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Brief Paper

A Predictive Controller with Artificial Lyapunov Function for Linear Systems with Input/State Constraints

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Key Words—Predictive control; constraints; Lyapunov function; set-point control; optimization problems; interior-point methods; quadratically constrained quadratic programming.

Abstract—This paper copes with the problem of satisfying input and/or state hard constraints in set-point tracking problems. Stability is guaranteed by synthesizing a Lyapunov quadratic function for the system, and by imposing that the terminal state lies within a level set of the function. Procedures to maximize the volume of such an ellipsoidal set are provided, and interior-point methods to solve on-line optimization are considered.

1. *Introduction*

The necessity of satisfying input/state constraints is a feature that frequently arises in control applications. Constraints are dictated, for instance, by physical limitations of the actuators or by the necessity to keep some plant variables within safe limits. In recent years, several control techniques have been developed which are able to handle hard constraints, (see e.g. Mayne and Polak, 1993, Sussmann *et al.*, 1994). In particular, in the last decades industry has been attracted by predictive controllers (Sanchez, 1976; Richalet *et al.*, 1978; Clarke *et al.*, 1987; Garcia *et al.*, 1989; Richalet, 1993; Mosca, 1995). These approaches are based on the so-called *receding horizon* strategy. This consists in determining a future control input sequence that optimizes an open-loop *performance index*, according to a prediction of the system evolution from the current time t. Then, the sequence is actually applied to the system, until another sequence based on more recent data is newly computed. The involved prediction depends on the current state and the selected control input. Several strategies (Keerthi and Gilbert, 1988; Zheng and Morari, 1995; Mayne and Michalska, 1990; Mosca *et al.*, 1990; Clarke and Scattolini, 1991) based on receding horizon have been developed during recent years; see e.g. the survey paper (Lee and Cooley, 1997).

More recently, Bemporad and Mosca (1994, 1998), Bemporad *et al.* (1997), Bemporad (1998) and Gilbert and Kolmanovsky (1995) have individually developed computationally efficient techniques for solving constrained problems, by adding to a precompensated system a reference governor, which enforces the constraints by manipulating the reference trajectory.

Besides (Polak and Yang, 1993), where a contraction constraint on the state norm is imposed, in order to prove stability, typically in the predictive control literature the value attained by the performance index at the minimizer, as a function of the current state, is used as a Lyapunov function for the overall system (Bemporad *et al.*, 1994; Keerthi and Gilbert, 1988). This method, however, traditionally requires either a terminal state constraint or an infinite output horizon. The former has been shown to lead to poor tracking performance, especially when a small control horizon is selected in order to reduce computations. The latter requires in principle the fulfillment of an infinite number of constraints, which however can be reduced to a (possibly large) finite number by adding a steady-state input constraint (Bemporad *et al.*, 1997; Bemporad and Mosca, 1998; Bemporad, 1998; Gilbert and Tan, 1991). In this paper, we "artificially" impose that a quadratic function is a Lyapunov function for the overall closed-loop system by introducing some additional constraints. A fixed output horizon is adopted, and the classical zero terminal-state constraint is relaxed in an *ellipsoid membership* constraint, where the ellipsoid is the level set of the Lyapunov function which has maximum volume within the set of feasible states. Off-line procedures are also provided to choose the Lyapunov function so as to "orient" its level sets along the set of feasible states.

The paper is organized as follows. In Section 2 we formulate the constrained predictive control problem and propose the receding horizon controller. The properties of the resulting closed-loop are analyzed in Section 3. Section 4 is devoted to the selection of the Lyapunov function. Computational issues are discussed in Section 5, and simulation results are reported in Section 6.

2. *Problem formulation*

Consider the following discrete-time system:

$$\begin{aligned} x(t+1) &= Ax(t) + Bu(t), \\ y(t) &= Cx(t) + Du(t), \\ c(t) &= Px(t) + Ru(t), \end{aligned} \tag{1}$$

along with a desired output reference $r(t) \in \mathbb{R}^p$, where $x(t) \in \mathbb{R}^n$ is the state vector, $y(t) \in \mathbb{R}^p$ the output, $u(t) \in \mathbb{R}^m$ the input, $c(t) \in \mathbb{R}^l$ the vector to be constrained within the given (possibly unbounded) polyhedron

$$\mathscr{P} \triangleq \{c \in \mathbb{R}^l : A_c c \le B_c\}, \quad B_c \in \mathbb{R}^q.$$

This is supposed to satisfy the following assumption.

Assumption 1. $\mathscr{P}$ has a nonempty interior.

By defining the polyhedral set

$$\mathscr{C} \triangleq \left\{ \begin{bmatrix} x \\ u \end{bmatrix} \in \mathbb{R}^{n+m} : A_c(Px + Ru) \le B_c \right\},$$

the problem is to generate the input $u(t)$ to system (1) so as to satisfy the constraint

$$\begin{bmatrix} x(t) \\ u(t) \end{bmatrix} \in \mathscr{C}, \tag{2}$$

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for all $t \geq 0$, without destabilizing the system, and minimize the tracking error $y(t) - r(t)$. We assume that equation (1) satisfies the following stability property

Assumption 2. A is asymptotically stable.

