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Brief Paper

Fulfilling Hard Constraints in Uncertain Linear Systems by Reference Managing*

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Key Words—Constraint satisfaction problems; robustness; nonlinear filters; optimization problems; reference input signals.

Abstract—A method based on conceptual tools of predictive control is described for tackling tracking problems of uncertain linear systems wherein pointwise-in-time input and/or state inequality constraints are present. The method consists of adding to a primal compensated system a nonlinear device called predictive reference filter which manipulates the desired reference in order to fulfill the prescribed constraints. Provided that an admissibility condition on the initial state is satisfied, the control scheme is proved to fulfill the constraints, as well as stability and set-point tracking requirements, for all systems whose impulse/step responses lie within given uncertainty ranges.

1. INTRODUCTION

In applications the design of feedback controllers is often complicated by the presence of physical constraints: saturating actuators, temperatures and pressures within safety margins, working space limited by constructive restrictions, etc. This issue has stimulated substantial theoretical advancements in the field of feedback control of dynamic systems subject to input/state constraints [see Mayne and Polak (1993), and Sussmann *et al.* (1994), which also include relevant references, for an account of pertinent results]. Most of this work has addressed the regulation problem in the presence of constraints—in particular input saturation—under the hypothesis that the plant model is exactly known. The main goal of the present paper is to address the constrained tracking problem for systems affected by *model uncertainties*, by using tools from *predictive control*. Handling of hard constraints is in fact one of the potential benefits of predictive control (Clarke, 1994; Keerthi and Gilbert, 1988; Mayne and Michalska, 1990; Mosca, 1995; Rawlings and Muske, 1993). Predictive control is based on the *receding horizon* control philosophy: a sequence a future control actions is chosen, by predicting the future evolution of the system, and applied to the plant until new measurements are available. Then, a new sequence is evaluated so as to replace the previous one. Each selected sequence is the result of an optimization procedure which takes into account two objectives: (i) maximize the tracking performance, and (ii) guarantee that the constraints are and will be fulfilled, i.e. no "blind-alley" is entered. Recently, Bemporad *et al.*, (1997) and Bemporad and Mosca (1994) have applied receding horizon tools to the reference trajectory rather than to the control input, with a consequent substantial eduction of computational complexity. In fact, in contrast to other predictive control approaches (Kothare *et al.*, 1996; Zheng and Morari, 1995), in Bemporad *et al.* (1997) and Bemporad and Mosca (1994) the constraint fulfillment problem is separated from stability, set-point tracking, and disturbance attenuation requirements, which in turn—in the absence of constraints—are supposed to be taken care of by a formerly designed compensator. By using the output maximal admissible sets theory, Gilbert *et al.* (1995) and Gilbert and Tan (1991) have developed a reference management technique for constraint fulfilment in the ideal noiseless case. An extension to the case of input disturbances has appeared in Gilbert and Kolmanovsky (1995), where no model mismatch is considered. The aim of the present paper is to lay down guidelines for synthesizing *predictive reference filters* (*PRF*) for systems whose impulse and step response are uncertain in that they are only known to lie within given sets. The system is supposed to be a standard feedback loop, designed according to available robust control techniques so as to perform satisfactorily in the absence of constraints. Whenever necessary, the filter alters on line the input to the primal control system so as to avoid constraint violation and possibly maximize the tracking performance, according to a worst-case criterion.

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The paper is arranged as follows. Section 2 introduces the problem and presents the PRF algorithm for the class of systems under consideration. A description of the adopted model uncertainty is given in Section 3. Section 4 studies how to reduce the infinite number of constraints involved in the problem formulation into a finite number. Stability, tracking and other properties of the PRF are investigated in Section 5, while Section 6 is devoted to computational aspects. Finally, a simulative example of application of the PRF, which provides some design guidelines, is described in Section 7.

2. PREDICTIVE REFERENCE FILTER DESIGN

Consider a family $\mathscr{S}$ of linear asymptotically stable systems. Each member Σ of $\mathscr{S}$ has a state-space description of the form

$$\Sigma: \begin{cases} x(\tau+1) = \Phi x(\tau) + Gg(\tau), \\ y(\tau) = Hx(\tau) + Dg(\tau), \\ c(\tau) = H_c x(\tau) + D_c g(\tau), \quad \Sigma \in \mathscr{S} \end{cases} \tag{1}$$

where $\tau \in \mathbb{Z}_+ \triangleq \{0, 1, \dots\}$, $x(\tau) \in \mathbb{R}^{n_\Sigma}$ is the state vector, $g(\tau) \in \mathbb{R}^p$ the command input, which in the absence of constraints would coincide with the desired output reference $r(\tau)$, $y(\tau) \in \mathbb{R}^p$ the output which is required to track $r(\tau)$, and $c(\tau) \in \mathbb{R}^q$ the vector to be constrained within a given set $\mathscr{C}$, which satisfies the following

Assumption 1. $\mathscr{C}$ is a convex polyhedron with nonempty interior.

Without loss of generality, we assume that $\mathscr{C}$ has the form

$$\mathscr{C} = \{c \in \mathbb{R}^q \colon c \le B_c\}. \tag{2}$$

In fact, a generic polyhedron described by inequalities of the form $A_c c \le B_c$ can be rewritten in the form (2) by defining a new vector $c^* = A_c c$ and, accordingly, new matrices $H_c^* = A_c H_c$, $D_c^* = A_c D_c$. Typically, equation (1) consists of an uncertain linear system under robustly stabilizing control. We take into account model uncertainties by assuming the true (unknown) plant is included in $\mathscr{S}$, where $\mathscr{S}$ will be characterized in Section 3. Furthermore, inside the $\mathscr{S}$, we choose a particular $\hat{\Sigma}$ called *nominal* system

$$\hat{\Sigma}: \begin{cases} \hat{x}(\tau+1) = \hat{\Phi}\hat{x}(\tau) + \hat{G}g(\tau), \\ c(\tau) = \hat{H}_c\hat{x}(\tau) + \hat{D}_c g(\tau), \end{cases} \tag{3}$$

with $\hat{x} \in \mathbb{R}^n$.

The aim of this paper is to design a PRF, a device finalized to transform the desired reference $r(\tau)$ to the command vector $g(\tau)$ so as to possibly enforce the prescribed constraints $c(\tau) \in \mathscr{C}$, $\tau \in \mathbb{Z}_+$, for all possible systems in $\mathscr{S}$, and make the tracking error $y(\tau) - r(\tau)$ small. The filtering action operates in a predictive manner: at time τ a *virtual* command sequence $\{g(\tau), g(\tau+1), \dots\}$ is selected in such a way that, for all systems $\Sigma \in \mathscr{S}$, the corresponding predicted c-evolution lies within $\mathscr{C}$. Then, according to a *receding horizon* strategy, only the first sample of the virtual sequence is applied at time τ, a new virtual command sequence being recomputed at time $\tau + 1$. Several criteria (Bemporad *et al.*, 1997; Bemporad and Mosca, 1994; Gilbert *et al.*, 1995) can be used to select the class of virtual commands. For reasons that will be clearer soon, we restrict our attention to the class of constant command sequences introduced in Gilbert *et al.* (1995). This class is parameterized by the scalar β, and each of its members defined by

$$g(t+\tau|\tau, \beta) \triangleq g(\tau-1) + \beta[r(\tau) - g(\tau-1)], \quad \forall t \in \mathbb{Z}_+. \tag{4}$$

