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Multiobjective model predictive control[☆]

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ABSTRACT

This paper proposes a novel model predictive control (MPC) scheme based on multiobjective optimization. At each sampling time, the MPC control action is chosen among the set of Pareto optimal solutions based on a time-varying, state-dependent decision criterion. Compared to standard single-objective MPC formulations, such a criterion allows one to take into account several, often irreconcilable, control specifications, such as high bandwidth (closed-loop promptness) when the state vector is far away from the equilibrium and low bandwidth (good noise rejection properties) near the equilibrium. After recasting the optimization problem associated with the multiobjective MPC controller as a multiparametric multiobjective linear or quadratic program, we show that it is possible to compute each Pareto optimal solution as an explicit piecewise affine function of the state vector and of the vector of weights to be assigned to the different objectives in order to get that particular Pareto optimal solution. Furthermore, we provide conditions for selecting Pareto optimal solutions so that the MPC control loop is asymptotically stable, and show the effectiveness of the approach in simulation examples.

1. Introduction

Multiobjective control problems are based on the optimization of multiple and often conflicting performance criteria that take into account different control specifications. Approaches to multiobjective control were proposed in the 1990s in Karbowski (1999), Li (1990), De Nicolao and Locatelli (1993) and Shtessel (1996), and problems such as mixed H_2/H_∞ control received much attention, in particular LMI-based design techniques (see Scherer, Gahinet, and Chilali (1997) and Shimomura (2000) and the references therein). More recently, in the context of model predictive control (MPC), a multiobjective controller was proposed in De Vito and Scattolini (2007), where the authors, rather than looking for Pareto optimal solutions in the standard multiobjective setting (Chinchuluun & Pardalos, 2007), look for the optimal control sequence that minimizes the max of a finite number of objectives.

This paper considers a multiobjective MPC problem in which the optimal control sequence corresponds to one of the Pareto optimal solutions. As multiple Pareto solutions may exist, we provide conditions for selecting a Pareto solution that is optimal for a desired weighted sum of the different objectives and that preserves closed-loop asymptotic stability. A related problem formulation in the context of MPC was done in Magni, Scattolini, and Tanelli (2008), where the authors consider a discrete set of performance indices that are switched depending on the value of the state vector, under stability constraints.

To address computational issues, in this paper we also investigate multiparametric multiobjective linear programming (mp-moLP) to handle multiobjective MPC problems with convex piecewise affine cost functions, and a special class of multiobjective quadratic programming (mp-moQP) to handle multiobjective MPC problems with a single quadratic and multiple convex piecewise affine cost functions. Multiparametric programming has been largely investigated in the last eight years as a technique for providing *explicit* MPC solutions, namely for characterizing the MPC command action as an explicit piecewise affine function of the state and reference vectors; see Bemporad, Morari, Dua, and Pistikopoulos (2002), Tøndel, Johansen, and Bemporad (2003a) and Seron, DeDoná, and Goodwin (2000) for the case of standard linear MPC problems with quadratic costs and linear constraints, and Alessio and Bemporad (2008) for a recent survey on explicit MPC.

For addressing the multiparametric multiobjective problem, in this paper we exploit the fact that Karush–Kuhn–Tucker (KKT) conditions for multiobjective optimization map into standard KKT conditions for a scalar optimization problem in which the cost

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function is the weighted sum of the objectives, and in which different weights provide different Pareto optimal solutions (Chinchuluun & Pardalos, 2007). By exploiting this result, mp-moLP (mp-moQP) can be rephrased as a multiparametric LP (QP) problem in which some of the parameters (the weights) appear in the cost function, and the rest of the parameters (the current states) in the right-hand side of the constraints.

Studies on parametric LP with a single parameter in the cost function date back to the 1950s with the work of Saaty and Gass (1954). This was extended in Gass and Saaty (1955) to cover the case of two parameters, also providing ideas for the general multiparametric LP (mp-LP) case. By duality, mp-LP problems with parameters only in the cost function are equivalent to mp-LP problems with parameters only in the right-hand side (rhs) of the constraints (Borrelli, Bemporad, & Morari, 2003; Gal & Nedoma, 1972). Multiparametric LP problems with parameters in both the cost function *and* the rhs of the constraints have been addressed in Barić, Baotić, and Morari (2005) and Barić, Jones, and Morari (2006). Alternatively, by looking at the KKT conditions, such problems can be treated as multiparametric linear complementarity (mp-LC) problems. Studies on parametric linear complementarity problems were done in the 1970s (Cottle, 1972; Kaneko, 1977; Maier, 1972; Murty, 1971); see also Murty and Yu (1988, Ch. 5), and later in Danao (1997) and Tammer (1998). Properties of mp-LC problems were studied in Xiao (1995) and algorithms for multiparametric LC problems were given in Gailly, Installe, and Smeers (2001) and, more recently, in Columbano, Fukuda, and Jones (in press) and Jones and Morari (2006). Multiparametric generalized LC problems were addressed in Palopoli, Bicchi, and Sangiovanni-Vincentelli (2002). In this paper, we adopt the mp-LC formulation to solve the multiobjective MPC problem for both the mp-moLP and mp-moQP cases.

The main contribution of the paper is twofold. First, the multiobjective MPC scheme is formulated so that, through a particular criterion for selecting Pareto-optimal solutions, closed-loop stability is guaranteed (Section 2). Second, the geometry of Pareto optimal solutions is explicitly characterized for the two cases considered (Sections 3 and 4), which allows us to recast the selection problem as a simple linear programming (mp-moLP) or convex programming (mp-moQP) problem with as many variables as the number of objectives.