We underline that Assumption 2 is non-restrictive. In fact, equation (1) might consist of a linear system which has been precompensated via standard control techniques *without considering the constraints*. In this case, because of feedback loops, the command input to the actuator becomes state-dependent. However, in equation (1) the constrained vector $c(t)$ can be any linear combination of inputs and states, and therefore possible saturating actuators can be still tackled.

Constraint (2) imposes limitations on the inputs u which can be supplied in steady state, and therefore on the set-points r which can be tracked. For reasons that will be clearer soon, we fix a small scalar $\delta > 0$ and define the set

$$C_\delta \triangleq \left\{ \begin{bmatrix} x \\ u \end{bmatrix} \in \mathbb{R}^{n+m} : A_c(Px + Ru) \leq B_c - \delta \mathbf{1} \right\}$$

where $\mathbf{1} \triangleq [1 \ldots 1]'$ has suitable dimensions. For a given set-point r, let

$$u_r \triangleq \begin{cases} \arg\min_{u \in \mathbb{R}^m} \|(CH + D)u - r\|^2 \\ \text{s.t.} \begin{bmatrix} Hu \\ u \end{bmatrix} \in C_\delta, \end{cases} \tag{3}$$

where $H \triangleq (I - A)^{-1}B$ is the input-to-state DC-gain, and $(CH + D)$ is assumed to be full rank. Note that the constraint in equation (3) imposes that the predicted steady-state constrained vector, corresponding to the constant input level u_r, lies inside $\mathscr{P}$ by at least a fixed distance away from the border.

We adopt the following receding horizon control law. Let

$$\mathbf{v} \triangleq \begin{bmatrix} v(N_u - 1) \\ \vdots \\ v(0) \end{bmatrix} \in \mathbb{R}^{N_u m} \tag{4}$$

be the future input sequence to be selected in order to minimize the performance function

$$J(\mathbf{v}, x(t), u_r) \sum_{k=0}^{N_y - 1} \|y(k|t) - CHu_r\|^2_{\Psi_y} + \sum_{k=0}^{N_u - 1} \|v(k) - u_r\|^2_{\Psi_u}, \tag{5}$$

where $\Psi_y \geq 0$ and $\Psi_u > 0$ are symmetric weight matrices, $\|y\|^2_\Psi \triangleq y'\Psi y$, $N_y < \infty$ is the output horizon, $N_u < \infty$ the input horizon, and $y(k|t)$ is the evolution of the output vector at time $t + k$ predicted at time t, according to model (1), initial state $x(t)$, and input (Rawlings and Muske, 1993)

$$u(t + k) = \begin{cases} v(k) & \text{if } 0 \leq k \leq N_u - 1, \\ v(N_u - 1) & \text{if } k \geq N_u \end{cases}.$$

(the same notation will be used hereafter for the predicted evolution of the state $x(k|t)$ and the constrained vector $c(k|t)$).

In order to prove later stability results in the case of constant references $r(t) \equiv r$, we impose that the function

$$V(\tilde{x}) \triangleq \tilde{x}'\mathscr{L}\tilde{x}, \tag{6}$$

(where $\tilde{x} \triangleq x - x_r$, $x_r \triangleq Hu_r$) is a Lyapunov function for the overall closed-loop system. This is achieved by introducing the constraints

$$\tilde{x}'(k + 1|t)\, \mathscr{L}\tilde{x}(k + 1|t) \leq \tilde{x}'(k|t)\, \mathscr{L}\tilde{x}(k|t) - \tilde{x}'(k|t)\, Q\tilde{x}(k|t), \quad \forall k = 0, \ldots, N_y - 1, \tag{7}$$

and

$$x(N_y|t) \in \Omega_{\gamma_r}, \tag{8}$$

where Ω_{γ_r} is the ellipsoid

$$\Omega_{\gamma_r} \triangleq \{x \in \mathbb{R}^n : (x - x_r)'\mathscr{L}(x - x_r) \leq \gamma_r\}, \tag{9}$$

and L satisfies the Lyapunov equation

$$\mathscr{L} = A'\mathscr{L}A + Q \tag{10}$$

for some matrix $Q = Q' > 0$, whose selection will be the topic of Section 4. Notice that the *ellipsoid membership* condition (8) replaces the more usual zero terminal-state constraint $x(N_r|t) = 0$.