At each time τ, the free parameter β is selected by the PRF via the optimization criterion

$$\beta(\tau) \triangleq \begin{cases} \arg\max_{\beta \in [0,1]} \beta \\ \text{subject to} \begin{cases} c(t+\tau \mid \tau, \Sigma, x(\tau), \beta) \in \mathscr{C}, \\ \forall t \ge 0, \forall \Sigma \in \mathscr{S} \end{cases} \end{cases} \tag{5}$$

where $c(t+\tau|\tau, \Sigma, x(\tau), \beta)$ denotes the predicted c-evolution at time $t + \tau$ which results by applying the constant input $g(t+\tau \mid \tau, \beta)$ to Σ from state $x(\tau)$ at time τ onwards. Then, according to the receding horizon strategy, at each time τ the PRF selects

$$g(\tau) = g(\tau|\tau, \beta(\tau)).$$

Notice that requiring $\beta(\tau)$ as close as possible to 1 corresponds to minimizing $\|g(\tau) - r(\tau)\|$, and consequently the norm of the tracking error $\|y(\tau) - r(\tau)\|$, depending on the tracking properties of system (1). A scalar β, or a constant command $g \in \mathbb{R}^p$ satisfying the constraints in equation (5) will be referred to as *admissible* at time τ.

Assumption 2 (*Feasible as initial condition*). There exists a vector $g(-1) \in \mathbb{R}^p$ such that at time $\tau = 0$ the virtual command $g(t|0, 0) = g(-1)$, $\forall t \in \mathbb{Z}_+$, is admissible.

For instance, Assumption 2 is satisfied for $x(0) = (I - \Phi)^{-1}Gg(-1)$, $H_c x(0) + D_c g(-1) \in \mathscr{C}$. Assumption 2, the particular structure of equation (4), and equation (5) ensure that $\beta = 0$ is admissible, and therefore the optimization problem (5) admits feasible solutions, at each time $\tau \in \mathbb{Z}_+$.

3. MODEL UNCERTAINTY DESCRIPTION

Uncertainty of dynamic systems models can be described in various ways. In the case at hand, frequency domain descriptions are not convenient because of the time-domain PRF design logic (4), (5). Furthermore, if uncertainties involving state-space realizations are adopted, the effect of matrix perturbations on the predicted evolutions become cumbersome to compute. Consider, for instance, a free response of the form $(\hat{\Phi} + \tilde{\Phi})^t x(0)$: this gives rise to prediction perturbations which are nonlinear in the uncertain parameter $\tilde{\Phi}$. On the contrary, uncertainties on the step-response or impulse-response samples provide a practical description in many applications, as they can be easily determined by experimental tests, and allow a reasonably simple way to compute predictions. Seemingly, step-response and impulse-response are equivalent, and one could be tempted to use either one or the other without distinction to describe model uncertainties. However, when used individually, both exhibit drawbacks. To show this, consider Fig. 1, which depicts perturbations expressed only in terms of the impulse response. The resulting step-response uncertainty turns out to be very large as $t \to \infty$. However, this is not the case when each Σ, for instance, contains an integrator in the feedback loop, which yields a unity DC-gain, and consequently vanishing step-response perturbations as $t \to \infty$. Conversely, as depicted in Fig. 2, uncertainty expressed only in terms of the step response could lead to nonzero impulse-response samples at large values of t, for instance when the DC-gain from g to c is uncertain; therefore, any *a priori* information about asymptotic stability properties would be wasted. In order to minimize the conservatism of the approach in equation (5), it is clear that the set Σ should be as small as possible. For

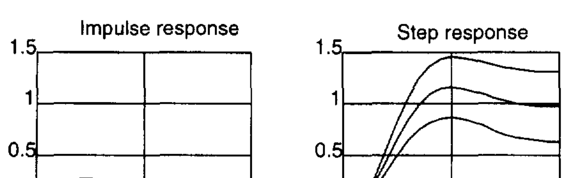

Fig. 1. Step-response interval ranges (right) arising from an impulse-response description (left).

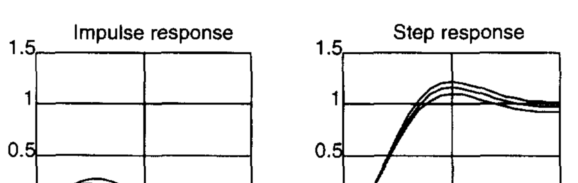

Fig. 2. Impulse-response interval ranges (left) arising from a step-response description (right).

this reason, in this paper we will jointly consider both step-response and impulse-response in order to describe model uncertainty.

Assumption 3. Let $\Sigma \in \mathscr{S}$ and let H be the impulse response from g to c. Then, there exist a matrix $M \in \mathbb{R}^{q \times p}$ and a scalar λ, $0 \le \lambda < 1$, such that, for all systems $\Sigma \in \mathscr{S}$,

$$|H_t^{ij}| \le M^{ij}\lambda^t, t \in \mathbb{Z}_+, \quad \forall i = 1, \dots, q, \quad \forall j = 1, \dots, p, \tag{6}$$

where H_t^{ij} is the impulse response at time t from the jth command input g_j to the ith constrained variable c_i.

Notice that, although in equation (1) we are considering asymptotically stable systems, condition (6) characterizes only stability properties of the subspace which is observable from c.

The impulse response H_t can be expressed as the sum of a nominal impulse response,

$$\hat{H}_t \triangleq \begin{cases} \hat{H}_c \hat{\Phi}^t \hat{G} & \text{if } t > 0, \\ \hat{D}_c & \text{if } t = 0, \end{cases}$$

and an additive perturbation $\tilde{H}_t$. We describe the range intervals of $\tilde{H}_t$ as

$$\begin{aligned} &\tilde{H}_t^{ij} \in [\underline{H}_t^{ij}, \bar{H}_t^{ij}] && \text{if } t = 0, 1, \dots, N-1, \\ &|\tilde{H}_t^{ij}| \le E^{ij}\lambda^t && \text{if } t \ge N, \end{aligned} \tag{7}$$

where $E \in \mathbb{R}^{q \times p}$, N is a fixed integer, and $i = 1, \dots, q$, $j = 1, \dots, p$. In the same way, the step-response from g to c can be expressed as the sum of a nominal response,

$$\hat{W}_t \triangleq \sum_{k=0}^{t-1} \hat{H}_c \hat{\Phi}^k \hat{G} + \hat{D}_c$$

and an additive perturbation $\tilde{W}_t$,

$$\begin{aligned} &\tilde{W}_t^{ij} \in [\underline{W}_t^{ij}, \bar{W}_t^{ij}] && \text{if } t = 0, 1, \dots, N-1 \\ &|\tilde{W}_t^{ij} - \tilde{W}_{t-1}^{ij}| \le E^{ij}\lambda^t && \text{if } t \ge N. \end{aligned} \tag{8}$$

4. REDUCTION TO A FINITE NUMBER OF CONSTRAINTS

Since the PRF operates over a semi-infinite prediction horizon, the optimization criterion in equation (5) involves an infinite number of constraints. In order to effectively solve equation (5), we need to reduce this infinite number to a finite one. This will be achieved by borrowing techniques presented in (Gilbert and Tan, 1991), as follows. Under some assumptions on the desired reference r and the past command inputs $g(-\tau)$ applied before the PRF was switched on at time $\tau = 0$, Lemma 1 will show that the command sequence $g(\tau)$ generated by the PRF is bounded. A new constraint on β will be introduced, which ensures that in steady state the predicted c-evolution stays away from the border of $\mathscr{C}$. Then, Theorem 1 will prove the existence of a finite *constraint horizon*, the shortest prediction interval over which constraints must be checked in order to assert admissibility of a given virtual command sequence. Finally, an algorithm to find such a constraint horizon will be provided.

