lambda \geqslant 0$, the set $\lambda\mathscr{S}$ is defined as $\{x \in \mathbb{R}^n \mid x = \lambda y \text{ for some } y \in \mathscr{S}\}$. The notation $\mathrm{Co}(\mathscr{S})$ denotes the convex hull of $\mathscr{S} \subseteq \mathbb{R}^n$. Similarly, for a set of points $\theta_0, \ldots, \theta_n$ of $\mathbb{R}^n$, $\mathrm{Co}(\theta_0, \ldots, \theta_n)$ denotes their convex hull, i.e.

$$\mathrm{Co}(\theta_0, \ldots, \theta_n) \triangleq \left\{ x \in \mathbb{R}^n \,\middle|\, x = \sum_{l=0}^{n} \mu_l \theta_l,\ \sum_{l=0}^{n} \mu_l = 1, \right. \left. \mu_l \geqslant 0 \text{ for } l = 0, 1, \ldots, n \right\}.$$

We call points $\theta_0, \ldots, \theta_k \in \mathbb{R}^n$ *affinely independent*, if $(1\theta_0^\top)^\top, \ldots, (1\theta_k^\top)^\top$ are linearly independent in $\mathbb{R}^{k+1}$. A simplex in $\mathbb{R}^n$ is defined as the convex hull of $(n+1)$ affinely independent points of $\mathbb{R}^n$. A polyhedron (or a polyhedral set) is a set obtained as the intersection of a finite number of open and/or closed half-spaces. A piecewise polyhedral set is the union of a finite number of polyhedra. The sets $\Omega_1, \ldots, \Omega_N$ form a polyhedral partition of $\mathbb{R}^n$, if $\Omega_i \subset \mathbb{R}^n$ is a polyhedron (not necessarily closed) for all $i = 1, \ldots, N$, $\Omega_i \cap \Omega_j = \emptyset$ for $i \neq j$, $\mathrm{int}(\Omega_i) \neq \emptyset$ for all $i = 1, \ldots, N$, and $\bigcup_{i=1,\ldots,N} \Omega_i = \mathbb{R}^n$.

A function $f : \mathbb{R}^n \to \mathbb{R}$ is a *quadratic function* if $f(x) \triangleq x^\top P x + Cx + \alpha$ for some $P \in \mathbb{R}^{n\times n}$, $C \in \mathbb{R}^{1\times n}$ and $\alpha \in \mathbb{R}$. A quadratic function f is *strictly convex* if and only if $P > 0$. A function $f : \mathbb{R}^n \to \mathbb{R}$ is called a *piecewise quadratic* (PWQ) *function* if there exists a polyhedral partition $\Omega_1, \ldots, \Omega_N$ of $\mathbb{R}^n$ such that $f(x) = x^\top P_i x + C_i x + \alpha_i$ when $x \in \Omega_i$, $i = 1, \ldots, N$. A function is called a PWA *function*, if there exists a polyhedral partition $\Omega_1, \ldots, \Omega_N$ of $\mathbb{R}^n$ such that $\bar{f} : \mathbb{R}^n \to \mathbb{R}$ with $\bar{f}(x) = H_i x + a_i$ when $x \in \Omega_i$, for some $H_i \in \mathbb{R}^{1\times n}$, $a_i \in \mathbb{R}$, $i = 1, \ldots, N$.

An *ellipsoid* (or an *ellipsoidal set*) $\mathscr{E}$ is defined as a sublevel set (corresponding to some constant level $f_0 \in \mathbb{R}_+$) of a strictly convex quadratic function, i.e. $\mathscr{E} \triangleq \{x \in \mathbb{R}^n \mid f(x) \leqslant f_0\}$. A *piecewise ellipsoidal set* is defined in this paper as a sublevel set of a piecewise quadratic function with matrices $P_i > 0$ for all $i = 1, \ldots, N$. Note that the sublevel set of PWA function is a piecewise polyhedral set.

2. Problem statement and proposed solution

Consider the discrete-time perturbed nonlinear system

$$x_{k+1} = G(x_k, w_k, v_k), \quad k \in \mathbb{Z}_+, \tag{1}$$

where $x_k \in \mathbb{R}^n$, $w_k \in \mathbb{W} \subset \mathbb{R}^p$, and $v_k \in \mathbb{V} \subset \mathbb{R}^q$ are the state, an unknown *parametric uncertainty* and an unknown *disturbance input*, respectively, and $\mathbb{W}$ and $\mathbb{V}$ are known, bounded sets. $G : \mathbb{R}^n \times \mathbb{R}^p \times \mathbb{R}^q \to \mathbb{R}^n$ is an arbitrary, possibly discontinuous, nonlinear function. We assume that the origin is an equilibrium of (1) for zero disturbance input, meaning that $G(0, w, 0) = 0$ for all $w \in \mathbb{W}$.

Definition 1. For a given $0 \leqslant \lambda \leqslant 1$, a set $\mathscr{P} \subseteq \mathbb{R}^n$ with $\lambda\mathscr{P} \subset \mathscr{P}$ and $0 \in \mathrm{int}(\mathscr{P})$ is called a (*robustly*) λ*-contractive set* for system (1) if for all $x \in \mathscr{P}$, it holds that $G(x, w, v) \in \lambda\mathscr{P}$ for

[1] A set $\mathscr{E}$ is a β-contractive set for an arbitrary discrete-time system, if for all initial conditions in $\mathscr{E}$, the state obtained after one discrete-time step lies in the set $\beta\mathscr{E}$.

all $w \in \mathbb{W}$ and all $v \in \mathbb{V}$. For $\lambda = 1$ a (robustly) λ-contractive set is called a (*robustly*) PI set.