In order to have equation (8) as less stringent as possible, the parameter γ_r is chosen on line as

$$\gamma_r \triangleq \sup_{\rho \geq 0} \{\rho : \Omega_\rho \subset \mathscr{X}_r\}, \tag{11}$$

where $\Omega_\rho \triangleq \{x \in \mathbb{R}^n : (x - x_r)'L(x - x_r) \leq \rho\}$, and

$$\mathscr{X}_r \triangleq \{x \in \mathbb{R}^n : A_c(Px + Ru_r) \leq B_c\} \tag{12}$$

is the section of $\mathscr{C}$ generated by the hyperplane $u = u_r$. By using the Lagrangian function and first-order conditions, the parameter γ_r can be easily computed as

$$\gamma_r = \min_{j=1,\ldots,q} \left\{ \frac{[B_c^j - A_c^j(Ru_r + Px_r)]^2}{(A_c^jP)\, \mathscr{L}^{-1}(A_c^jP)'} \right\}, \tag{13}$$

where $(\cdot)^j$ denotes the jth row of $(\cdot)$. Note that the condition $\delta > 0$ implies that each section $\mathscr{X}_r$ has a nonempty interior, and therefore $\gamma_r > 0$. Vice versa, $\delta = 0$ and $Px_r + Ru_r \in \partial\mathscr{P}$ (boundary of $\mathscr{P}$) would lead to $\gamma_r = 0$ and the ellipsoid membership condition would degenerate in a zero-terminal state condition. Moreover, if the problem is non trivial (i.e. $\mathscr{P} \neq \mathbb{R}^l$, which implies rank $A_c > 0$), and the constraints involve state components such that $\text{rank}(A_cP) \geq 1$, then $\mathscr{X}_r \neq \mathbb{R}^n$, which implies $\gamma < +\infty$.

Finally, constraint fulfillment $c(t) \in \mathscr{P}$ is enforced by adding the constraints

$$A_c[Px(j|t) + Rv(j)] \leq B_c, \ \forall j = 0, \ldots, N_y - 1. \tag{14}$$

Define

$$\Gamma(t) \triangleq \{\mathbf{v} \in \mathbb{R}^{N_u m} : \text{equations (7), (8), and (14) are satisfied}\}$$

and, for $\Gamma(t) \neq \emptyset$,

$$\mathbf{v}^*(t) \triangleq \arg\min_{\mathbf{v} \in \Gamma(t)} J(v, x(t), u_{r(t)}).$$

Then, at each time t, the optimal vector $\mathbf{v}(t) = [v_t'(N_u - 1), \ldots, v_t'(0)]'$ is selected as

$$\mathbf{v}(t) \triangleq \begin{cases} \mathbf{v}^*(t) & \text{if } \Gamma(t) \neq \emptyset, \\ \mathbf{v}_1(t) & \text{otherwise}, \end{cases} \tag{15}$$

where $\mathbf{v}_1(t)$ is a one-step shifted version of $\mathbf{v}(t - 1)$ and is defined as

$$\mathbf{v}_1(t) \triangleq \begin{bmatrix} u_{r(t-1)} \\ v_{t-1}(N_u - 1) \\ \vdots \\ v_{t-1}(1) \end{bmatrix}. \tag{16}$$

Finally, according to the receding horizon strategy described above,

$$u(t) \triangleq [0 \ldots 0\ I_m]\mathbf{v}_t. \tag{17}$$

The entire procedure is then repeated at time $t + 1$. The above scheme is completed by the following hypothesis on $x(0)$.

Assumption 3. At time $t = 0$ there exists an input sequence $\mathbf{v}(-1)$ and a reference r_{-1} such that the sequence $\mathbf{v}_1(0)$ obtained by equation (16) satisfies equations (7), (8), and (14) from the initial state $x(0)$.

For instance, Assumption 3 is verified for $x(0) = 0$, $\mathbf{v}(-1) = 0$, $r_{-1} = 0$, $0 \in \mathscr{P}$.

3. *Feasibility and stability*

In principle, any symmetric positive-definite matrix $\mathscr{L}$ might be used to define the artificial Lyapunov function (6). However, the particular structure (10) allows the proof of the following feasibility lemma.

Lemma 1. Let Assumption 3 hold, $\mathscr{L}$ satisfy equation (10), and $r(t) \equiv r$. Then $\Gamma(t) \neq \emptyset$, $\forall t \geq 0$.

Proof. The lemma can be easily proved by induction. Suppose that at time $t-1$ a feasible $\mathbf{v}(t-1)$ exists, and consider at time t the sequence $\mathbf{v} \triangleq \mathbf{v}_1(t)$ as in equation (16). Since, by equation (17), $\tilde{x}(t) = \tilde{x}(1|t-1)$, for the chosen $\mathbf{v}$ it holds that $\tilde{x}(k|t) = \tilde{x}(k+1|t-1)$, $\forall k = 0, 1, \ldots, N_y - 2$. Then, constraints (7) and (14) are satisfied $\forall k = 0, 1, \ldots, N_y - 2$. Since by equation (10)

$$\begin{aligned} \tilde{x}'(N_y|t)\, \mathscr{L}\tilde{x}(N_y|t) &= \tilde{x}'(N_y-1|t)\, A'\mathscr{L}A\tilde{x}(N_y-1|t) \\ &= \tilde{x}'(N_y|t-1)\, L\tilde{x}(N_y|t-1) \\ &\quad - \tilde{x}'(N_y|t-1)\, Q\tilde{x}(N_y|t-1), \end{aligned} \tag{18}$$

constraints (8) is satisfied, as well as equation (7) for $k = N_y - 1$. Because the ellipsoid $\Omega_{\gamma_r} \subset X_r$, fulfillment of equation (14) for $k = N_y - 1$ follows. □

Remark 1. Lemma 1 proved that, for constant reference trajectories $r(t)$, a feasible solution exists at each time $t \geq 0$. However, besides the case of nonconstant $r(t)$, the second instance in equation (15), might be verified because of numerical issues. This will be discussed in Remark 3.