Assumption (*Set-Point Conditioning*). The reference signal $r(\cdot)$ satisfies $r(\tau) \in \mathscr{R}$ for all $\tau \in \mathbb{Z}_+$, where $\mathscr{R}$ is compact and convex.

This amounts to assuming either that the class of references to be tracked is bounded, or that a clamping device is artificially added to the PRF mechanism so as to satisfy Assumption 4. In practice, this is not a restriction since bounds on the reference are often dictated by the physical application.

Assumption 5. For all $\tau > 0$, $g(-\tau) \in \mathscr{R}$.

Lemma 1. Provided that Assumptions 4 and 5 are satisfied, $g(\tau) \in \mathscr{R}$, $\forall \tau \in \mathbb{Z}_+$.

Proof. By equation (5), $\beta(\tau) \in [0, 1]$. Then, as depicted in Fig. 3, $g(\tau)$ lies on the segment whose vertices are $g(\tau - 1)$, $r(\tau)$. By convexity of $\mathscr{R}$, the result straightforwardly follows by induction.

In order to proceed further, we impose an additional constraint on the optimization (5). By equation (7), we can define $\tilde{W}_\infty \triangleq \lim_{t\to\infty} \tilde{W}_t$, along with the relation

$$\tilde{W}_\infty \in [\underline{W}_\infty, \bar{W}_\infty], \tag{9}$$

wherein $\underline{W}_\infty \triangleq \underline{W}_{N-1} - [\lambda^N/(1-\lambda)]E$, $\bar{W}_\infty \triangleq \bar{W}_{N-1} + [\lambda^N/(1-\lambda)]\,E$, and by Assumption 3, $\hat{W}_\infty \triangleq \lim_{t\to\infty} \hat{W}_t$.

Notice that equation (9) is not overconservative in that there exist systems $\Sigma \in \mathscr{S}$ for which $\tilde{W}_\infty = \underline{W}_\infty$ or $\bar{W}_\infty$. For an arbitrarily small $\delta > 0$, consider the following set:

$$\mathscr{G}_\delta = \left\{ g \in \mathscr{R}: (\hat{W}_\infty + \tilde{W}_\infty) g \le B_c - \underline{\delta} \quad \forall \tilde{W}_\infty \in [\underline{W}_\infty, \bar{W}_\infty],\ \underline{\delta} = \begin{bmatrix} \delta \\ \vdots \\ \delta \end{bmatrix} \in \mathbb{R}^q \right\}. \tag{10}$$

For all constant command inputs $g \in \mathscr{G}_\delta$, the corresponding steady-state constrained vector $c_g \triangleq W_\infty g$ is located in $\mathscr{C}$ at a distance from the border greater than or equal to a fixed quantity, which depends on δ. The constraint $g \in \mathscr{G}_\delta$ adds the following q additional constraints:

$$(\hat{W}_\infty + \tilde{W}_\infty)\, g \le B_c - \underline{\delta}, \quad \forall \tilde{W}_\infty \in [\underline{W}_\infty, \bar{W}_\infty]. \tag{11}$$

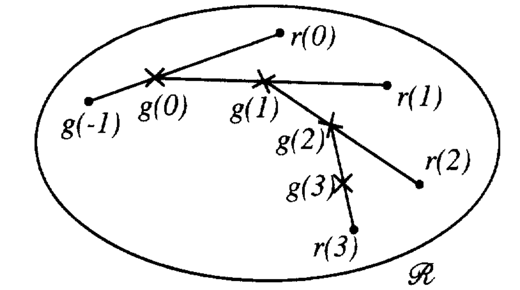

Fig. 3. Reference set.

Hereafter, the constraints (11) will be added in equation (5) to determine $\beta(\tau)$.

In order to simplify the notation, in the next theorem we consider $\tau = 0$. Moreover, we replace $x(0)$ with the sequence of past commands $\mathscr{X}_- \triangleq \{g(k)\}_{k=-1}^{-\infty}$, and define $\mathscr{R}^* \triangleq \{\mathscr{X}_-: \mathscr{X}_- \subseteq \mathscr{R}\}$. Accordingly, $c(t + 0\,|\,0, \Sigma, x(0), g)$ is denoted by $c(t, \Sigma, \mathscr{X}_-, g)$, or, when the remaining arguments will be clear from the context, by $c(t)$.

Theorem 1. Suppose that Assumptions 4 and 5 hold. Then there exists a finite time t^* such that

$$c(t) \in \mathscr{C}, \forall t \le t^* \Leftrightarrow c(t) \in \mathscr{C}, \quad \forall t \in \mathbb{Z}_+$$
$$\forall \Sigma \in \mathscr{S}, \quad \forall \mathscr{X}_- \in \mathscr{R}^*, \quad \forall g \in \mathscr{R}, \tag{12}$$

Proof. See Appendix.

Theorem 1 proves that in the optimization problem (5) it is sufficient to take into account only the constraints up to time t^*. Since we are interested in the smallest time t^*, we define the following.

Definition 1 (*Constraint horizon*).

$$t_0 = \min_{t^*} \{t^*: (12) \text{ holds}\}. \tag{13}$$

Theorem 1 provides an existence result, as it proves that t_0 in equation (13) is well defined. However, in general, t_0 is smaller than the quantity t^* estimated by Theorem 1 [see equations (32) and (33) for details]. In order to find out an algorithm which enables us to compute t_0, define $\mathscr{Q}_t \triangleq \{[\Sigma, \mathscr{X}_-, g] \in \mathscr{S} \times \mathscr{R}^* \times \mathscr{R}$: $c(k, \Sigma, \mathscr{X}_-, g) \le B_c$, $\forall k = 0, \ldots, t$, $\lim_{j\to\infty} c(j, \Sigma, \mathscr{X}_-, g) \le B_c - \underline{\delta}\}$, where, clearly, $\mathscr{Q}_{t+k} \subseteq \mathscr{Q}_t$.

Theorem 2. Suppose that Assumptions 4 and 5 hold. Then,

$$\mathscr{Q}_t \subseteq \mathscr{Q}_{t+1} \Rightarrow \mathscr{Q}_{t+k} = \mathscr{Q}_t, \quad \forall k \geq 0 \tag{14}$$