For a set $\mathscr{P} \subseteq \mathbb{R}^n$, let $\mathscr{Q}_1(\mathscr{P}) \triangleq \{x \in \mathbb{R}^n \mid G(x, w, v) \in \mathscr{P}, \forall w \in \mathbb{W}, \forall v \in \mathbb{V}\}$ denote the (robustly) one-step controllable set for system (1), with respect to $\mathscr{P}$.

Problem 2. *Suppose that a (piecewise) ellipsoidal β-contractive set with $\beta \in [0, 1)$ is known for system* (1). *Construct a (piecewise) polyhedral λ-contractive set with $\lambda \in [0, 1]$ for system* (1).[2]

Systematic solutions to obtain β-contractive (piecewise) *ellipsoidal* sets are available in the literature for many relevant subclasses of (1), such as linear systems subject to input saturation (Hu, Lin, & Chen, 2002), perturbed linear systems (Kolmanovsky & Gilbert, 1998), piecewise affine systems (Ferrari-Trecate, Cuzzola, Mignone, & Morari, 2002), et cetera. Typically, they are obtained as sublevel sets of quadratic (PWQ) Lyapunov functions, which can be calculated efficiently via semi-definite programming.

An alternative solution to Problem 2 can be obtained via the existing algorithms for computing maximal λ-contractive (PI) sets (also named maximal admissible sets (MAS) in some works). The articles (Blanchini, 1994; Blanchini, Mesquine, & Miani, 1995; Dorea & Hennet, 1999; Kolmanovsky & Gilbert, 1998) provide recursive algorithms for calculating polyhedral invariant sets for linear systems. Although these algorithms do not require that an ellipsoidal contractive set is known, existence of a quadratic Lyapunov function is often used to prove finite termination of the algorithms.

Remark 3. The algorithm developed in the present paper does not produce a maximal PI set, but just a polyhedral PI set. However, while the above-mentioned methods for computing MAS are only applicable to *linear systems*, the procedure presented here is independent of the system dynamics and can be applied to a wider class of systems, including *linear systems*, *piecewise linear systems* and *nonlinear systems* that are quadratically stabilizable. Furthermore, the polyhedral PI set obtained via the developed algorithm can be used as a starting point for computing the MAS, using the backward procedure of Blanchini et al. (1995).

In this paper we generalize results from Blanchini (1995) to obtain a novel solution to Problem 2. In Lemmas 4.1 and 4.2 in Blanchini (1995), where perturbed linear systems are considered, it was shown that a polyhedral set contained in between two *convex* sublevel sets of a Lyapunov function is invariant and λ-contractive. The result of Blanchini (1995) is extended in the theorem presented next to a wide class of systems, which includes, for example, any stable (closed-loop) system allowing a PWQ Lyapunov function.

[2] Notice that $\lambda = 1$ corresponds to a PI set.

Theorem 4. *Consider system* (1) *and let $\mathscr{E} \subseteq \mathbb{R}^n$ be a β-contractive set for system* (1), *for some $\beta \in (0, 1)$, that contains the origin in its interior. Let $\beta\mathscr{E} \subset \lambda\mathscr{P} \subset \mathscr{P} \subset \mathscr{E}$ for some*[3] *$\lambda \in (0, 1]$. Then, $\mathscr{P}$ is a (robustly) λ-contractive set for system* (1) *and $0 \in \operatorname{int}(\mathscr{P})$. Moreover, $\mathscr{Q}_1(\lambda\mathscr{P})$ is a (robustly) λ-contractive set for system* (1) *and $\mathscr{E} \subset \mathscr{Q}_1(\lambda\mathscr{P})$.*

Proof. For any $x \in \mathscr{P} \subset \mathscr{E}$ it follows that $G(x, w, v) \in \beta\mathscr{E} \subset \lambda\mathscr{P}$ for any $w \in \mathbb{W}$ and any $v \in \mathbb{V}$ due to the fact that $\mathscr{E}$ is a β-contractive set for system (1). Hence, $\mathscr{P}$ is a (robustly) λ-contractive set for system (1) and $0 \in \operatorname{int}(\mathscr{P})$ as $\beta\mathscr{E} \subset \lambda\mathscr{P}$. Moreover, from the fact that for any $x \in \mathscr{E}$ it holds that $G(x, w, v) \in \beta\mathscr{E} \subset \lambda\mathscr{P}$ for any $w \in \mathbb{W}$ and any $v \in \mathbb{V}$, it follows that $\mathscr{E} \subset \mathscr{Q}_1(\lambda\mathscr{P})$. Since $\mathscr{P} \subset \mathscr{E}$, we have that $\mathscr{P} \subset \mathscr{Q}_1(\lambda\mathscr{P})$ and thus, $\lambda\mathscr{P} \subset \lambda\mathscr{Q}_1(\lambda\mathscr{P})$. Then, for any $x \in \mathscr{Q}_1(\lambda\mathscr{P})$ we have that $G(x, w, v) \in \lambda\mathscr{P} \subset \lambda\mathscr{Q}_1(\lambda\mathscr{P})$ for any $w \in \mathbb{W}$ and any $v \in \mathbb{V}$. Hence, $\mathscr{Q}_1(\lambda\mathscr{P})$ is a (robustly) λ-contractive set for system (1) and $\mathscr{E} \subset \mathscr{Q}_1(\lambda\mathscr{P})$. □

From the above theorem we get the following corollary for the case $\lambda = 1$ which is related to (robust) positive invariance.