Next Theorem 1 describes the asymptotical behavior of the overall control scheme.

Theorem 1. Consider system (1) along with Assumptions 1–3, and let $r(t) \equiv r$, $\forall t > t_r \geq 0$. Then, the control strategy (15)–(17), based on the optimization of the performance function (5) in the presence of constraints (7), (8), and (14), guarantees stability of the overall control loop, in that

(i) $\lim_{t\to\infty} x(t) = x_r$;
(ii) $\lim_{t\to\infty} u(t) = u_r$;

Moreover, if $\Psi_y = 0$, there exists a finite time $t_s \geq t_r$ such that

(iii) $u(t) \equiv u_r$, $\forall t \geq t_s$.

Proof. (i) By Lemma 1 the solution $\mathbf{v}(t) = \mathbf{v}^*(t)$ in equation (15) exists at each time $t \geq 0$, and hence the trajectories $x(t)$ are defined, $\forall t \geq 0$, for all the initial states $x(0)$ which satisfy Assumption 3. Since discrete-time trajectories are bounded in any finite interval, without loss of generality assume $t_r = 0$, $x(0) = x(t_r)$. The function V defined in equation (6) is a Lyapunov function for $\tilde{x}(t) = x(t) - x_r$. Since $\mathscr{L} > 0$, V is radially unbounded. Hence, by (LaSalle 1986, Proposition 2.3), the trajectories $\tilde{x}(t)$ are bounded. Since $\Delta V(t) \triangleq V(t) - V(t-1) = 0$ iff $\tilde{x}(t) = 0$, by (LaSalle, 1986, Proposition (2.6)) x_r is asymptotically stable, the domain of attraction being the set of initial states $x(0)$ such that Assumption 3 holds.

(ii) Since $\gamma_r > 0$, by (i) after a finite time t_s $x(t) \in \Omega_{\gamma_r}$, $\forall t \geq t_s$. Consider the vector $\mathbf{v}_r \triangleq [u_r' \ldots u_r']'$ at time $t \geq t_s$. By applying $v(k) = u_r$ for $k = 0, 1, \ldots, N_u - 1$, constraint (7) is satisfied, and hence the predicted trajectory $x(k|t) \in \Omega_{\gamma_r}$, $\forall k = 0, 1, \ldots, N_y$. In particular, equation (8) is fulfilled, and since $\Omega_{\gamma_r} \times \{u_r\} \subseteq \mathscr{C}$, constraints (14) are fulfilled as well. Then, $\mathbf{v}_r$ is admissible at each time $t \geq t_s$. Consequently, $J(\mathbf{v}(t), x(t), u_r) \leq J(\mathbf{v}_r, x(t), u_r)$. Since J is continuous in $x(t)$, by (i) it follows $\lim_{t\to\infty} J(\mathbf{v}_r, x(t), u_r) = J(\mathbf{v}_r, x_r, u_r) = 0$, and hence, $\lim_{t\to\infty} J(\mathbf{v}(t), x(t), u_r) = 0$. Therefore, $\lim_{t\to\infty} \sum_{k=0}^{N_u-1} \|v_t(k) - u_r\|^2_{\Psi_u} = 0$, and since $\Psi_u > 0$ it follows $\lim_{t\to\infty} \mathbf{v}(t) \to \mathbf{v}_r$.

(iii) Consider again $t \geq t_s$. The performance function (5) is minimized by $\mathbf{v}_r$, which is feasible. Hence, $\mathbf{v}(t) = \mathbf{v}_r$, and $u(t) = u_r$, $\forall t \geq t_s$. □

Remark 2. When $\Psi_y = 0$, (iii) implies that the original asymptotical properties of system (1) are not modified by the predictive controller (15)–(17).

4. *Selection of the Lyapunov function*

As mentioned above, we relax the classical zero terminal state constraint in the ellipsoid membership condition (9). Therefore, for each set-point r to be tracked, we wish to provide a terminal ellipsoid Ω_{γ_r} which is as large as possible. The basic idea is to select off line matrices Q, $\mathscr{L}$ so as to "orient" Ω_{γ_r} as the sections $\mathscr{X}_r$ of the constraint set $\mathscr{C}$, which are defined in equation (12). This is achieved by choosing the matrices Q, $\mathscr{L}$ which provide the maximum volume ellipsoid centered in x_r and inscribed in X_r.