and

$$t_0 = \min_t \{t: \mathscr{Q}_t = \mathscr{Q}_{t+1}\}.$$

Proof. Let t such that equation (14) holds, and consider a generic $[\Sigma, \mathscr{X}_-, g] \in \mathscr{Q}_{t+1}$. By shifting $\mathscr{X}_- = \{g(-1), g(-2), \dots\}$ in the new sequence $\mathscr{X}^*_- \triangleq \{g, g(-1), g(-2), \dots\}$, it follows that $c(t+1, \Sigma, \mathscr{X}_-, g) = c(t, \Sigma, \mathscr{X}^*_-, g)$. Moreover, by Lemma 1, $\mathscr{X}^*_- \in \mathscr{R}^*$. Then, $[\Sigma, \mathscr{X}^*_-, g] \in \mathscr{Q}_t$, and being $\mathscr{Q}_t \subseteq \mathscr{Q}_{t+1}$, $[\Sigma, \mathscr{X}^*_-, g] \in \mathscr{Q}_{t+1}$, which implies $[\Sigma, \mathscr{X}_-, g] \in \mathscr{Q}_{t+2}$. Hence, $\mathscr{Q}_{t+1} \subseteq \mathscr{Q}_{t+2}$, or $\mathscr{Q}_{t+2} = \mathscr{Q}_{t+1} = \mathscr{Q}_t$. By induction, $\mathscr{Q}_{t+k} = \mathscr{Q}_t$, $\forall k \geq 0$. Let now $t_m \triangleq \min_t\{t: \mathscr{Q}_t = \mathscr{Q}_{t+1}\}$. If $[\Sigma, \mathscr{X}_-, g] \in \mathscr{Q}_{t_0}$, then $[\Sigma, \mathscr{X}_-, g] \in \mathscr{Q}_t$, $\forall t \geq t_0$, and hence $t_0 \geq t_m$. If, by contradiction, $t_m \leq t_0$, then, by minimality of t_0, there exists $[\Sigma, \mathscr{X}_-, g] \in \mathscr{Q}_{t_m}$ such that $c(t_m + 1, \Sigma, \mathscr{X}_-, g) \notin \mathscr{C}$, which implies $\mathscr{Q}_{t_m} \notin \mathscr{Q}_{t_m+1}$.

Following an approach similar to the one used in Bemporad *et al.* (1997) and Gilbert and Tan (1991), t_0 can be determined by the following algorithm:

Algorithm 1 (*Determination of* t_0)
(1) $t \leftarrow -1$
(2) $\mathscr{Q}_{-1} \leftarrow \{[\Sigma, \mathscr{X}_-, g]: c(-1, \Sigma, \mathscr{X}_-, g) \leq B_c$, $\lim_{j \to \infty} c(j, \Sigma, \mathscr{X}_-, g) \leq B_c - \underline{\delta}$, $\Sigma \in \mathscr{S}$, $\mathscr{X}_- \in \mathscr{R}^*$, $g \in \mathscr{R}\}$
(3) $m_t^i \leftarrow \max_{[\Sigma, \mathscr{X}_-, g] \leftarrow \mathscr{Q}_t} \{c^i(t+1, \Sigma, \mathscr{X}_-, g) - B_c^i\}$, $i = 1, \dots, q$
(4) If $m_t^i \leq 0$, for all $i = 1, \dots, q$, go to 7
(5) $t \leftarrow t + 1$
(6) Go to 3
(7) $t_0 \leftarrow t$
(8) Stop.

Observe that Algorithm 1 involves optimizations with respect to an infinite dimensional vector which contains $\mathscr{X}_-$ and the impulse (or step) response coefficients of system Σ. However, by virtue of Assumption 3 (asymptotic stability), these can be approximated with arbitrary precision by finite-dimensional optimizations. In fact, once a precision ε_p has been fixed, we can express the evolution of vector c as

$$c(t, \Sigma, \mathscr{X}_-, g) = W_t g + \sum_{k=t+1}^{M} H_k g(t-k) + \sum_{k=M+1}^{\infty} H_k g(t-k) \tag{15}$$

with M such that

$$\left| \sum_{k=M+1}^{\infty} \sum_{j=1}^{p} H_k^{ij} g^i(t-k) \right| \leq \varepsilon_p^i, \quad \forall i = 1, \dots, q,$$

$$\forall \Sigma \in \mathscr{S}, \quad \forall \mathscr{X}_- \in \mathscr{R}^*.$$

This allows one to implement Algorithm 1 by solving quadratically constrained quadratic programs (QCQP) with respect to $[\{g(-1), \dots; g(-M)\}, \{\tilde{H}_0, \dots, \tilde{H}_M\}, g]$. Notice that M, and, consequently, the complexity of Algorithm 1, is related to the estimate λ in equation (6).

5. MAIN PROPERTIES OF PRF

We investigate how the PRF affects system stability when the reference to be tracked becomes constant. Since all systems $\Sigma \in \mathscr{S}$ are asymptotically stable, next Lemma 2 first guarantees system stability by simply showing that $g(\tau)$ converges to a vector g_∞. By using the viability result of Lemma 4, Lemma 5 will prove that, amongst the admissible command inputs, g_∞ is the vector which is closest to r. Lemma 6 will show that this limit is reached in a finite time. Finally, Theorem 3 will summarize the overall properties of the PRF.

Lemma 2. Suppose $r(\tau) = r$ for all $\tau \geq \tau_0$. Then there exists $g_\infty = \lim_{\tau \to \infty} g(\tau)$.

Proof. If $g(\tau_0) = r$, it follows that $\beta(\tau) = 1$ is admissible for all $\tau \geq \tau_0$, and therefore $g_\infty = r$. Suppose $g(\tau_0) \neq r$. Since, by equation (4), $g(\tau_0 + k)$ lies on a segment whose vertices are $g(\tau_0)$ and r, by setting $d(\tau) \triangleq \|g(\tau) - r\|$ one has $g(\tau) = r + (d(\tau)/\|g(\tau_0) - r\|)[g(\tau_0) - r]$ and $0 \leq d(\tau) \leq d(\tau - 1)$. Hence, since $d(\tau)$ is monotonically non-increasing and lower-bounded, there exists $d_\infty \triangleq \lim_{\tau \to \infty} d(\tau)$, and, as a consequence, $g_\infty = r + (d_\infty/\|g(\tau_0) - r\|)[g(\tau_0) - r]$.

Lemma 3. Convexity of $\mathscr{C}$ implies convexity of the set $\mathscr{G}_\delta$ defined in equation (10), $\forall \delta > 0$.

Proof. Consider δ such that $\mathscr{G}_\delta \neq \emptyset$ ($\mathscr{G}_\delta = \emptyset$ is trivially convex), two set-points g_0, $g_1 \in \mathscr{G}_\delta$, and $g_\alpha \triangleq g_0 + \alpha[g_1 - g_0]$, $0 < \alpha < 1$. Define $W_\infty \triangleq \bar{W}_\infty + \tilde{W}_\infty$. Being $\mathscr{C}$ convex, the set $\mathscr{C}_\delta = \{c \in \mathbb{R}^q: c \leq B_c - \underline{\delta}\}$ is convex as well. Since $W_\infty g_\alpha = W_\infty g_0 + \alpha[W_\infty g_1 - W_\infty g_0]$, it follows that $W_\infty g_\alpha \in \mathscr{C}_\delta$, $\forall \Sigma \in \mathscr{S}$.

We show now that, given an admissible set-point $g_0 \in \mathscr{G}_\delta$ and a new set-point $g_1 \in \mathscr{G}_\delta$, there always exists a finite settling time after which a new admissible set point g_γ is found by moving from g_0 towards g_1. In other words, as $x(\tau)$ approaches the equilibrium state respective to g_0, a new set-point in the direction of g_1 becomes admissible.

Lemma 4 (*Viability*). Let $g_0, g_1 \in \mathscr{G}_\delta$. At each time τ there exist two positive reals γ and ε such that the command $g_\gamma = g_0 + \gamma(g_1 - g_0)$ is admissible for all the past input sequences $\mathscr{X}_- = \{g(\tau-1), g(\tau-2), \dots\} \in \mathscr{R}^*$ satisfying the condition

$$|g^i(\tau - t) - g_0^i| \le \varepsilon\lambda^{-t/2}, \quad \forall t > 0, \qquad \forall i = 1, \dots, p, \tag{16}$$

where g^i denotes the ith component of g.