Corollary 5. *Consider system* (1) *and let $\mathscr{E} \subseteq \mathbb{R}^n$ be a β-contractive set for system* (1), *for some $\beta \in (0, 1)$, with $0 \in \operatorname{int}(\mathscr{E})$. Suppose there exists a set $\mathscr{P} \subseteq \mathbb{R}^n$ that satisfies $\beta\mathscr{E} \subset \mathscr{P} \subset \mathscr{E}$. Then, $\mathscr{P}$ is a (robustly) PI set for system* (1) *and $0 \in \operatorname{int}(\mathscr{P})$. Moreover, $\mathscr{Q}_1(\mathscr{P})$ is a (robustly) PI set for system* (1) *and $\mathscr{E} \subset \mathscr{Q}_1(\mathscr{P})$.*

Theorem 4 assumes the satisfaction of the triple inclusion $\beta\mathscr{E} \subset \lambda\mathscr{P} \subset \mathscr{P} \subset \mathscr{E}$. The corollary below provides a way to reduce it to a double inclusion at the price of having tighter inclusions.

Corollary 6. *Consider system* (1) *and let $\mathscr{E} \subseteq \mathbb{R}^n$ be a β-contractive set for system* (1), *for some $\beta \in (0, 1)$, with $0 \in \operatorname{int}(\mathscr{E})$. Suppose there exists a set $\mathscr{P} \subseteq \mathbb{R}^n$ that satisfies $\sqrt{\beta}\mathscr{E} \subset \mathscr{P} \subset \mathscr{E}$. Then, $\mathscr{P}$ is a (robustly) λ-contractive set for system* (1) *for $\lambda = \sqrt{\beta}$ and $0 \in \operatorname{int}(\mathscr{P})$.*

Proof. From $\sqrt{\beta}\mathscr{E} \subset \mathscr{P} \subset \mathscr{E}$ one obtains

$$\beta\mathscr{E} \subset \sqrt{\beta}\mathscr{P} \subset \mathscr{P} \subset \mathscr{E}.$$

The result then follows from Theorem 4. □

Notice that the above results apply to certain types of *non-convex* sets $\mathscr{E}$ and $\mathscr{P}$, i.e. for instance piecewise ellipsoidal and piecewise polyhedral sets, respectively (see Section 4 for an illustrative example).

Remark 7. The fact that $\mathscr{E} \subset \mathscr{Q}_1(\lambda\mathscr{P})$ in Theorem 4 is relevant when the state of system (1) is constrained in a compact polyhedral set $\mathbb{X} \subset \mathbb{R}^n$ with $0 \in \operatorname{int}(\mathbb{X})$. Then, given the

[3] The result also holds when $\beta = 0$ and $\lambda = 0$ except that in this case $\mathscr{P}$ does not necessarily contain the origin in its interior.

largest β-contractive piecewise *ellipsoidal* set contained in $\mathbb{X}$, we construct a *larger* λ-contractive set, i.e. $\mathscr{Q}_1(\lambda\mathscr{P}) \cap \mathbb{X}$.

The case of interest in this paper is, as stated in Problem 2, when $\mathscr{E}$ is a piecewise *ellipsoidal* set and $\mathscr{P}$ is a piecewise *polyhedral* set. By Theorem 4, it is sufficient to construct a piecewise polyhedral set $\mathscr{P}$ that lies between the piecewise ellipsoidal sets $\beta\mathscr{E}$ (or $\sqrt{\beta}\mathscr{E}$) and $\mathscr{E}$ to obtain a PI or $\sqrt{\beta}$-contractive solution, respectively to Problem 2. In the next section we present an algorithm for solving this problem of computational geometry and also indicate in Remark 8 how one can solve the triple inclusion $\beta\mathscr{E} \subset \lambda\mathscr{P} \subset \mathscr{P} \subset \mathscr{E}$ of Theorem 4.

3. Squaring the circle

In this section we present a solution to the problem of fitting a piecewise polyhedral set $\mathscr{P}$ between two piecewise ellipsoidal sets where one is contained in the interior of the other, i.e. $\beta\mathscr{E} \subsetneq \mathscr{E}$, with β a real number[4] in (0, 1), by adapting the algorithm of Alessio, Bemporad, Addis, and Pasini (2005). In case $\mathscr{E}$ is an ellipsoid, the main idea is to treat the sets $\mathscr{E}$ and $\beta\mathscr{E}$ as sublevel sets of two *quadratic* functions $f_{\mathscr{E}}(x)$ and $f_{\beta\mathscr{E}}(x)$, respectively, that correspond to the same constant (level) $f_0 \in \mathbb{R}_+$, i.e. $\mathscr{E} \triangleq \{x \in \mathbb{R}^n \mid f_{\mathscr{E}}(x) \leqslant f_0\}$ and $\beta\mathscr{E} \triangleq \{x \in \mathbb{R}^n \mid f_{\beta\mathscr{E}}(x) \leqslant f_0\}$. Then, we compute a PWA function $\bar{f}$ that satisfies $f_{\beta\mathscr{E}}(x) > \bar{f}(x) \geqslant f_{\mathscr{E}}(x)$ for all $x \in \mathbb{R}^n$. The desired piecewise polyhedral set is obtained as $\overline{\mathscr{P}} \triangleq \{x \in \mathbb{R}^n \mid \bar{f}(x) \leqslant f_0\}$.