By setting $S \triangleq A_c P$, $T \triangleq B_c - A_c[PH + R]u_r$, it results

$$\mathscr{X}_r = \{\tilde{x} \in \mathbb{R}^n : S^i \tilde{x} \leq T^i, i = 1, \ldots, q\}.$$

Since $[H'\ I]'u_r \in C_\delta$, it follows that $0 \in \operatorname{Int} \mathscr{X}_r$, and hence $T^i > 0$, $\forall i = 1, \ldots, q$. Therefore, by setting $W^i \triangleq S^i/T^i$, it results

$$\mathscr{X}_r = \{\tilde{x} \in \mathbb{R}^n : W^i \tilde{x} \leq 1, i = 1, \ldots, q\}. \tag{18}$$

Then, chosen a scalar $\varepsilon > 0$, $\mathscr{L}$ can be determined by solving the following optimization problem (Khanchiyan and Todd, 1993):

$$L \triangleq \begin{cases} \arg\min_L \log(\det(\mathscr{L})) \\ \mathscr{L} - A'\mathscr{L}A > \varepsilon I \\ \text{s.t. } (W^i)'\mathscr{L}^{-1}\, W^i \leq 1, \ \forall i = 1, \ldots, q \\ \mathscr{L} = \mathscr{L}' > 0. \end{cases} \tag{19}$$

The inscribed ellipsoid provided by equation (19) has the form

$$\Omega_{\gamma_r} = \{\tilde{x} : \tilde{x}'\mathscr{L}\tilde{x} \leq 1\}, \tag{21}$$

The constraint $Q > \varepsilon I$ in equation (19) determines a lower bound for the desired rate of decreasing of the Lyapunov function (6), and prevents that the inscribed ellipsoid degenerates in a cylinder in the case that X_r is unbounded in some direction. Moreover, the constraint in equation (19) can be rewritten in the form of linear matrix inequalities (LMIs) (Boyd *et al.*, 1993)

$$\mathscr{L} - A'\mathscr{L}A - \varepsilon I > 0,$$

$$\begin{bmatrix} 1 & (W^i)' \\ W^i & \mathscr{L} \end{bmatrix} > 0, \ \forall i = 1, \ldots, q. \tag{22}$$

Note that the strict positivity required in the second constraint in equation (21) amounts to inscribe the ellipsoid in the open set $\operatorname{Int}\mathscr{X}_r = \{\tilde{x} : W^i\tilde{x} < 1, i = 1, \ldots, q\}$, and consequently there are no numerical differences with the results provided by equation (19).

The solution of equations (19) and (21) provides matrices Q, $\mathscr{L}$ which depend on u_r. Therefore, in principle one should solve the optimization problem on line for each current reference $r(t)$. In order to shift off-line the selection of Q, $\mathscr{L}$, we define an "average" vector u_0 and evaluate Q, $\mathscr{L}$ for such a vector. Then, the same matrices Q, $\mathscr{L}$ will be used for any u_r, the volume of the ellipsoid being maximized by choosing γ_r in equation (11) according to equation (13). The selection of u_0 proceeds as follows. Consider a section $\mathscr{X}_u$ of $\mathscr{C}$ at a fixed u, the hyperplane $x = Hu$, and the polytope $\mathscr{X}_R \triangleq HR$, where R is the hyperrectangle

$$\mathscr{R} \triangleq \{u \in \mathbb{R}^m : |u^i| \leq M^i \ i = 1, \ldots, m\},$$

and the components M^i of the vector $M \in \mathbb{R}^m$ are arbitrarily large and fixed. The situation is depicted in Fig. 1. We define the vector u_0 in such a manner that the vector $z_0 \triangleq [H'I_m]'u_0$ is "as far as possible" from the boundary $\partial\mathscr{C}$ of $\mathscr{C}$, namely by requiring that the minimum distance of z_0 from the faces of $\mathscr{C}$ is maximized. Accordingly,

$$u_0 \triangleq \begin{cases} \arg\min\limits_{u \in \mathbb{R}^m} \min\limits_{i=1,\ldots,q} \dfrac{(A_u^i u - B_c^i)^2}{\|A_c^i P\|^2 + \|A_c^i R\|^2} \\ \text{s.t. } A_u u \leq B_c, \begin{bmatrix} I_m \\ I_m \end{bmatrix} u \leq \begin{bmatrix} M \\ M \end{bmatrix}, \end{cases} \tag{22}$$

where $A_u \triangleq A_c(PH + R)$. The constraint $u \in \mathscr{R}$ guarantees that u_0 is finite. Notice that this constraint is not kept into account in determining the optimal sequence $\mathbf{v}(t)$ in equation (15).

5. *Computations*

In this section we shall consider computational issues involved in the minimization of equation (5) subject to the constraints (7), (8), and (14) with respect to the optimization vector $\mathbf{v}$ defined in equtiion (4). This will be referred to as *on-line optimization problem* (OLOP). In general, OLOP is a nonconvex

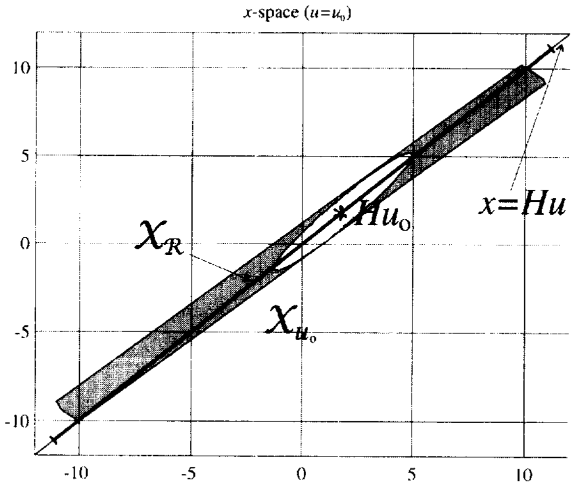

Fig. 1. Sets $\mathcal{X}_{u0}$, $\mathcal{X}_R$ and hyperplane $x = Hu$; point u_0 and ellipsoid generated, respectively by equations (22) and (19)–(21).