A preliminary version of this paper focusing in more detail on the mp-moLP case has appeared in Bemporad and Muñoz de la Peña (2009).

2. Problem formulation

Consider the problem of regulating a process modeled by the following linear discrete-time system:

$$x(t+1) = Ax(t) + Bu(t) \tag{1}$$

under the linear input and state constraints

$$x(t) \in \mathcal{X}, \qquad u(t) \in \mathcal{U}, \tag{2}$$

where $x \in \mathbb{R}^n$ is the state vector, $u \in \mathbb{R}^m$ is the input vector, and $t \in \mathbb{N}$ denotes the time step. We assume that $\mathcal{X} \subseteq \mathbb{R}^n$ and $\mathcal{U} \subseteq \mathbb{R}^m$ are convex full-dimensional polyhedral sets containing the origin in their interior.

The standard MPC approach is based on minimizing a performance criterion repeatedly at each time step t over a finite prediction horizon of N steps, $[t, t+N]$. The solution of such an optimal control problem depends on the current state $x(t)$ and leads to an optimal control sequence of length N whose first sample $u(t)$ is applied to process (1), the remaining future optimal moves are discarded, the optimization is repeated at the next time step $t+1$ over a shifted time horizon, and so on (for this reason, the procedure is also known as *receding horizon* control). In this paper, instead of minimizing a single performance criterion, we consider the case of having $l+1$ different performance indices, $l \in \mathbb{N}$, and follow a multiobjective optimization approach.

Multiobjective optimization is the process of simultaneously optimizing two or more (possibly) conflicting objectives subject to certain constraints. We will show that multiobjective optimization in the context of MPC leads to the design of a controller whose performance index is allowed to vary in time and even to depend on the state vector $x(t)$, hence increasing the tuning versatility with respect to standard MPC.

Consider the following multiobjective optimal control problem:

$$\min_U \quad J(U, x) \tag{3a}$$

subject to

$$\begin{aligned} & x_{k+1} = Ax_k + Bu_k, \qquad x_0 = x \\ & x_k \in \mathcal{X}, \quad k = 1, \ldots, N \\ & u_k \in \mathcal{U} \quad k = 0, \ldots, N-1 \\ & x_N \in \Omega, \end{aligned} \tag{3b}$$

where $J(U, x) = [J_0(U, x), J_1(U, x), \ldots, J_l(U, x)]' : \mathbb{R}^s \times \mathbb{R}^n \to \mathbb{R}^{l+1}$ is a vector function, $l \geq 1$, $s = Nm$, $U = [u'_0, \ldots, u'_{N-1}]'$ is the sequence of future control moves to be optimized, x_k is the k-steps ahead predicted state from the initial state $x = x(t)$, and Ω is a terminal polyhedral set containing the origin in its interior. Each performance index is defined as

$$J_i(U, x) = \sum_{k=0}^{N-1} L_i(x_k, u_k) + F_i(x_N), \tag{4}$$

where the stage costs $L_i : \mathbb{R}^{n+m} \to \mathbb{R}$ and the terminal costs $F_i : \mathbb{R}^n \to \mathbb{R}$, $i = 0, \ldots, l$, satisfy the following assumption.

Assumption 1. For all $i = 0, \ldots, l$, function $L_i : \mathbb{R}^{n+m} \to \mathbb{R}$ is jointly convex with respect to (x, u), function $F_i : \mathbb{R}^n \to \mathbb{R}$ is convex with respect to x, $L_i(0, 0) = 0$, $F_i(0) = 0$, and there exist $\mathcal{K}$-functions[1] σ_1, σ_2 such that $L_i(x, u) \geq \sigma_1(\|x\|)$ for all $u \in \mathcal{U}$, $F_i(x) \geq \sigma_1(\|x\|)$ and $F_i(x) \leq \sigma_2(\|x\|)$ for some norm $\|\cdot\|$.

In general, the performance indices $J_i(U, x)$ are conflicting and it is not possible to obtain a solution that optimizes all the objectives at the same time. In order to obtain an optimal input trajectory U, an additional decision criterion must be taken into account that provides a trade-off between the different performance indices. In this work we propose to choose the optimal input trajectory among the set of Pareto optimal solutions of (3). Pareto optimality (Pareto, Bousquet, & Busino, 1964) is a measure of efficiency in multiobjective optimization:

Definition 2 (*Chinchuluun & Pardalos, 2007*). A feasible point U^p is Pareto optimal if and only if there exists no other feasible point U such that $J_i(U, x) \leq J_i(U^p, x)$, $\forall i = 0, \ldots, l$ and $J_j(U, x) < J_j(U^p, x)$ for at least one index $j \in \{0, \ldots, l\}$.

In other words, at a Pareto optimal point no objective can be further decreased without increasing at least another objective.