In case of a *piecewise ellipsoidal* set $\mathscr{E}$ we assume that the polyhedral partitioning $\{\Omega_j \mid j \in \mathscr{S}\}$ ($\mathscr{S}$ is a finite set of indexes) consists of cones, which ensures that $\beta\Omega_j \subseteq \Omega_j$. We write $\mathscr{E}$ as

$$\mathscr{E} = \bigcup_{j \in \mathscr{S}} (\mathscr{E}_j \cap \Omega_j) \quad \text{with } \mathscr{E}_j \triangleq \{x \in \mathbb{R}^n \mid f_{\mathscr{E}_j}(x) \leqslant f_0\},$$

where $f_{\mathscr{E}_j}(x) \triangleq x^\top P_j x + C_j x + \alpha_j$ is a strictly convex quadratic function for all $j \in \mathscr{S}$. Then, we construct for each $j \in \mathscr{S}$ a PWA function $\bar{f}_j(x)$, as in the quadratic (ellipsoidal) case mentioned above, such that $f_{\beta\mathscr{E}_j}(x) > \bar{f}_j(x) \geqslant f_{\mathscr{E}_j}(x)$ for all $x \in \mathbb{R}^n$. Then, a piecewise polyhedral set $\mathscr{P}$ that satisfies $\beta\mathscr{E} \subset \mathscr{P} \subset \mathscr{E}$ is simply obtained as

$$\mathscr{P} = \bigcup_{j \in \mathscr{S}} (\mathscr{P}_j \cap \Omega_j) \quad \text{with}$$

$$\mathscr{P}_j \triangleq \mathrm{Co}(\overline{\mathscr{P}}_j) = \mathrm{Co}(\{x \in \mathbb{R}^n \mid \bar{f}_j(x) \leqslant f_0\}).$$

Indeed, as $\mathscr{P}_j$ is a polyhedral set that satisfies $\beta\mathscr{E}_j \subset \mathscr{P}_j \subset \mathscr{E}_j$, $j \in \mathscr{S}$, we obtain

$$\mathscr{P} = \bigcup_{j \in \mathscr{S}} (\mathscr{P}_j \cap \Omega_j) \subset \bigcup_{j \in \mathscr{S}} (\mathscr{E}_j \cap \Omega_j) = \mathscr{E}.$$

[4] The case $\beta = 0$ is trivial: any $\mathscr{P} \subset \mathscr{E}$ with $0 \in \mathrm{int}(\mathscr{P})$ works.

Since $\beta\mathscr{E}_j \subset \mathscr{P}_j$ and $\beta\Omega_j \subseteq \Omega_j$ for all $j \in \mathscr{S}$, we have that

$$\beta\mathscr{E} = \beta\left(\bigcup_{j \in \mathscr{S}} (\mathscr{E}_j \cap \Omega_j)\right) = \bigcup_{j \in \mathscr{S}} \beta(\mathscr{E}_j \cap \Omega_j)$$
$$= \bigcup_{j \in \mathscr{S}} (\beta\mathscr{E}_j \cap \beta\Omega_j) \subseteq \bigcup_{j \in \mathscr{S}} (\mathscr{P}_j \cap \Omega_j) = \mathscr{P}.$$

As the PWQ case can be split into a finite number of quadratic instances of the problem, we consider only the ellipsoidal case, i.e. when the set $\mathscr{E}$ is a sublevel set of a strictly convex quadratic function $f_{\mathscr{E}}$.

Next, choose $P \in \mathbb{R}^{n \times n}$ (with $P > 0$) and $f_0, \alpha_{\mathscr{E}} \in \mathbb{R}$ (with $f_0 > \alpha_{\mathscr{E}}$) such that $\mathscr{E}$ is the sublevel set of[5] $f_{\mathscr{E}}(x) \triangleq x^\top P x + \alpha_{\mathscr{E}}$, corresponding to the level f_0. Then, we have that $\beta\mathscr{E}$ is the sublevel set of $f_{\beta\mathscr{E}}(x) \triangleq x^\top P x + \alpha_{\beta\mathscr{E}}$, corresponding to the level f_0, where

$$\alpha_{\beta\mathscr{E}} \triangleq (1-\beta^2) f_0 + \beta^2 \alpha_{\mathscr{E}} > \alpha_{\mathscr{E}}.$$

Consider now an initial polyhedron $\mathscr{P}_0 \subset \mathbb{R}^n$ that contains $\mathscr{E}$. Let $(\tilde{\theta}_0, \ldots, \tilde{\theta}_m)$, with $m \geqslant n$, be the vertices of $\mathscr{P}_0$. An initial set of simplices $S_1^0, \ldots, S_{l_0}^0$ that contains these points is determined by Delaunay triangulation (Yepremyan & Falk, 2005). Then, for every simplex $S_i^0 \triangleq \mathrm{Co}(\theta_{0i}^0, \ldots, \theta_{ni}^0)$, $i = 1, \ldots, l_0$, the following operations are performed.

Algorithm 1. (1) *Let* $k = 0$.

(2) *For every simplex* S_i^k, $i = 1, \ldots, l_k$, *construct the matrix* $M_i^k \triangleq \begin{bmatrix} 1 & 1 & \ldots & 1 \\ \theta_{0i}^k & \theta_{1i}^k & \ldots & \theta_{ni}^k \end{bmatrix}$.