quadratically constrained quadratic program (QCQP). In fact, despite equation (5) is a (strictly) convex quadratic function, equation (14) are linear, and equation (8) are convex quadratic, constraints (7) are in general nonconvex. An exception is the case $N_u = 1$, which renders the controller described in the previous sections close to the *reference governor* technique developed in Bemporad *et al.* (1997), Bemporad and Mosca (1998) and Bemporad (1998). In this case, equation (7) is convex quadratic as well, and OLOP can be recast as a convex *second-order cone programming* (SOCP) problem (Mehrotra and Sun, 1991), for which efficient solvers are available (Lobo *et al.*, 1997a; Alizadeh *et al.*, 1997).

For the general case $N_u > 1$, we address two alternative computational tools, viz. *sequential quadratic programming* (SQP) and SOCP, being the investigation of other branch-and-bound algorithms for nonconvex QCQP beyond the scope of this paper (the reader may refer for instance to (Al-Khayyal *et al.*, 1995).

5.1. *Sequential quadratic programming (SQP) methods.* By solving OLOP via SQP methods, which are the most commonly used algorithms for generic nonlinear constrained optimization, as the one adopted in the Matlab Optimization Toolbox (Grace, 1992), finite available computational time may lead to suboptimal *local* minima. However, in this paper, the stability proof in Theorem 1 is not affected by local minima, since the existence of a Lyapunov function is guaranteed by equation (7), and feasibility at each time $t > 0$ is ensured. Therefore, as soon as Assumption 3 is satisfied, SQP-based solutions to the OLOP problems preserve stability and feasibility, despite they might lead to tracking-performance deterioration.

5.2. *Interior-Point Methods.* In very recent years, as reliable public-domain and commercial software packages became available, interior-point methods (Nesterov and Nemirovskii, 1994) have gained popularity as efficient tools to solve certain classes of convex problems.

In order to solve OLOP via interior point methods, let us investigate the nature of equation (7) for $N_u > 1$. By equation (19), $B'\mathcal{L}B$ is positive definite, and hence admits a Cholesky factorization $B'\mathcal{L}B = S'S$, with S nonsingular. Consider $k \geq 1$ and let $z_k \triangleq Sv(k)$, $s_k \triangleq [v'(k-1) \; \dots \; v'(0)]'$. By equation (10), constraint equation (7) is equivalent to

$$v'(k)\, B'\mathcal{L}Bv(k) + 2\tilde{x}(k|t)\, A'\mathcal{L}Bv(k) \leq 0,$$

and hence can be rewritten in the form

$$z'z + M'z + 2z'Ns \leq 0, \tag{23}$$

where the index k has been omitted for clarity, and M, N are constant matrices dependent of A, B, and $\mathcal{L}$. equation (23) is a *bilinear* (or *biaffine*) inequality (Goh *et al.*, 1994), and makes the global minimization OLOP an NP-hard problem (Boyd and Vandenberghe, 1997). In fact, by considering for the sake of simplicity the SISO case, and by introducing a new scalar variable w, equation (23) can be rewritten in the form

$$\left\| \begin{bmatrix} 2w \\ 2z + M + 2Ns \end{bmatrix} \right\| \leq \begin{cases} -M - 2Ns & \text{for } 0 \leq z \leq -(M + 2Ns), \\ M + 2Ns & \text{for } -(M + 2Ns) \leq z \leq 0. \end{cases} \tag{24}$$

Then, each constraint (7) gives rise to two cases, leading the global minimization of OLOP to the propagation through a decision binary-tree which, in the worst case, involves the solution of $2(2^{N_r - 1} - 1)$ SOCP problems.

In alternative, we propose a modified version of OLOP which can be solved by only one SOCP. Consider the inequality

$$z'z + M'z + 2z'Ns \leq -s'N'Ns, \tag{25}$$

which is a stronger condition than equation (23). Equation (25) can be rewritten as

$$(z + Ns)'(z + Ns) + M'z \leq 0$$

or, equivalently, as the second-order cone constraint

$$\left\| \begin{bmatrix} 2(z + Ns) \\ 1 + M'z \end{bmatrix} \right\| \leq 1 - M'z. \tag{26}$$

Then, by changing equation (7) in the form

$$\tilde{x}'(k+1|t)\,\mathcal{L}\tilde{x}(k+1|t) \leq \tilde{x}'(k|t)\,\mathcal{L}\tilde{x}(k|t) - \tilde{x}'(k|t)\,Q\tilde{x}(k|t) - s_k'R_k'R_k s_k, \quad \forall k = 1, \dots, N_u - 1, \tag{27}$$

where

$$R_k \triangleq B'\mathcal{L}A[I_m|A|\dots|A^{k-1}]B,$$

the resulting modified OLOP can be solved via one SOCP. Notice that the proof of Theorem 1 still holds with the new constraints (27). However, implementation simplicity is obtained at the price of a diminished feasibility, and hence of performance, (27) being more stringent than equation (7).