Finding the set of Pareto optimal solutions of a multiobjective optimization problem (i.e., solving the multiobjective optimization problem) can be a hard task. For the class of problems considered here it is possible to use the so-called *weighting method* to solve (3) (Boyd & Vandenberghe, 2004; Chinchuluun & Pardalos, 2007). The weighting method is based on assigning a different weight to each performance index, hence obtaining a scalar objective function. Consider the following *scalarization* of the multiobjective problem (3) defined as

[1] A $\mathcal{K}$ function $\sigma : \mathbb{R} \to \mathbb{R}$ is a strictly increasing continuous function of its argument and satisfies $\sigma(0) = 0$.

$$U^*(x,\alpha) = \begin{array}{ll} \arg\min_U & \alpha' J(U,x) \\ \text{s.t.} & (3b) \end{array} \tag{5}$$

where $\alpha = [\alpha_0, \ldots, \alpha_l]' \in \mathbb{R}^{l+1}$ is a weight vector, $\alpha_i \geq 0$, $\forall i = 0, \ldots, l$, $\sum_{i=0}^{l} \alpha_i = 1$.

As described in Boyd and Vandenberghe (2004, Chapter 4.7.4), for each given $\alpha > 0$[2] the solution $U^*(x,\alpha)$ of (5) is also a Pareto optimal solution of (3), usually different for different values of $\alpha \in \mathbb{R}^{l+1}$. For convex vector optimization problems as in (3), it is also true that for every Pareto optimal point U^p there exists a vector $\alpha \geq 0$ such that $U^p = U^*(x,\alpha)$. Hence, the corresponding solutions of (5) for all possible weight vectors α cover the whole set of Pareto optimal solutions of (3). Note that when some of but not all the components of α are zero, the solution may not be Pareto optimal. We can either restrict $\alpha > 0$ in (5) or, alternatively, tolerate possibly non-Pareto optimal solutions by leaving $\alpha \geq 0$. Both choices are not harmful, as will be discussed in Remark 5.

In order to uniquely define a Pareto optimal solution to the multiobjective MPC problem (3), at each time step t a weight vector $\alpha(t)$ must be selected. The optimal future input trajectory associated with the MPC controller is then given by the optimizer of (5) for $\alpha = \alpha(t), x = x(t)$. The decision criterion for $\alpha(t)$ clearly affects the overall stability and performance of the closed loop. In what follows we present an optimization-based decision method that guarantees that the closed-loop system is asymptotically stable.

2.1. Proposed multiobjective MPC scheme

In MPC design the performance index is used to tune the properties of the closed-loop system (stability, robustness, speed of convergence to the target state, etc.). In general, different choices of weights in the performance index provide a different closed-loop response. In this paper we propose to use a time-varying and possibly state-dependent target weight $\alpha_d(t) \in \mathbb{R}^{l+1}$ at each time step t as an additional tuning parameter. The reference weight vector α_d may take into account different priorities of the objectives depending on time and on the value of the state vector; for example, one may penalize the command input less when the state is far from the origin and more when the state is near the origin to ensure good noise rejection properties in the steady state. On the other hand, arbitrary switching of $\alpha(t)$ may lead to instability, so the objective of the proposed MPC controller is to choose $\alpha(t)$ as close as possible to the desired $\alpha_d(t)$ at each sampling time t in a way that closed-loop stability is guaranteed. To this end, let

$$\alpha^*(x, \alpha_d, J_a) = \arg\min_\alpha f(\alpha - \alpha_d) \tag{6a}$$

$$\text{s.t. } V^*(x,\alpha) \leq J_a \tag{6b}$$

$$\sum_{i=0}^{l} \alpha_i = 1 \tag{6c}$$

$$\alpha_i \geq 0, \quad i = 0, \ldots, l, \tag{6d}$$

where $V^* : \mathbb{R}^{n+l+1} \to \mathbb{R}$, $V^*(x,\alpha) = \alpha' J(U^*(x,\alpha), x)$, is the *value function* associated with Problem (5), $f : \mathbb{R}^{l+1} \to \mathbb{R}$ is a convex function that penalizes the deviation of α from the target weight vector α_d, and J_a is a value that depends on the optimal solution of the MPC at the previous time step, defined in Eq. (7) below.

The proposed multiobjective MPC algorithm is summarized by Algorithm 1, which is executed at each time step t. In Algorithm 1, with a slight abuse of notation, we have set $\alpha^*(t) = \alpha^*(x(t), \alpha_d(t), J_a(t))$ and $U^*(t) = U^*(x(t), \alpha^*(t))$. Note that the multiobjective MPC controller can be thought as a stabilizing MPC controller with a time-varying and possibly state-dependent performance index.

[2] Vector inequalities denote the corresponding set of element-wise comparisons.

Algorithm 1 Multiobjective MPC algorithm

1. Get $x(t)$, $\alpha_d(t)$;
2. Let $U^*(t-1) = [u_0', u_1', \ldots, u_{N-1}']'$ be the optimal sequence predicted at time $t-1$ starting from $x(t-1)$, and let x_N be the corresponding optimal state for time step $t-1+N$;
3. Set $U_s(t) = [u_1', u_2', \ldots, u_{N-1}', (Kx_N)']'$;
4. Evaluate
$J_a(t) = \alpha^*(t-1)' J(U_s(t), x(t))$; (7)
5. Compute $\alpha^*(t)$ by solving (6) for $x = x(t)$, $\alpha_d = \alpha_d(t)$, $J_a = J_a(t)$;
6. Compute $U^*(t)$ by solving (5) for $x = x(t)$, $\alpha = \alpha^*(t)$;
7. Set $u(t)$ equal to the first optimal move in the sequence $U^*(t)$;
8. End.