(3) *Set* $v_i^k \triangleq [f_{\mathscr{E}}(\theta_{0i}^k)\ f_{\mathscr{E}}(\theta_{1i}^k)\ \ldots\ f_{\mathscr{E}}(\theta_{ni}^k)]^\top$ *and construct the function* $\bar{f}_i^k(x) \triangleq (v_i^k)^\top (M_i^k)^{-1} \begin{bmatrix} 1 \\ x \end{bmatrix}$.

(4) *Solve the QP problem*:

$$J_i^{k*} \triangleq \min_{x \in S_i^k} \{J_i^k(x) \triangleq f_{\beta\mathscr{E}}(x) - \bar{f}_i^k(x)\}. \tag{2}$$

(5) *If* $J_i^{k*} > 0$ *for all* $i = 1, \ldots, l_k$, *then Stop. Otherwise, for all* S_i^k, $i = 1, \ldots, l_k$, *for which* $J_i^{k*} \leqslant 0$ *build two new simplices* $\overline{S}_0^i, \overline{S}_1^i$ *defined by the vertices* $(\theta_{0i}^k, \ldots, \theta_{ti}^k, \tilde{x}_i^k, \ldots, \theta_{ni}^k)$, $(\theta_{0i}^k, \ldots, \tilde{x}_i^k, \theta_{si}^k, \ldots, \theta_{ni}^k)$, *where* $(\theta_{ti}^k, \theta_{si}^k)$ *are the vertices of the maximal length edge of the simplex* S_i^k, *and* $\tilde{x}_i^k = (\theta_{ti}^k + \theta_{si}^k)/2$. *Increment* k *by one, add the new simplices* $\overline{S}_0^i, \overline{S}_1^i$ *to the set of simplices* $\{S_i^k\}_{i=1,\ldots,l_k}$ *and repeat the algorithm recursively from Step* 2.

Algorithm 1 computes a simplicial partition of a given initial polyhedral set $\mathscr{P}_0$ that contains the ellipsoidal set $\mathscr{E}$, by splitting a single simplex S_i^k into $n+1$ simplices. This is done by fixing a new vertex $\tilde{x}_i^k$ which is obtained by solving the QP problem (2), and by calculating a new PWA approximation over the new set of simplices.

The steps of Algorithm 1 are repeated for all resulting simplices, until $J_i^{k*} > 0$ for all simplices. At every iteration k, a

[5] Note that by a suitable change of coordinates we can always take $C = 0$.

tighter PWA approximation of the quadratic function $f_{\mathscr{E}}$ is obtained. Algorithm 1 proceeds in a typical branch and bound way, i.e. *branching* on a new vertex $\tilde{x}_i^k$, and *bounding* whenever it finds a simplex S_i^k for which it holds that $J_i^{k*} > 0$.

Suppose Algorithm 1 stops at the $\bar{k}$th iteration[6] for some $\bar{k} \in \mathbb{Z}_+$. The following PWA function is generated:

$$\begin{aligned}\bar{f}(x) &\triangleq \bar{f}_i^{\bar{k}}(x) \quad \text{when } x \in S_i^{\bar{k}}, \quad i = 1, \ldots, l_{\bar{k}} \\ &\triangleq H_i^{\bar{k}} x + a_i^{\bar{k}} \quad \text{when } x \in S_i^{\bar{k}}, \quad i = 1, \ldots, l_{\bar{k}},\end{aligned} \tag{3}$$

where $l_{\bar{k}}$ is the number of simplices obtained at the end of Algorithm 1 and $H_i^{\bar{k}} x + a_i^{\bar{k}} = (v_i^{\bar{k}})^\top (M_i^{\bar{k}})^{-1} \begin{bmatrix} 1 \\ x \end{bmatrix}$. The PWA function $\bar{f}$ constructed via Algorithm 1 is a continuous function (Alessio et al., 2005). Moreover, for $x = \sum_{j=0}^{n} \mu_j \theta_{ji}$ with $\sum_{j=0}^{n} \mu_j = 1$, the corresponding functions $\bar{f}_i^{\bar{k}}$ satisfy

$$\bar{f}_i^{\bar{k}}(x) = \bar{f}_i^{\bar{k}} \left(\sum_{j=0}^{n} \mu_j \theta_{ji} \right) = \sum_{j=0}^{n} \mu_j f_{\mathscr{E}}(\theta_{ji}),$$

which, by convexity of $f_{\mathscr{E}}$, implies that $\bar{f}_i^{\bar{k}}(x) \geqslant f_{\mathscr{E}}(x)$ for all $x \in S_i^{\bar{k}}$ and all $i = 1, \ldots, l_{\bar{k}}$. Hence, $\bar{f}(x) \geqslant f_{\mathscr{E}}(x)$ for all $x \in \mathscr{P}_0$. Since the stopping criterion defined in Step 4 of Algorithm 1 assures that at the end of the entire procedure the optimal value $J_i^{\bar{k}*}$ of the QP problem (2) will be greater than zero in every simplex $S_i^{\bar{k}}$, $i = 1, \ldots, l_{\bar{k}}$, it follows that

$$f_{\mathscr{E}}(x) \leqslant \bar{f}(x) < f_{\beta\mathscr{E}}(x), \quad \forall x \in \bigcup_{i=1,\ldots,l_{\bar{k}}} S_i^{\bar{k}} = \mathscr{P}_0.$$