6. *Simulation Results*

Example 1. The proposed control strategy has been investigated by simulations on the following second order discrete-time SISO system

$$\begin{aligned} x(t+1) &= \begin{bmatrix} 1.5910 & -0.7261 \\ 1 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t), \\ y(t) &= [0.1351 \quad 0]\, x(t), \\ c(t) &= \begin{bmatrix} -1.5952 & 1.7303 \\ -0.1384 & -0.1595 \\ 0.2 & 0.1 \end{bmatrix} x(t), \end{aligned} \tag{28}$$

and the reference

$$r(t) = \begin{cases} 1 & \text{if } t < 20, \\ 0 & \text{if } t \geq 20. \end{cases}$$

The y- and c-responses of system (28) in the absence of constraints are depicted in Fig. 2. The transfer function from the input u to the constrained variable c is underdamped, and nonminimum phase for the first component c^1. In order to compress the dynamics of c within the set

$$\mathcal{P} = \{c \in \mathbb{R}^3\colon -1.5 \leq c^1 \leq 2, \; -3 \leq c^2 \leq 3, \; -3.1 \leq c^3 \leq 3.1\}$$

and improve the tracking output properties, we adopt the control law described in the previous sections along with the parameters $\Psi_y = \Psi_u = 1$, $N_y = N_u = 10$, $\varepsilon = 10^{-6}$, $\delta = 10^{-8}$, and solve on-line optimization by using SQP methods. Figure 3 shows the resulting trajectories. Equation (22) generates the "average" vector $u_0 = 0.25$, for which the Lyapunov matrix $\mathcal{L}$ is

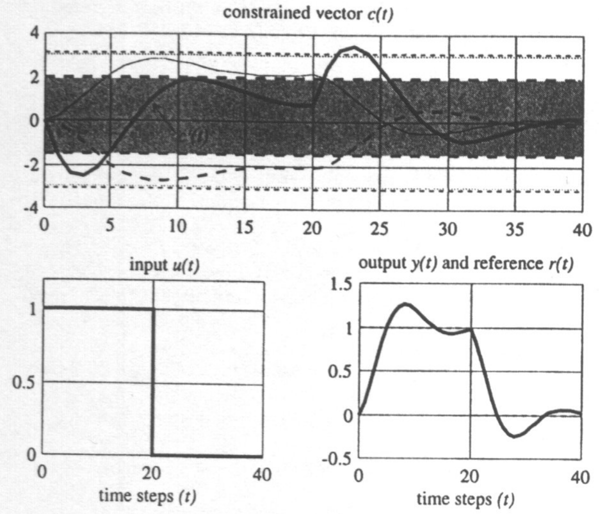

Fig. 2. Unconstrained response of system (28). Thick solid line: $c^1(t)$; dashed line: $c^2(t)$; thin solid line: $c^3(t)$; shaded area: admissible range for $c^1(t)$.

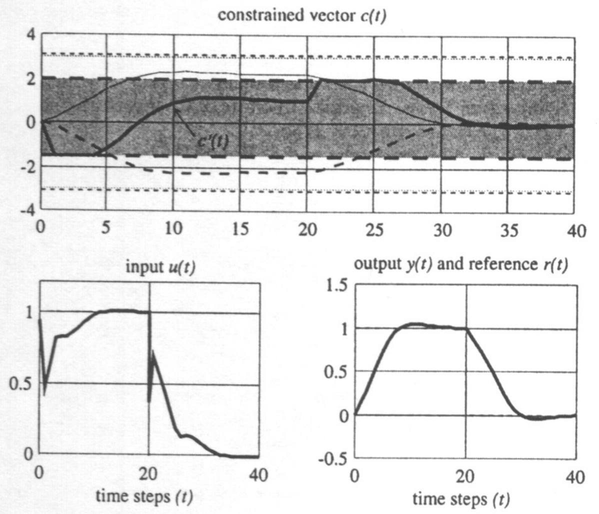

Fig. 3. Response of the system (28) with constraints. Same line styles as in Fig. 2.

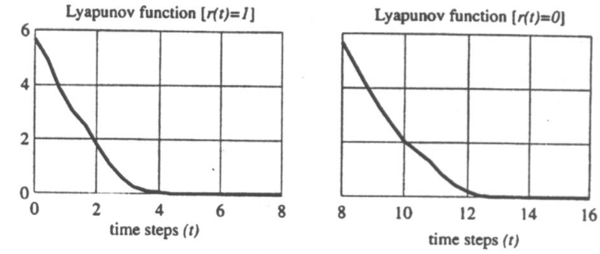

Fig. 4. Lyapunov function for the evolution in Fig. 3.