Solving Problem (6) may be time consuming due to the presence of the value function $V^*(x,\alpha)$ in (6b). In this paper we focus on two instances in which Problem (6) can be solved efficiently, due to the fact that a multiparametric solution of Problem (5) can be obtained. We study the case in which L_i and F_i in (4) are convex piecewise affine functions in Section 3, and then extend in Section 4 to the case in which one of the objective functions is convex and quadratic. In the following section we study the stability properties of the proposed scheme.

2.2. Stability properties

Closed-loop stability properties are guaranteed by following a standard terminal region/terminal constraint approach (Mayne, Rawlings, Rao, & Scokaert, 2000).

Theorem 3. *Let L_i and F_i, $i = 0, \ldots, l$, satisfy Assumption 1. Assume that there exists a linear feedback $u = Kx$ such that the following conditions hold:*

$$F_i((A+BK)x) - F_i(x) + L_i(x, Kx) \leq 0, \quad i = 0, \ldots, l \tag{8a}$$

$$x \in \Omega \to (A+BK)x \in \Omega, \tag{8b}$$

$$Kx \in U, \quad \forall x \in \Omega. \tag{8c}$$

If Problem (3) is feasible for $x = x(0)$, then Problems (6), (5) are feasible at all time steps $t \geq 0$ and system (1) in a closed loop with the MPC controller defined by Algorithm 1 is asymptotically stable.

Proof. The proof consists of two parts. We first prove the recursive feasibility of Problems (5) and (6) under the assumption that Problem (3) is feasible at time $t = 0$. Then we prove that, under the stated assumptions, $v(t) = V^*(x(t), \alpha^*(t)) = \alpha^*(t) J(U^*(t), x(t))$ is a decreasing sequence of values that implies asymptotic stability of system (1) in a closed loop with the proposed MPC scheme of Algorithm 1.

Part 1. By taking into account (8b)–(8c), it is easy to see that if $U^*(t-1)$ satisfies (3b) for $x = x(t-1)$, then $U_s(t)$ satisfies (3b) for $x = x(t)$. As Problem (5) is feasible at $t = 0$, it can be proved recursively that Problem (5) is feasible at all time steps $t \geq 0$. Feasibility of Problem (6) follows because, by definition, the optimal cost obtained by solving (5) for $x = x(t)$ and $\alpha = \alpha^*(t-1)$ is not greater than $J_a(t)$. This implies that $\alpha = \alpha^*(t-1)$ is always a feasible solution of Problem (6).

Part 2. The contractive constraint included in (6) guarantees that $v(t) \leq J_a(t)$. We prove next that $J_a(t) \leq v(t-1)$. By taking into account the definitions of $v(t)$ and $J_a(t)$, we have that $J_a(t) - v(t-1) = \alpha^*(t-1)'(J(U_s(t), x(t)) - J(U^*(t-1), x(t-1)))$. By letting $\alpha^*(t-1) = [\alpha_0^*(t-1), \ldots, \alpha_l^*(t-1)]' \in \mathbb{R}^{l+1}$, we have $J_a(t) - v(t-1) = \sum_{i=0}^{l} \alpha_i^*(t-1)(J_i(U_s(t), x(t)) - J_i(U^*(t-1), x(t-1)))$. Let $x_{[t+N-1]}$ and $x_{[t+N]}$ be the optimal state x_N for Problem (5) computed at time $t-1$ and t, respectively. By taking into account the optimality of $U^*(t-1)$ and the definition of $U_s(t)$, it follows that

$J_i(U_s(t), x(t)) - J_i(U^*(t-1), x(t-1)) = F_i(x_{[t+N]}) - F_i(x_{[t+N-1]}) + L_i(x_{[t+N-1]}, Kx_{[t+N-1]}) - L_i(x(t-1), u(t-1))$, which by (8) implies that

$$J_i(U_s(t), x(t)) - J_i(U^*(t-1), x(t-1)) \leq -L_i(x(t-1), u(t-1)) \leq 0 \tag{9}$$

$\forall i = 0, \dots, l$, $\forall t \geq 1$. Hence, by taking convex combinations with coefficients $\alpha_i^*(t-1) \geq 0$, $\forall i = 0, \dots, l$ of the terms in (9) and recalling (6b), we obtain $\nu(t) \leq J_a(t) \leq \nu(t-1)$. Hence, $\nu(t)$ is a non-increasing sequence lower bounded by zero, which admits a limit $\nu_\infty \in \mathbb{R}$ as $t \to 0$. By (9) this also proves that $\lim_{t\to\infty} \sum_{i=0}^{l} \alpha_i^*(t) L_i(x(t), u(t)) = 0$. Since $\sum_{i=0}^{l} \alpha_i^*(t) L(x(t), u(t)) \geq \sigma_1(\|x(t)\|) \to 0$ for $t \to \infty$ and all $u(t) \in \mathcal{U}$, convergence of $x(t)$ to zero is easily proved. Taking into account Assumption 1 and following the same lines of thought as in Lazar, Heemels, Weiland, and Bemporad (2006), stability can also be proved. □

The most restrictive assumption is that the stability conditions (8) must be satisfied simultaneously by each of the considered performance indices for the same feedback $u = Kx$, which is a key requirement to guarantee the stability properties of the closed-loop system. In order to satisfy these constraints, different terminal cost functions F_i can be used in order to increase the degrees of freedom of the design.