Then, the sublevel set of $\bar{f}$ given by

$$\overline{\mathscr{P}} \triangleq \bigcup_{i=1,\ldots,l_{\bar{k}}} \{x \in S_i^{\bar{k}} | H_i^{\bar{k}} x + a_i^{\bar{k}} \leqslant f_0\},$$

satisfies $\beta\mathscr{E} \subset \overline{\mathscr{P}} \subset \mathscr{E}$. Indeed, note that for $x \in \overline{\mathscr{P}}$ it holds that

$$\bar{f}(x) \leqslant f_0 \Rightarrow f_{\mathscr{E}}(x) \leqslant \bar{f}(x) \leqslant f_0 \Rightarrow x \in \mathscr{E}$$

and for $x \in \beta\mathscr{E}$ it holds that

$$f_{\beta\mathscr{E}}(x) \leqslant f_0 \Rightarrow \bar{f}(x) < f_{\beta\mathscr{E}}(x) \leqslant f_0 \Rightarrow x \in \overline{\mathscr{P}}.$$

The desired polyhedral set $\mathscr{P}$ (see Fig. 1) satisfying $\beta\mathscr{E} \subset \mathscr{P} \subset \mathscr{E}$, is obtained as the convex hull of $\overline{\mathscr{P}}$. Indeed $\beta\mathscr{E} \subset \overline{\mathscr{P}} \subset \mathrm{Co}(\overline{\mathscr{P}}) \triangleq \mathscr{P}$ and, by the convexity of $\mathscr{E}$, it holds that $\mathscr{P} = \mathrm{Co}(\overline{\mathscr{P}}) \subseteq \mathrm{Co}(\mathscr{E}) = \mathscr{E}$. Notice that the computation of the vertices of $\overline{\mathscr{P}}$ and of their convex hull can be performed efficiently using, for instance, the Geometric Bounding Toolbox (GBT) (Veres, 1995).

Remark 8. The following identity

$$\begin{aligned}\min_i J_i^{k*} &= \min_{x \in \mathscr{P}_0} [f_{\beta\mathscr{E}}(x) - \bar{f}^k(x)] \\ &= \varepsilon_{\max} - \max_{x \in \mathscr{P}_0} [\bar{f}^k(x) - f_{\mathscr{E}}(x)]\end{aligned} \tag{4}$$

[6] The existence of a finite $\bar{k}$ will be established in Section 3.1.

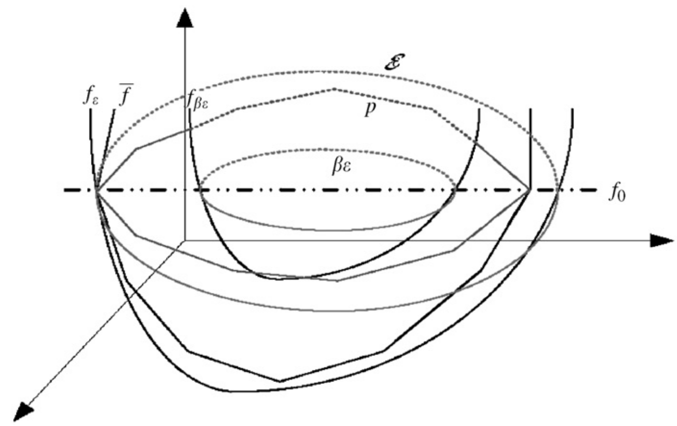

Fig. 1. Illustration of the proposed solution for constructing the polyhedral invariant set $\mathscr{P}$.

is an immediate consequence of $f_{\beta\mathscr{E}}(x) - f_{\mathscr{E}}(x) = \varepsilon_{\max} \triangleq \alpha_{\beta\mathscr{E}} - \alpha_{\mathscr{E}} > 0$. Hence, the error $\bar{\varepsilon} \triangleq \max_{x \in \mathscr{P}_0} [\bar{f}(x) - f_{\mathscr{E}}(x)]$ obtained at the end of Algorithm 1 is upper bounded by the allowed maximum error $\varepsilon_{\max} = \max_{x \in \mathscr{P}_0} [f_{\beta\mathscr{E}}(x) - f_{\mathscr{E}}(x)]$. Thus, the Stop criterion of Algorithm 1 can be set as $J_i^{k*} > \delta$ for some $\delta \in (0, \varepsilon_{\max})$, instead of just $J_i^{k*} > 0$, to create a gap between $\mathscr{P}$ and $\beta\mathscr{E}$. A larger δ will result in a smaller $\lambda \in (0, 1)$ for which it holds that $\beta\mathscr{E} \subset \lambda\mathscr{P} \subset \mathscr{P} \subset \mathscr{E}$. Note that if δ tends to $\varepsilon_{\max}$, then the number of vertices of $\mathscr{P}$ tends to infinity, $\mathscr{P}$ recovers the ellipsoidal set $\mathscr{E}$ and λ tends to β.