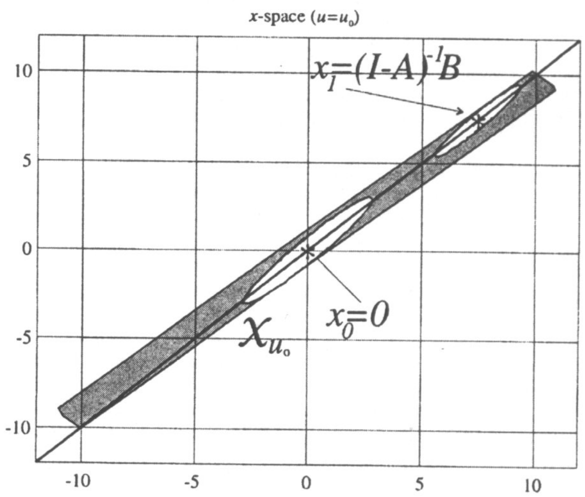

Fig. 5. Ellipsoids computed on-line via equation (13) for $r = 0$ and $r = 1$.

selected. The behavior of the Lyapunov function V defined in equation (6) is shown in Fig. 4. Finally, Fig. 5 depicts the ellipsoid generated on line for $r = 1$ and $r = 0$. Note that since c does not depend on u, the sections $\mathscr{X}_r$ coincide for any u_r.

6.1. *Terminal ellipsoid vs zero terminal state constraint.* In order to assess the improvements which derive by having a terminal ellipsoidal constraint rather than a zero terminal state constraint, consider again the previous example. For the set-point $r(t) \equiv 0.75$ and the constraint horizon $N_u = 3$, the constraint $\tilde{x}(N_u|t) = 0$ produces infeasibility at each $t \geq 0$. On the other hand, for the same setup, the on-line optimization problem (OLOP) (5), (7), (8), (14) is feasible at each $t \geq 0$. For large constraint horizons N_u, both methods are feasible and behave in a comparable manner. In fact, as N_u increases, during the transient the active constraints become equation (14), which are common to both procedures.

6.2. *Off-line vs on-line selection of $\mathscr{L}$.* In order to investigate the degradation of performance caused by moving the solution of equation (19) off line (via the definition of u_0 in equation (22)), simulations have been run by solving equation (19) on line for each u_r. The results are indistinguishable from those ones where L is computed off-line. This can be explained by two reasons. First, it is clear that the "orientation" in the x-space of the maximal ellipsoid obtained by equation (19) is related on the shape of the admissible set $\mathscr{X}_r$ defined in equation (12), which for $R = 0$ does not depend on u_r, and in general mainly depends on the matrix A_cP. Second, the size of the ellipsoid is always maximized on-line, and therefore the sensitivity with respect to the orientation is low, at least when the condition number of L is small (i.e. the ellipsoid is not flattened out in any direction).

6.3. *SQP vs SOCP.* Simulations have been performed to compare sequential quadratic programming (SQP) against second order cone programming (SOCP) interior-point methods. For the first, the standard CONSTR.M routine provided in the Matlab Optimization Toolbox has been used, while SOCP has been solved by using the package [Lobo *et al.*, 1997a]. Simulation tests have been run on a 486DX4/100 with Matlab 4.2 and Simulink 1.3. No appreciable feasibility and performance differences were noticed by replacing constraints (7) by equation (27), as described in Section 5, which is needed to implement SOCP. Figure 6 reports average computational times for different values of N_u (same system and control parameters of Example 1). We point out that computational times should not be directly compared, since SQP was executed by interpreted code, while SOCP by compiled code.

Remark 3. During some simulations, we observed that SOCP interior-point methods were failing, while SQP were performing correctly. The reason for this might be that interior-point methods much more suffer from "flat" domains. In fact, despite no equality constraint is present, the feasible domain might become flatter, for instance when the only admissible control

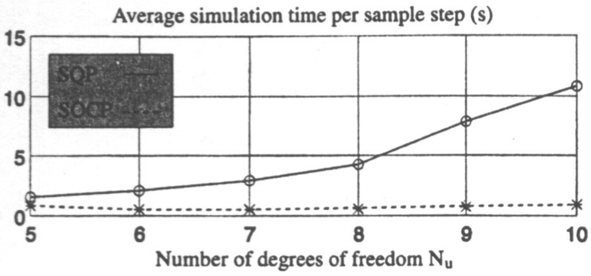

Fig. 6. SQP vs SOCP solver. Average simulation time per sample step (s).

moves at time t are obtained by shifting the previous optimal solution $\mathbf{v}(t-1)$. Note that in this case a failure of the interior-point method leads to the second case in equation (15), and hence $\mathbf{v}(t) = \mathbf{v}_1(t)$ is safely applied (constraint fulfillment follows by the proof of Lemma 1), without even sacrificing optimality.

7. *Conclusions*

In this paper we have presented a predictive control approach to satisfy input and/or state hard constraints. Stability is guaranteed by imposing an ellipsoid membership constraint, and that a certain quadratic function is a Lyapunov function for the system. Procedures are provided to optimally select such a function. Future research will concern criteria to select control horizons N_u such that the system can be moved to the desired equilibrium point from any initial state within a given set, and when this is possible. Moreover, an extension of the method to linear systems with disturbances and nonlinear systems will be considered, as constraint (7) might be generalized to different settings.

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