The closed-loop stability properties of the multiobjective MPC controller are not affected by $\alpha_d(t)$, as the feasibility of Problem (6) is independent of $\alpha_d(t)$, whose choice only determines the closed-loop performance. Hence, $\alpha_d(t)$ does not need to be decided a priori, but it can be changed arbitrarily during online operation of the controller without loosing stability properties.

The stability result of Theorem 3 holds for all costs satisfying Assumption 1.

3. Multiparametric multiobjective LP

Let Assumption 1 hold and assume that $L_i : \mathbb{R}^{n+m} \to \mathbb{R}$ and $F_i : \mathbb{R}^n \to \mathbb{R}$ are convex and piecewise affine functions. In particular, as in Bemporad, Borrelli, and Morari (2002), let

$$L_i(x, u) = \|Q_i x\|_\infty + \|R_i u\|_\infty, \qquad F_i(x) = \|P_i x\|_\infty \tag{10}$$

Assumption 1 is satisfied if $Q_i \in \mathbb{R}^{q_i \times n}$, $R_i \in \mathbb{R}^{r_i \times m}$ are matrices with full column-rank, $\forall i = 0, \dots, l$ (Lazer, Muñoz de la Peña, Heemels, & Alamo, 2008). It is easy to enforce the assumptions of Theorem 3 when using stage and terminal costs as in (10). In fact, in the special case of matrix A stable, one can apply the techniques reported in Bemporadet al. (2002) to find a common matrix P that satisfies condition (8) for $K = 0$. More generally, as proposed in Lazar et al. (2006), one can use nonlinear optimization to find a set of matrices P_i and a gain K satisfying $\|P_i(A+BK)P_i^{-L}\|_\infty + \|Q_i P_i^{-L}\|_\infty + \|R_i K P_i^{-L}\|_\infty \leq 1$, where $P_i^{-L} = [P_i' P_i]^{-1} P_i'$.

By following an approach similar to the one in Bemporadet al. (2002), Problem (3) can be recast into the multiparametric multiobjective linear program

$$\begin{aligned} \min_z \quad & Cz \\ & Gz \leq b + Sx \end{aligned} \tag{11}$$

where $z \in \mathbb{R}^d$ is the vector of optimization variables, $x \in \mathbb{R}^n$ is a vector of parameters, and $C \in \mathbb{R}^{(l+1)\times d}$ defines the linear vector function of dimension $l+1$, where each row of matrix C defines a different scalar objective function $C = [c_0 \ \dots \ c_l]'$, $c_i \in \mathbb{R}^d$, $i = 0, \dots, l$. Vector z has dimension $d = s + (l+1)(2N-1)$, as it includes the s components of U and one set of $(2N-1)$ additional non-negative variables per objective (4), each one upperbounding the corresponding piecewise affine term as in (10) of the stage cost; see Bemporadet al. (2002).

According to the weighting method, the set of Pareto optimal points of Problem (11) can be fully characterized from the corresponding solutions of the following optimization problem:

$$\begin{aligned} \min_z \quad & \alpha' Cz \\ & Gz \leq b + Sx \end{aligned} \tag{12}$$

for all possible weight vectors α such that $\alpha = [\alpha_0, \dots, \alpha_l]' \in \mathbb{R}^{l+1}$, $\alpha_i \geq 0$, $\forall i = 0, \dots, l$, $\sum_{i=0}^{l} \alpha_i = 1$. Problem (12) is equivalent to

$$\begin{aligned} \min_z \quad & (c_0' + \mu' C_\mu) z \\ & Gz \leq b + Sx, \end{aligned} \tag{13}$$

where, in order to get rid of the equality constraint $\sum_{i=0}^{l} \alpha_i = 1$, we have expressed $\alpha_0 = 1 - \sum_{i=1}^{l} \alpha_i$, $\mu = [\alpha_1 \ \dots \ \alpha_l] \in \mathbb{R}^l$, and $C_\mu = [(c_1 - c_0) \ \dots \ (c_l - c_0)]' \in \mathbb{R}^{l\times d}$.

Most of the multiparametric LP solvers only handle parameters either in the cost function or in the rhs of the constraints (which, by duality, is totally equivalent). By exploiting the KKT conditions of Problem (13), in Bemporad and Muñoz de la Peña (2009) we characterized the explicit solution of this class of problems, whose properties are summarized below. By assuming that all the components of z are lower bounded[3] by a quantity $z_{\min}$, Problem (13) can be recast as the multiparametric linear complementarity problem (mp-LCP)

$$\begin{aligned} & \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} - \begin{bmatrix} 0 & -G \\ G' & 0 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} b - Gz_{\min} \\ c_0 \end{bmatrix} + \begin{bmatrix} 0 & S \\ C_\mu' & 0 \end{bmatrix} \begin{bmatrix} \mu \\ x \end{bmatrix} \\ & \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}' \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = 0, \end{aligned} \tag{14}$$

where $z_2 = z - z_{\min}$, $z_1 = \lambda$, w_2 are the Lagrange multipliers associated with the constraints $z \geq z_{\min}$, and w_1 is the vector of slack variables satisfying $Gz + w_1 = b + Sx$. Problem (14) can be solved by existing mp-LCP solvers (Columbano et al., in press; Jones & Morari, 2006); or in a less efficient way, by exploiting equivalence results between linear complementarity and mixed-integer problems (Heemels, De Schutter, & Bemporad, 2001), by multiparametric mixed-integer linear programming solvers (Dua & Pistikopoulos, 2000).