3.1. An estimate of the computational complexity

Algorithm 1 computes at every iteration k a tighter PWA approximation $\bar{f}^k$ of the given strictly convex quadratic function $f_{\mathscr{E}}$. It stops when the approximation error obtained at the kth iteration of the algorithm satisfies

$$0 < \min_{x \in \mathscr{P}_0} [f_{\beta\mathscr{E}}(x) - \bar{f}^k(x)]$$

or equivalently (due to (4)),

$$\varepsilon_k \triangleq \max_{x \in \mathscr{P}_0} [\bar{f}^k(x) - f_{\mathscr{E}}(x)] < \varepsilon_{\max}, \quad k \in \mathbb{Z}_+.$$

The algorithm builds recursively a binary tree, where in each node it stores the vertices of the current simplex S_i^k and the pairs (H_i^k, a_i^k) such that $\bar{f}^k(x) = H_i^k x + a_i^k$, for all $x \in S_i^k$, $i \geqslant 1$, $k \in \mathbb{Z}_+$. If the value of J_i^{k*} for the current simplex is less than zero, then Algorithm 1 splits S_i^k in 2 simplices and adds a new level to the tree. The height of the tree can be easily computed once the values of the allowed maximum error $\varepsilon_{\max}$ is known, which yields the following result.

Theorem 9 (*Alessio et al., 2005*). *Suppose that the initial polyhedral set $\mathscr{P}_0$ and the desired final approximation error $\varepsilon_{\max}$ are known. Then, Algorithm 1 has complexity:*

$$O\left(2^{\frac{n(n-1)}{2}} n \log n \left(d_0 \sqrt{\frac{\lambda_{max}(P)}{\varepsilon}} \right)^{\gamma} \right),$$

where $\gamma = 1/(1 - log\sqrt{3})$ and d_0 is the maximal length edge of the initial simplex (Alessio et al., 2005).

Remark 10. The number of facets of the resulting polyhedron depends on the contraction factor β and on the size of the problem data. As it depends on the number of vertices and simplices generated, in most cases the number of facets is tractable. However, as the contraction factor tends to one, the number of facets of the resulting polyhedron may become very large.

Remark 11. By exploiting symmetries of the problem, it is always possible to reduce the overall computational time of the algorithm. If the axes of symmetry of the ellipsoids are used as a new orthogonal basis for the system,[7] the algorithm does not need to be run over the entire polyhedron $\mathscr{P}_0$, but only over a reduced closed convex set $\widehat{\mathscr{P}}_0$, i.e.

$$\widehat{\mathscr{P}}_0 \triangleq \{x \in \mathbb{R}^n \mid x \in \mathscr{P}_0, x_i \geqslant 0\}. \tag{5}$$

The polyhedron $\widehat{\mathscr{P}}_0$ so obtained, is then used to compute polyhedron $\overline{\mathscr{P}}$, through n reflections of $\widehat{\mathscr{P}}_0$ along appropriate hyperplanes. The desired polyhedral set $\mathscr{P}$ is obtained as the convex hull of $\overline{\mathscr{P}}$.

4. Illustrative examples

In this section we present two examples that illustrate the potential of the algorithm.

4.1. Perturbed linear systems

Consider the perturbed discrete-time triple integrator

$$x_{k+1} = Ax_k + Bu_k + v_k, \quad k \in \mathbb{Z}_+, \tag{6}$$

where, $A = \begin{bmatrix} 1 & T_s & \frac{T_s^2}{2} \\ 0 & 1 & T_s \\ 0 & 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} \frac{T_s^3}{3!} \\ \frac{T_s^2}{2} \\ T_s \end{bmatrix}$, $T_s = 0.8$, $v_k \in \mathbb{V}$ is the additive disturbance input, and $\mathbb{V} = [-0.1,\ 0.1] \times [-0.1,\ 0.1] \times [-0.1,\ 0.1]$. We employed the method of Lazar and Heemels (2006) to calculate a robust stabilizing state-feedback control law for system (6), i.e. $u_k = Kx_k$, with $K = [-1.1739\ \ -2.4071\ \ -2.0888]$, together with a robust quadratic Lyapunov function $V(x) = x^\top Px$ with $P = \begin{bmatrix} 14.4684 & 13.5850 & 4.0221 \\ 13.5850 & 17.4375 & 5.4581 \\ 4.0221 & 5.4581 & 2.5328 \end{bmatrix}$. The procedure presented in this paper was employed to calculate a polyhedral set $\mathscr{P}$ such that $\beta\mathscr{E} \subset \mathscr{P} \subset \mathscr{E}$, where $\mathscr{E}$ is the sublevel set of V, corresponding to the level $f_0 = 20$, and the contraction factor is $\beta = 0.8$. The resulting set $\mathscr{P}$ is λ-contractive with $\lambda = 0.9$ and has 56 vertices. A plot of $\mathscr{P}$ is given in Fig. 2 together with a plot of the closed-loop system state trajectory obtained for $x_0 = [-3\ 2\ 2]^\top$ and randomly generated additive disturbance inputs.

[7] This operation is realized by a simple change of coordinates.

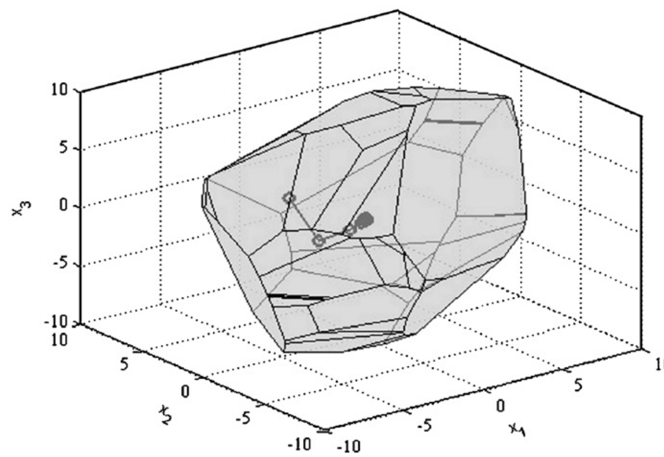

Fig. 2. Polyhedral invariant set and state trajectory for system (6) in closed-loop with $u_k = Kx_k$, $k \in \mathbb{Z}_+$, and randomly generated disturbances v in $\mathbb{V}$.