Proof. See Appendix.

Lemma 5. Suppose $r(\tau) = r \in \mathscr{R}$, $\forall \tau \ge \tau_0$. Then $g_\infty = g_r$, where

$$g_r = \begin{cases} \arg\min\limits_{d \in \mathbb{R}} \|g_d - r\|^2 \\ \text{subject to } g_d \triangleq g(\tau_0 - 1) \\ \quad + d[r - g(\tau_0 - 1)] \in \mathscr{G}_\delta. \end{cases} \tag{17}$$

Proof. Suppose by contradiction that $g_\infty \ne g_r$. We can apply Lemma 4 to the pair g_∞, g_r. In fact, since $\mathscr{R}$ is bounded and $g(\tau) \in \mathscr{R}$, $\forall \tau \in \mathbb{Z}$, then there exist ε, τ_1 such that $|g^i(\tau_1 - k) - g_\infty| \le \varepsilon\lambda^{-k/2}$, $\forall k \ge 0$, with ε given by equation (36). By defining γ as in equation (36), by Lemma 4 the command $g_\gamma = g_\infty + \gamma(g_r - g_\infty)$ is admissible. Since $d_\gamma < d_\infty$ and $d(\tau)$ is monotonically not increasing, $d(\tau) \le d(\tau_1) = d_\gamma < d_\infty$, $\forall \tau \ge \tau_1$, which contradicts $d_\infty = \lim_{\tau\to\infty} d(\tau)$.

Lemma 6 (*Finite stopping time*). If $r(\tau) = r \in \mathscr{R}$, $\forall \tau \ge \tau_0$, then there exists a finite stopping time τ_s such that $g(\tau) = g_r$, $\forall \tau \ge \tau_s$, with g_r as in equation (17).

Proof. By Lemma 5, $\lim_{\tau\to\infty} g(\tau) = g_r$, and therefore Lemma 4 can be applied to the pair $g(\tau_1)$, g_r, with ε given by equation (36) and τ_1 such that $\gamma = 1$ satisfies equation (36).

Next Theorem 3 summarizes the properties of PRF.

Theorem 3 (*PRF Properties*). Suppose that Assumptions 1–5 hold, and that $r(\tau) = r \in \mathscr{R}$, $\forall \tau \ge \tau_0$. Then, once the integer t_0 is computed off-line via Algorithm 1, the optimization problem

$$\beta(\tau) = \begin{cases} \arg\max\limits_{\beta \in [0,1]} \beta \\ \text{subject to } \begin{cases} c(t+\tau\,|\,\tau, \Sigma, x(\tau), \beta) \in \mathscr{C}, \\ \quad \forall t \ge t_0, \forall \Sigma \in \mathscr{S} \\ g(\tau-1) + \beta[r(\tau) - g(\tau-1)] \in \mathscr{G}_\delta \end{cases} \end{cases} \tag{18}$$

can be solved for all $\tau \ge 0$. By setting

$$g(\tau) = g(\tau-1) + \beta(\tau)[r(\tau) - g(\tau-1)], \quad \forall \tau \ge 0,$$

the constraints $c(\tau) \in \mathscr{C}$ are fulfilled for all $\tau \ge 0$, and for all $\Sigma \in \mathscr{S}$. Moreover, after a finite stopping time τ_s,

$$g(\tau) = g_r, \quad \forall \tau \ge \tau_s$$

where g_r is defined in equation (17). In particular, if $r \in \mathscr{G}_\delta$, the PRF behaves as an all-pass filter for all $\tau \ge \tau_s$.

Remark 1. Since after a finite time $g(\tau) = g_r$, the asymptotical properties of the original system 1 remain unaltered, in particular $x(\tau) \to (I - \Phi)^{-1} G g_r$ as $\tau \to \infty$.

6. PREDICTIONS AND COMPUTATIONS

To lighten the notation, assume $\tau = 0$, and consider the predicted evolution $c(t)$ determined by a past command sequence $\mathscr{X}_- \subseteq \mathscr{R}$ and a future constant command $g(t) = g$, $\forall t \ge 0$, $c(t) = \sum_{k=0}^\infty H_k g(t-k) = \hat{c}(t) + \tilde{c}(t)$, $\hat{c}(t) \triangleq \sum_{k=0}^\infty \hat{H}_k g(t-k)$, $\tilde{c}(t) \triangleq \sum_{k=0}^\infty \tilde{H}_k g(t-k)$. Equivalently, the nominal prediction $\hat{c}(t)$ can be expressed in the computationally more preferable form,

$$\hat{c}(t) = \hat{H}_c \hat{\Phi}^t \hat{x}(0) + \hat{W}_t g, \tag{19}$$

for a consistent initial state $\hat{x}(0)$. For the sake of simplicity, suppose that the quantity N in equation (7) is such that $N > t_0$, where t_0 is the constraint horizon computed via Algorithm 1. Then, the prediction error $\tilde{c}(t)$ can be rewritten as $\tilde{c}(t) = \tilde{W}_t g + \sum_{k=t+1}^{N-1} \tilde{H}_k g(t-k) + \tilde{c}_p(t)$, $\tilde{c}_p(t) \triangleq \sum_{k=N}^\infty \tilde{H}_k g(t-k)$, $0 \le t \le t_0 < N$.

We wish to obtain a recursive formula to determine the range of $\tilde{c}_p(t)$ without requiring the storage of all past commands $g(t-k)$. Consider now a generic time $\tau \in \mathbb{Z}_+$ and define

$$\bar{c}_p^i(t+\tau\,|\,\tau) \triangleq \max_{\{\tilde{H}_k \in [-E\lambda^k, E\lambda^k]\}_{k=N}^\infty} \tilde{c}_p^i(t+\tau\,|\,\tau) = \sum_{j=1}^p \sum_{k=N}^\infty E^{ij}\lambda^k |g^j(t-k+\tau)|.$$

At time $\tau + 1$, at the same prediction step t, one has

$$\bar{c}^i_p(t + (\tau + 1) \mid \tau + 1) = \lambda \bar{c}^i_p(t + \tau \mid \tau) + \sum_{j=1}^{p} E^{ij} \lambda^N \, |g^j(t - N + \tau + 1)|. \quad (20)$$

Assuming that at time $\tau = 0$, system (1) is in an equilibrium condition, corresponding to $g(-t) \equiv g_0$ for all $t > 0$, equation (20) can be initialized with

$$\bar{c}^i_p(t \mid 0) = \frac{\lambda^N}{1 - \lambda} \sum_{j=1}^{p} E^{ij} |g^j_0| \quad (21)$$