Lemma 4. *Consider the multiparametric linear problem (13) with parameters $\mu \in \mathbb{R}^l$ in the cost function and $x \in \mathbb{R}^n$ in the rhs of the constraints. Then the set F^* of parameters (μ, x) for which (13) has a solution is a convex polyhedron, the value function $V^* : F^* \to \mathbb{R}$ is continuous w.r.t. (μ, x), convex and piecewise affine w.r.t. μ for any given x and w.r.t. x for any given μ. Moreover, there exists a piecewise affine optimizer function $z^* : F^* \to \mathbb{R}^d$ of (μ, x) defined as*

$$z^*(\mu, x) = \begin{array}{ll} \phi_i x + \gamma_i & \text{if } \begin{bmatrix} H_i^\mu & 0 \\ 0 & H_i^x \end{bmatrix} \begin{bmatrix} \mu \\ x \end{bmatrix} \leq \begin{bmatrix} K_i^\mu \\ K_i^x \end{bmatrix}, \\ i = 1, \dots, n_r. & \end{array} \tag{15}$$

Proof. See Bemporad and Muñoz de la Peña (2009). □

Remark 5. As discussed earlier, non-strictly positive values of μ may lead to non-Pareto optimal solutions. However, we can either restrict $\mu > 0$ in (17) or, alternatively, tolerate possibly non-Pareto optimal solutions by leaving $\mu \geq 0$. In the first case, the stability result of Theorem 3 still holds.

3.1. Online selection of the weight vector

We consider now the online selection problem (6) of the weight vector $\alpha^*(x, \alpha_d, J_a)$ for the particular case of f convex and

[3] This is always the case when the set $\mathcal{U}$ of admissible inputs is bounded.

piecewise affine:

$$f(\alpha-\alpha_d)=\max\{f_j^{\alpha}(\alpha-\alpha_d)+f_j^0\},\quad j=1,\ldots,n_f. \tag{16}$$

A possible choice for f in (16) is $f(\alpha-\alpha_d)=\|\alpha-\alpha_d\|_\infty$.

Theorem 6. *Let $I(x)\subseteq\{1,\ldots,n_r\}$ be the set of indices i of the regions $CR_{xi}=\{x\in\mathbb{R}^n:\ H_i^x x\le K_i^x\}$ to which x belongs. Given $\alpha_d=[\alpha_{d0},\mu_{d1},\ldots,\mu_{dl}]\in\mathbb{R}^{l+1}$, with $\alpha_{d0}=1-\sum\mu_{di}$, the solution to Problem (6) is $\alpha^*(x,\alpha_d,J_a)=[1-\sum_{i=1}^{l}\mu_i^*,\mu_1^*,\ldots,\mu_l^*]'$, where μ^* can be determined by solving the linear programming problem*

$$\begin{aligned}\min_{\mu,\beta}\quad & \beta\\ \text{s.t.}\quad & \beta\ge f_j^{\alpha}\begin{bmatrix}0\\ \mu-\mu_d\end{bmatrix}+f_j^0,\quad j=1,\ldots,n_f\\ & (\phi_i x+\gamma_i)'(c_0'+C_\mu'\mu)\le J_a,\quad \forall i\in I(x)\\ & \sum_{i=1}^{l}\mu_i\le 1,\ \mu_i\ge 0,\quad i=1,\ldots,l,\end{aligned} \tag{17}$$

with $l+1$ variables and $n_f+card(I(x))+2$ constraints.

Proof. For a fixed x, the value function $V^*(\mu,x)$ is a piecewise affine and convex function of μ that, by the structure of the regions CR_i proved in Lemma 4, is defined over the regions $CR_{\mu i}=\{\mu\in\mathbb{R}^l:\ H_i^{\mu}\mu\le K_i^{\mu}\}$ indexed by $i\in I(x)$. Hence, thanks to the result of Schechter (1987) for convex piecewise affine functions, for every fixed x the value function $V^*(\mu,x)$ by Lemma 4 can be evaluated as the maximum of the affine functions $\{(\phi_i x+\gamma_i)'(c_0'+C_\mu'\mu)\}_{i\in I(x)}$. □

Unfortunately, Problem (17) in general is not jointly convex with respect to (μ,β) and (x,μ_d,J_a), due to the fact that $V^*(\mu,x)$ may not be a jointly convex function of (μ,x). Henceforth, multiparametric convex programming approaches like the one suggested in Bemporad and Filippi (2006) and Muñoz de la Peña, Bemporad and Filippi (2006) cannot be applied to determine μ^* as an (approximated) function of (x,μ_d,J_a), and Problem (17) needs to be solved online for the given values of $x(t),\mu_d(t),J_a(t)$ and the corresponding set of constraints indexed by $I(x(t))$.