4.2. Piecewise linear systems

Consider the following open-loop unstable PWL system:

$$x_{k+1} = \begin{cases} A_1x_k + Bu_k & \text{if } E_1x_k > 0, \\ A_2x_k + Bu_k & \text{if } E_2x_k \geqslant 0, \\ A_3x_k + Bu_k & \text{if } E_3x_k > 0, \\ A_4x_k + Bu_k & \text{if } E_4x_k \geqslant 0, \end{cases} \tag{7}$$

subject to the constraints $x_k \in \mathbb{X} = [-10,\ 10] \times [-10,\ 10]$, $u_k \in \mathbb{U} = [-1, 1]$, where $A_1 = \begin{bmatrix} 0.5 & 0.61 \\ 0.9 & 1.345 \end{bmatrix}$, $A_2 = \begin{bmatrix} -0.92 & 0.644 \\ 0.758 & -0.71 \end{bmatrix}$, $B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $A_3 = A_1$ and $A_4 = A_2$. The state-space partition of the system is given by $E_1 = -E_3 = \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix}$, $E_2 = -E_4 = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$. The following PWQ Lyapunov function $V(x) = x^\top P_j x$ when $x \in \Omega_j$, $j = 1, 2, 3, 4$, feedback gains and contraction factor β were calculated in Lazar, Heemels, Weiland, and Bemporad (2005):

$$\begin{aligned} &P_1 = \begin{bmatrix} 12.9707 & 10.9974 \\ 10.9974 & 14.9026 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 7.9915 & -5.5898 \\ -5.5898 & 5.3833 \end{bmatrix}, \\ &P_3 = P_1, \quad P_4 = P_2, \\ &K_1 = [-0.7757\ \ -1.0299], \quad K_2 = [0.6788\ \ -0.4302], \\ &K_3 = K_1, \quad K_4 = K_2, \quad \beta = 0.9378. \end{aligned} \tag{8}$$

Let $\mathbb{X}_{\mathbb{U}} \subseteq \mathbb{X}$ denote the set of states for which the feedback control law given in (7)–(8) satisfies the state and input constraints. A contractive piecewise polyhedral set $\mathscr{P}$ was computed for system (7) in closed-loop with the feedbacks given in (8) using the approach of Theorem 4 and Algorithm 1 for the sublevel sets $\mathscr{E} \triangleq \{x \in \mathbb{X} \mid V(x) \leqslant 14\} \subseteq \mathbb{X}_{\mathbb{U}}$ and $\beta\mathscr{E}$. The resulting set $\mathscr{P}$ is the union of four polyhedra and it is a λ-contractive set with $\lambda = 0.9286$. The closed-loop state trajectories with the vertices of $\mathscr{P}$ as initial conditions are plotted in Fig. 3 together with a plot of the safe set $\mathbb{X}_{\mathbb{U}}$. The trajectories of the closed-loop system remain inside $\mathscr{P}$ at all times and converge to zero.

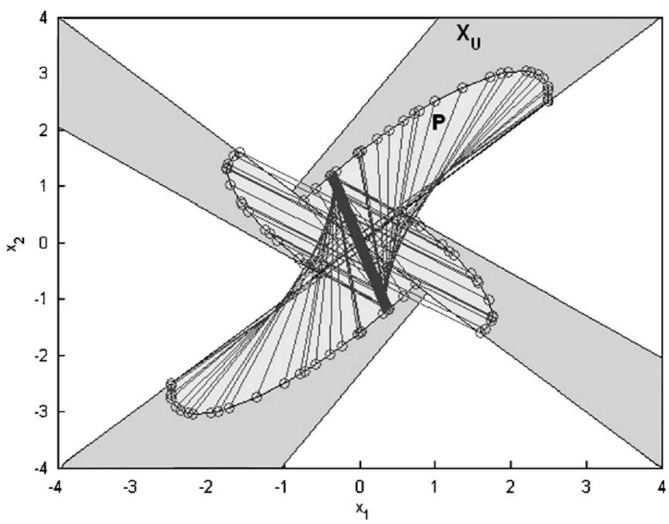

Fig. 3. Piecewise polyhedral invariant set $\mathscr{P}$ (light grey) and $\mathbb{X}_{\mathbb{U}}$ (dark grey and light grey) for system (7).

5. Conclusions

A new method for computing (piecewise) *polyhedral* (robustly) PI and contractive sets was developed based on a geometrical argument. The novelty of the proposed approach consists of formulating the problem of computing polyhedral invariant sets as solving a number of QP problems. This was achieved by observing that any polyhedral set that lies between two ellipsoidal sets $\beta\mathscr{E}$ and $\mathscr{E}$ with $\mathscr{E}$ β-contractive for some $\beta \in (0, 1)$ is contractive and thus, PI. A new algorithm based on QP was developed to construct the desired polyhedral set. A guarantee that the number of QP problems that need to be solved is always finite was also given. This fact establishes finite termination for the algorithm. Two examples illustrated the wide applicability of the method.

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