The constraint $c(t + \tau | \tau, \Sigma, x(\tau), g) \in \mathscr{C}$ can be finally expressed as the following constraint on g:

$$(\hat{W}_t + \tilde{W}_t) g \le B_c - \hat{H}_c \hat{\Phi}^k \hat{x}(\tau) - \bar{c}_p(t + \tau \mid \tau) - \sum_{k=t+1}^{N-1} (\tilde{W}_k - \tilde{W}_{k-1}) g(t - k + \tau) \quad (22)$$

or equivalently, by equation (4), as the following constraint on β:

$$(a_t + y_t)\beta \le b_t + x_t, \quad t = 0, \dots, t_0, \quad (23)$$

where

$$a_t \triangleq \hat{W}_t [r(\tau) - g(\tau - 1)],$$

$$b_t \triangleq B_c - \hat{H}_c \hat{\Phi}^t \hat{x}(\tau) - \hat{W}_t g(\tau - 1),$$

$$x_t \triangleq \bar{c}_p(t + \tau \mid \tau) - \sum_{k=t+2}^{N-1} (\tilde{W}_k - \tilde{W}_{k-1}) g(t - k + \tau) - \tilde{W}_{t+1} g(\tau - 1),$$

and

$$y_t \triangleq \tilde{W}_t [r(\tau) - g(\tau - 1)].$$

In the same manner, constraints (11) can be rearranged in the form (23). While vectors a_t and b_t are known, vectors x_t and y_t are linear functions of the uncertainties $\tilde{W}_k$. Observe that the particular form (4) has allowed to make x_t, y_t independent; in fact, y_t is a function of $\tilde{W}_t$, while x_t depends on $\{\tilde{W}_k\}_{k=t+1}^{N-1}$. Therefore, $x^i_t \in [\underline{x}^i_t, \bar{x}^i_t]$, $y^i_t \in [\underline{y}^i_t, \bar{y}^i_t]$, for each component $i = 1, \dots, q$, and for all $t = 0, \dots, t_0$, with

$$\underline{x}^i_t = \min \left\{ \bar{c}^i_p(t) - \sum_{k=t+2}^{N-1} \sum_{j=1}^{p} (\tilde{W}^{ij}_k - \tilde{W}^{ij}_{k-1}) g^j(t - k) - \sum_{j=1}^{p} \hat{W}^{ij}_{t+1} g^j(-1) \right\}$$

$$\text{subject to} \begin{cases} \hat{W}^{ij}_k \in [\underline{W}^{ij}_k, \bar{W}^{ij}_k], \\ \hat{W}^{ij}_k - \hat{W}^{ij}_{k-1} \in [\underline{H}^{ij}_k, \bar{H}^{ij}_k], \end{cases} \quad (24)$$

$$\underline{y}^i_t = \min \left\{ \sum_{j=1}^{p} \hat{W}^{ij}_k [r^j(0) - g^j(t - k)] \right\}$$

$$\text{subject to } \hat{W}^{ij}_k \in [\underline{W}^{ij}_k, \bar{W}^{ij}_k] \quad (25)$$

and $\bar{x}^i_t$, $\bar{y}^i_t$ defined analogously.

6.1. *β-parameter selection*

The constraints involved in the optimization problem (18) can be rewritten as

$$(a^i_t + y^i_t)\beta \le b^i_t + x^i_t, \quad \forall x^i_t \in [\underline{x}^i_t, \bar{x}^i_t],$$

$$\forall y^i_t \in [\underline{y}^i_t, \bar{y}^i_t], t = 0, \dots, t_0 + 1, i = 1, \dots, q, \quad (26)$$

where $\underline{x}^i_t, \bar{x}^i_t, \underline{y}^i_t, \bar{y}^i_t$ are obtained by equations (24) and (25) for $t \le t_0$, and the index $t = t_0 + 1$ has been defined for the constraints deriving by equation (11). Consider the generic constraint

$$(a + y)\beta \le b + x. \quad (27)$$

By defining

$$f(x, y) \triangleq \frac{b + x}{a + y}, \quad x \in [\underline{x}, \bar{x}], \quad y \in [\underline{y}, \bar{y}] \backslash \{a\}$$

and recalling the result in Bazaraa and Shetty (1979, pp. 101, 473), $f(x, y)$ assumes global minima and maxima on the extreme points of its domain. Consequently, it is easy to show that equation (23) has the following solution:

$$\beta \le \min \left\{ \frac{b + \underline{x}}{a + \underline{y}}, \frac{b + \underline{x}}{a + \bar{y}} \right\} \quad \text{if } \underline{y} > -a,$$

$$\frac{b + \underline{x}}{a + \underline{y}} \le \beta \le \frac{b + \underline{x}}{a + \bar{y}} \quad \text{if } \underline{y} < -a < \bar{y}, \quad (28)$$

$$\beta \ge \min \left\{ \frac{b + \underline{x}}{a + \underline{y}}, \frac{b + \underline{x}}{a + \bar{y}} \right\} \quad \text{if } \bar{y} < -a.$$

Constraints (26) can be summarized as

$$\max_{\substack{i=1,\dots,q \\ t=0,\dots,t_0+1}} \underline{\beta}^i_t \le \beta \le \min_{\substack{i=1,\dots,q \\ t=0,\dots,t_0+1}} \bar{\beta}^i_t,$$

where, possibly, $\underline{\beta}^i_t = -\infty$, $\bar{\beta}^i_t = +\infty$.

Remark 2. Recalling equation (24), the bounds $\underline{x}^i_t, \bar{x}^i_t$ are evaluated as solutions of linear programs. The computational burden can be hugely lightened if uncertainty ranges are given only on $\tilde{W}_t$, $t = 0, \dots, N - 1$. In this case, equation (24) can be solved as trivially as equation (25) without the need of linear programs, at the cost of a more conservative description of system uncertainty, as pointed out in Section 3.

Remark 3. In the present formulation, the state $\hat{x}(\tau)$ is calculated in an open-loop manner by iterating equation (3). Apparently, this would lead to high sensitivity w.r.t. sensor noise, which in this paper is not taken into account. In fact, no feedback from Σ takes part in the selection of $g(\tau)$. However, one should remind that system Σ, in general, represents a feedback linear loop. For example, for a constant reference trajectory $r(\tau) \equiv r \in \mathscr{G}_\delta$, after a finite stopping time τ_s system Σ receives $g(\tau) \equiv r$. Then, $y(\tau)$ will track r with the error rejection properties deriving from the primal linear controller.

7. AN EXAMPLE

The PRF is applied in connection with the position servomechanism schematically described in Fig. 4. This consists of a DC-motor, a gear-box, an elastic shaft and an uncertain load. Technical specifications involve bounds on the shaft torsional torque T as well as on the input voltage V. System parameters are reported in Table 1. Model uncertainties originate from the moment of inertia J_L of the load, which is determined by the specific task. Denoting by θ_M, θ_L, respectively, the motor and the load angle, and by setting $x_p \triangleq [\theta_L \dot{\theta}_L\ \theta_M \dot{\theta}_M]'$, the model can be described by the following state-space form:

$$\dot{x}_p = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\dfrac{k_\theta}{J_L} & -\dfrac{\beta_L}{J_L} & \dfrac{k_\theta}{\rho J_L} & 0 \\ 0 & 0 & 0 & 1 \\ \dfrac{k_\theta}{\rho J_M} & 0 & -\dfrac{k_\theta}{\rho^2 J_M} & -\dfrac{\beta_M + k_T^2/R}{J_M} \end{bmatrix} x_p + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \dfrac{k_T}{RJ_M} \end{bmatrix} V,$$

$$\theta_L = [1 \quad 0 \quad 0 \quad 0]\, x_p,$$

$$T = \left[k_\theta \quad 0 \quad -\frac{k_\theta}{\rho} \quad 0\right] x_p.$$

Since the steel shaft has finite shear strength, determined by a maximum admissible $\tau_{adm} = 50\ \text{N/mm}^2$, the torsional torque T must satisfy the constraint

$$|T| \leq 78.5398\ \text{Nm}. \tag{29}$$

Moreover, the input DC voltage V has to be constrained within the range

$$|V| \leq 220\ \text{V}. \tag{30}$$

The model is transformed in discrete time by sampling every $T_s = 0.1$ s and using a zero-order holder on the input voltage. A robust digital controller is designed by pole-placement techniques, and has the following transfer function from $e = (r - \theta_L)$ to V:

$$G_c(z) = 1000 \times \frac{9.7929z^3 - 2.1860z^2 - 7.2663z + 2.5556}{10z^4 - 2.7282z^3 - 3.5585z^2 - 1.3029z - 0.0853}, \tag{31}$$

The resulting closed-loop system exhibits a very fast response but inadmissible voltage inputs and torsional torques for the references of interest, as shown in Fig. 5 for a set-point $r = 30°$.