4. Multiparametric multiobjective QP with a single quadratic objective

If all the objective functions were quadratic, Problem (3) would be a multiparametric multiobjective quadratic program. By applying the weighting method to find Pareto optimal solutions, one would get a multiparametric quadratic program with parameters in the quadratic part of the objective function, besides parameters in the rhs of the constraints and in the affine part of the objective function, which makes the multiparametric solution very difficult to obtain. However, if only one of the objective functions is quadratic and the remaining functions are piecewise affine it is possible to cast (5) as a (non-strictly) convex multiparametric quadratic program, for which multiparametric solvers are available (Jones & Morari, 2006; Tøndel, Johansen, & Bemporad, 2003b). It may be very convenient from a practical viewpoint to allow one of the objectives to be quadratic, as when constraints are not active the corresponding optimal control action would be a standard linear state feedback. Without loss of generality, let us choose

$$\begin{aligned}L_0(x,u)&=x'Q_0x+u'R_0u\\ F_0(x)&=x'P_0x\end{aligned} \tag{18}$$

and let L_i,F_i be as in (10), $i=1,\ldots,l$. According to the standard transformations in linear MPC (Bemporadet al., 2002) and the techniques reviewed in the previous section, Problem (5) can be cast into the following multiparametric quadratic programming problem:

$$\begin{aligned}\min_z\quad & \alpha_0\left(\frac{1}{2}z'Hz+x'F'z+\frac{1}{2}x'Yx\right)+\sum_{i=1}^{l}\alpha_i c_i'z\\ & Gz\le b+Sx\end{aligned} \tag{19}$$

where $H\succeq 0$, the parameters are x and $\alpha=[\alpha_0,\ldots,\alpha_l]'\in\mathbb{R}^{l+1}$, $\alpha_i\ge 0$, $i=0,\ldots,l$, and the optimization vector $z=\begin{bmatrix}U\\ e\end{bmatrix}\in\mathbb{R}^{s+(2N-1)l}$, $e\in\mathbb{R}^{(2N-1)l}$. We assume without loss of generality that the first optimal move $u=[I\ 0\ \ldots\ 0]z$. We denote the optimal solution of this problems as $V^*(\alpha,x)$.

Theorem 7. *Given $x(t),J_a(t)$ and*

$$\alpha_d(t)=[\alpha_{d0}(t),\alpha_{d1}(t),\ldots,\alpha_{dl}(t)]\in\mathbb{R}^{l+1}$$

with $\alpha_{d0}(t)=1-\sum_{i=1}^{l}\alpha_{di}(t)$, the multiobjective MPC law defined by Algorithm 1 with $L_0(x,u)$, $F_0(x)$ as in (18) and L_i, F_i as in (10), $i=1,\ldots,l$ can be computed by solving:

(i) the multiparametric quadratic programming problem offline

$$\begin{aligned}V_\mu^*(\mu,x)=\min_z\quad & \frac{1}{2}z'Hz+x'F'z+\frac{1}{2}x'Yx+\mu'C_\mu z\\ & Gz\le b+Sx,\end{aligned} \tag{20}$$

where $\mu=[\mu_1\ldots,\mu_l]'\in\mathbb{R}^l$, $\mu_i\ge 0$, and $C_\mu=[c_1,\ldots,c_l]'$;

(ii) the convex programming problem online with optimizer $\mu^*(t)=\mu^*(x(t),\mu_d(t),J_a(t))$

$$\begin{aligned}\min_\mu\quad & f\left(\begin{bmatrix}1\\ \mu\end{bmatrix}-\begin{bmatrix}1\\ \mu_d(t)\end{bmatrix}\right)\\ \text{s.t.}\quad & V_\mu^*(x(t),\mu)\le J_a(t)\left(1+\sum_{i=1}^{l}\mu_i\right)\\ & \mu_i\ge 0,\ i=1,\ldots,l,\end{aligned} \tag{21}$$

for $\mu_d(t)=\frac{1}{\alpha_{0d}(t)}[\alpha_{d1}(t),\ldots,\alpha_{dl}(t)]\in\mathbb{R}^l$; and

(iii) by setting $u(t)=[I\ 0\ \ldots\ 0]z^*(\mu^*(t),x(t))$, where $z^*(\mu,x)$ is an optimizer function of (20) and is piecewise affine.

Proof. (i) As the quadratic term of the objective function in (19) only depends on α_0, we can introduce the change of weights $\alpha_0=\frac{1}{1+\sum_{i=1}^{l}\mu_i}$, $\alpha_i=\frac{\mu_i}{1+\sum_{j=1}^{l}\mu_j}$, $i=1,\ldots,l$, with $\mu_i\ge 0$. Hence, $V^*(\alpha,x)=\frac{1}{1+\sum_{i=1}^{l}\mu_i}V_\mu^*(\mu,x)$. (ii) By the results of Mangasarian and Rosen (1964) for convex multiparametric programming it follows that $V_\mu^*(\mu,x)$ is a jointly convex function of (μ,x), and hence a convex function of μ for any fixed $x\in\mathbb{R}^n$, which proves that (21) is a convex programming problem with l variables, one convex constraint, and l non-negativity constraints. (iii) Easily follows, as an optimizer function $z^*(\mu,x)$ is also an optimizer function for Problem (19), whose parametric solution is piecewise affine (Tøndel et al., 2003b).