The PRF is applied in order to fulfill equations (29) and (30) for the uncertainty range which

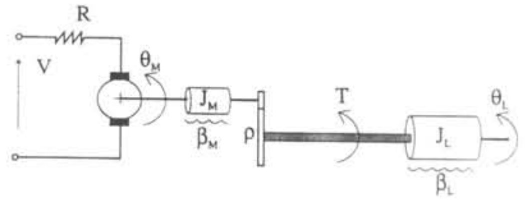

Fig. 4. Servomechanism model.

Table 1. Model parameters

Symbol	Value (MKS)	Meaning
L_S	1.0	Shaft length
d_S	0.02	Shaft diameter
J_S	Negligible	Shaft inertia
J_M	0.5	Motor inertia
β_M	0.1	Motor viscous friction coefficient
R	20	Resistance of armature
K_T	10	Motor constant
ρ	20	Gear ratio
k_θ	1280.2	Torsional rigidity
$\bar{J}_L$	$20J_M$	Nominal load inertia
β_L	25	Load viscous friction coefficient
T_s	0.1	Sampling time

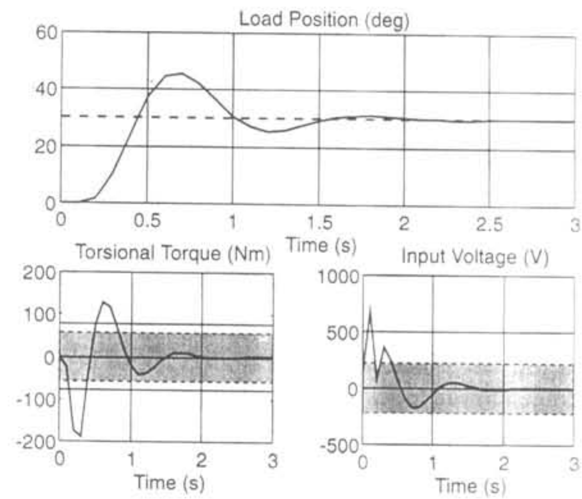

Fig. 5. Unconstrained linear response. The shadowed area represents the admissible range.

derives by an unknown load J_L, $10J_M \le J_L \le 30J_M$ (the corresponding impulse/step response uncertainty limits were obtained by maximizing $\pm H_t^{ij}$, $\pm W_t^{ij}$ with respect to J_L). Figure 6 shows the resulting uncertainty set for both the impulse and step responses. The constraint horizon is $t_0 = 15$. A nominal load inertia $\hat{J}_L = 20J_M$ is selected, along with $N = 17$, $E = 1000[1\ 1\ 1\ 1]'$, $\lambda = 0.8$, $\delta = 10^{-6}$. As a design rule of thumb, in order to have a description of the family of plant $\mathscr{S}$ as less conservative as possible, N should be approximately equal to the "length" of the impulse response in terms of time steps: sufficiently large to describe accurately the range of variation of each sample when this is perceptibly nonzero, but also small enough to minimize computational complexity. Figure 7 shows the resulting trajectories for a set-point $r = 30°$ and a load $J_L = 25J_M$. These were obtained in 112 s by using Matlab 4.2 on a 486 DX2/66 personal computer, with no particular care of code optimization. The standard Matlab LP.M routine was used to solve linear programs. Figure 8 describes the effect of the width of the uncertainty interval. The larger the uncertainty range, the more conservative the PRF action, and hence the slower the output response.

In order to make comparisons, constraint fulfilment is also achieved by linear control. The gain of controller (31) is reduced by a factor 16.9802 in order to have a maximum admissible set-point of 180° for the nominal plant $\hat{J}_L = 20J_M$ (with such a gain the linear loop reaches the maximum admissible torque T during the transient). The resulting trajectories are depicted in Fig. 9 (thin line). In the same figure, the trajectories produced by applying the PRF together with the fast controller (31) are shown (thick lines). Figure 9 shows also the resulting trajectories for $r = 90°$. While for the

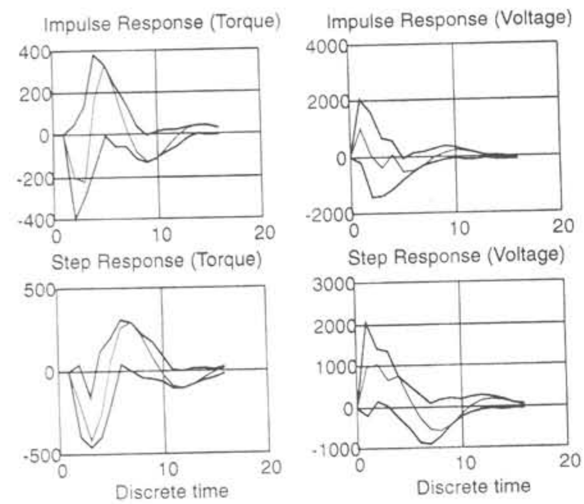

Fig. 6. Uncertainty ranges for $10J_M \le J_L \le 30J_M$ (thick lines) and nominal $J_L = 20J_M$ response (thin lines).

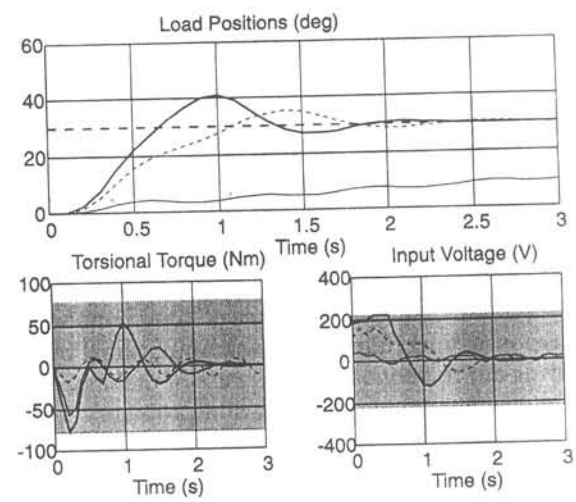

Fig. 8. Response for $\hat{J}_L = J_L = 20J_M$ and different uncertainty ranges: no uncertainty (thick solid line), $[15J_M, 25J_M]$ (dashed line), and $[2J_M, 40J_M]$ (thin solid line).

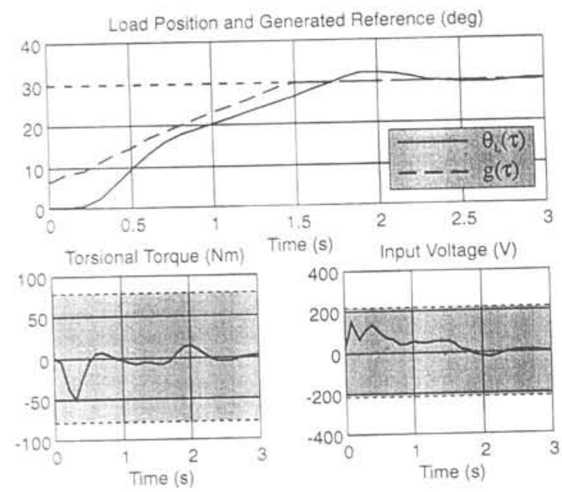

Fig. 7. Response for $J_L = 25J_M$, $10J_M \le J_L \le 30J_M$, and a real $J_L = 25J_M$.

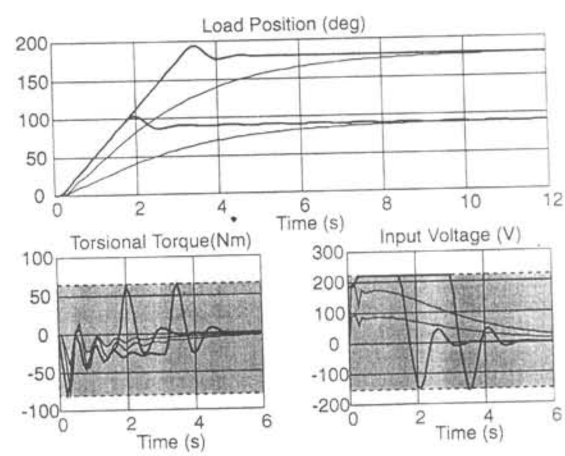

Fig. 9. Set-point $r = 90, 180°$, $J_L = 29J_M$, no uncertainty. Fast controller + PRF (thick lines) linear controller (thin lines).

linear closed loop the rise time is the same for both $r = 180$ and $90°$ (thin lines), this is no more true for responses obtained by applying the nonlinear PRF (thick lines). In general, even if a linear controller gives the same performance of the PRF *for the maximum admissible set-point*, when the desired reference sequence is nonconstant a better tracking is provided by using a nonlinear reference filter.

Remark 5. As a general rule of thumb to design controllers which will be used in connection with a PRF, in order to maximize the properties of tracking one should select a robust controller which provides fast closed-loop response for all the systems of the considered family. This usually corresponds to large violations of the constraints, which therefore can be enforced by inserting a PRF. On the other hand, this cannot improve poor tracking properties of the original system because, as observed in Remark 1, the PRF becomes an all-pass filter when the constraints are inactive.

8. CONCLUSIONS

This paper has addressed the robust PRF problem, viz., the one of filtering the desired reference trajectory in such a way that an uncertain primal compensated control system can operate in a stable way with satisfactory tracking performance and no constraint violation in the face of plant impulse/step responses uncertainties. The computational burden turns out to be moderate because of the underlying simple constrained optimization problem.

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APPENDIX: PROOFS OF THEOREM 1 AND LEMMA 4

Proof of Theorem 1. Consider the c-evolution at a generic time $t \geq N$ to commands $g(-t) \in \mathscr{R}$. $g(t) \equiv g \in \mathscr{R}$, $\forall t \in \mathbb{Z}_+$,

$$c(t) = \sum_{k=-\infty}^{-1} H_{t-k} g(k) + \hat{W}_\infty g + \tilde{W}_\infty g - \sum_{k=t+1}^{\infty} H_t g.$$

Since $\mathscr{R}$ is bounded, all references $r \in \mathscr{R}$ satisfy inequalities of the form $|r^j| \leq \bar{r}^j$, $\forall j = 1, \ldots, p$, where r^j denotes the jth component of r. Therefore, for $t \geq N - 1$,

$$|c^i(t) - [\hat{W}_\infty + \tilde{W}_\infty g]^i| \leq 2 \frac{\lambda^{t+1}}{1-\lambda} \sum_{j=1}^{p} M^{ij} \bar{r}^j.$$

By setting

$$t_i = \frac{\log\left[\frac{1}{2}\delta(1-\lambda)\sum_{j=1}^{p} M^{ij}\bar{r}^j\right]}{\log \lambda} - 1 \tag{A1}$$

and

$$t^* = \max_{i=1,\ldots,q} t_i, \tag{A2}$$

one has

$$|c^i(t) - \hat{c}^i_g - [\tilde{W}_\infty g]^i| \leq \delta, \ \forall t \geq t^*, \ \forall i = 1, \ldots, q. \tag{A3}$$

Since $g \in \mathscr{G}_\delta$, it follows that $c(t) \in \mathscr{C}$, $\forall t > t^*$. This proves the "$\Rightarrow$" part. The "$\Leftarrow$" part is obvious.

Proof of Lemma 4. Without loss of generality, assume $\tau = 0$. Define $\tilde{g}(t) \triangleq g(t) - g_0$ and consider the predicted evolution of vector c obtained by supplying system Σ with the constant command $g(t) = g_\gamma$, $\forall t \in \mathbb{Z}_+$,

$$\begin{aligned} c(t) &= \sum_{k=-\infty}^{-1} H_{t-k}[g_0 + \tilde{g}(k)] + \sum_{k=0}^{t} H_{t-k} g_\gamma \\ &= (\hat{W}_\infty + \tilde{W}_\infty)[g_0 + \gamma(g_1 - g_0)] \\ &\quad + \sum_{k=-\infty}^{-1} H_{t-k}[\tilde{g}(k) - \gamma(g_1 - g_0)]. \end{aligned} \tag{A5}$$

By Lemma 3, $g_\gamma \in \mathscr{G}_\delta$. In order to prove that g_γ is admissible, we must show that $c(t) \in \mathscr{C}$, $\forall \Sigma \in \mathscr{S}$, and $\forall t \geq 0$, or, equivalently, by equation (A5), $\sum_{k=-\infty}^{-1} H_{t-k}[\tilde{g}(k) - \gamma(g_1 - g_0)] \leq \underline{\delta}$, $\forall t \geq 0$.

Then,

$$\sum_{k=-\infty}^{-1} H_{t-k}[\tilde{g}(k) - \gamma(g_1 - g_0)] = \sum_{k=t+1}^{N-1} H_k\tilde{g}(k) + \sum_{k=\max\{N,t+1\}}^{\infty} H_k\tilde{g}(k) + \gamma \sum_{k=t+1}^{\infty} H_k(g_1 - g_0),$$

where the first sum in the second term is equal to zero for $t \geq N - 1$. By letting

$$\alpha_1^j \triangleq \begin{cases} \max\limits_{\substack{i=1,\dots,q \\ t=0,\dots,N-2}} \left\{ \begin{array}{l} |\hat{W}^i_{N-1} - \hat{W}_t| + \\ \max\{|\bar{W}^{ij}_{N-1} - \underline{W}^{ij}_t|, \\ |\underline{W}^{ij}_{N-1} - \bar{W}^{ij}_t|\} \end{array} \right\}, & \text{if } 0 \leq t \leq N-2, \\ 0, & \text{if } t \geq N-1, \end{cases} \quad E^{ij} \quad \alpha_2^j \triangleq \max_{i=1,\dots,q} \frac{\lambda^{N/2}}{1-\lambda}$$

it follows that

$$c(t) \leq B_c - \underline{\delta} + \sum_{h=1}^{2} \sum_{j=1}^{p} \alpha_h^j[\varepsilon + (g_1^j - g_0^j)\gamma] \leq B_c$$

for

$$\varepsilon \leq \frac{\delta}{2\sum_{h=1}^{2}\sum_{j=1}^{p}\alpha_h^j}, \qquad \gamma \leq \frac{\delta}{2\sum_{h=1}^{2}\sum_{j=1}^{p}\alpha_h^j(g_1^j - g_0^j)}. \tag{A6}$$