Problem (20) can be solved by multiparametric quadratic programming algorithms that handle non-strictly convex Hessian matrices, as in Tøndel et al. (2003b). Alternatively, by exploiting the KKT conditions of optimality for Problem (20) this can be recast as the mp-LCP

$$\begin{aligned}&\begin{bmatrix}w_1\\ w_2\end{bmatrix}-\begin{bmatrix}0 & -G\\ G' & H\end{bmatrix}\begin{bmatrix}z_1\\ z_2\end{bmatrix}=\begin{bmatrix}b-Gz_{\min}\\ Hz_{\min}\end{bmatrix}+\begin{bmatrix}0 & S\\ C_\mu' & F\end{bmatrix}\begin{bmatrix}\mu\\ x\end{bmatrix}\\ &\begin{bmatrix}w_1\\ w_2\end{bmatrix}'\begin{bmatrix}z_1\\ z_2\end{bmatrix}=0,\end{aligned} \tag{22}$$

where, exactly as in the mp-moLP case (Bemporad & Muñoz de la Peña, 2009), $z_2=z-z_{\min}$, $z_1=\lambda$, w_2 are the Lagrange multipliers associated with the constraints $z\ge z_{\min}$, w_1 is the vector of slack variables satisfying $Gz+w_1=b+Sx$, and solved by multiparametric linear complementarity algorithms like the one in Jones and Morari (2006).

Note that Problem (21) is a (piecewise-smooth) convex optimization problem of small scale (only l optimization variables are involved), for which several solution algorithms are available (Boyd & Vandenberghe, 2004). In particular, the gradients and the Hessians of the constraints in (21), where defined, are easily obtained from the explicit solution of Problem (20).

4.1. Fully offline solution

An alternative method to the weighting approach taken in the previous sections is to impose $\alpha_0 = 1$ and drop the constraint $\sum_{i=0}^{l} \alpha_i = 1$. Both weighting methods are equivalent in that the corresponding optimization problems span the same set of Pareto optimal solutions, except the one in which the quadratic objective has zero weight. Then, Problem (19) can be solved offline directly by multiparametric quadratic or linear complementarity solvers with respect to the parameter vector $(x, \alpha_1, \dots, \alpha_l)$.

Problem (21) can be therefore solved *offline* as the multiparametric convex programming problem

$$\begin{aligned} \alpha^*(x, \alpha_d, J_a) = \arg\min_{\alpha} \quad & f(\alpha - \alpha_d) \\ \text{s.t.} \quad & V^*(x, \alpha) \leq J_a \\ & \alpha_i \geq 0,\ i = 1, \dots, l \\ & \alpha_0 = \alpha_{0d} = 1, \end{aligned} \tag{23}$$

where $(x, \alpha_{d1}, \dots, \alpha_{dl}, J_a) \in \mathbb{R}^{n+l+1}$ is the vector of parameters and $\alpha_1, \dots, \alpha_l$ the optimization variables, after $\alpha_0 = 1$ is substituted. Problem (23) can be solved in approximate explicit form with arbitrary precision using for instance the approach of Bemporad and Filippi (2006) and Muñoz de la Peña et al. (2006). The limitation of the approach (Bemporad & Filippi, 2006; Muñoz de la Peña et al., 2006) is the curse of dimensionality due to simplicial partitions. However, such a limitation is mitigated by the fact that, from a practical viewpoint, there is no need for excessive precision in solving (23), as small deviations of the weights α are not likely to change the MPC action $u(t)$ significantly, especially when constraints are active.

In summary, the posed multiobjective MPC problem can be solved by getting $\alpha^*(t)$ as an approximate explicit solution to the convex multiparametric programming problem (23), and by getting $u(t)$ from the exact explicit solution to the multiparametric quadratic programming problem (19) with $\alpha_0 = 1$.

5. Example

Consider a linear system (1) defined by matrices $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 0.5 \\ 1 \end{bmatrix}$ subject to the operating constraints $|x_1(t)|, |x_2(t)| \leq 10$, $|u(t)| \leq 10$. Consider the following objective functions: J_0 of the form (18) defined by $Q_0 = \begin{bmatrix} 0.1 & 0 \\ 0 & 1 \end{bmatrix}$, $R_0 = 0.2$, $P_0 = \begin{bmatrix} 0.5649 & 0.4054 \\ 0.4054 & 1.6027 \end{bmatrix}$, and J_1 of the form (10) defined by $Q_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0.1 \end{bmatrix}$, $R_1 = 0.1$, $P_1 = \begin{bmatrix} 9.6085 & 1.1401 \\ -0.2965 & 9.4107 \end{bmatrix}$. For $K = [-0.5 - 1.4]$ the constraint $(A + BK)'P_0(A + BK) - P_0 < -Q_0 - K'R_0K$ is satisfied, and $1 - \|P_1(A + BK)P_1^{-L}\|_\infty - \|Q_1P_1^{-L}\|_\infty - \|R_1KP_1^{-L}\|_\infty \geq 0$ is also satisfied, with $P_1^{-L} = [P_1'P_1]^{-1}P_1'$. Hence, (8a) is satisfied for the common local controller $u = Kx$ for $i = 0$ and for $i = 1$ (Lazar et al., 2006). Also, consider the terminal region

$$\Omega = \left\{ x \in \mathbb{R}^2 \left| \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & -1 \\ -1 & 0 \\ -0.5 & -1.4 \\ 0.5 & 1.4 \end{bmatrix} x \leq \begin{bmatrix} 10 \\ 10 \\ 10 \\ 10 \\ 10 \\ 10 \end{bmatrix} \right. \right\}$$

defined by the positive invariant set of the system in a closed loop with the local controller K. The set Ω satisfies (8b) and (8c) for the common local controller $u = Kx$. In summary, the cost functions J_0, J_1 and the terminal region Ω satisfy the assumptions of Theorem 3.

We compare a multiobjective MPC controller based on the piecewise affine cost J_1 and quadratic cost J_0 with the corresponding standard MPC controllers. For a comparison of a multiobjective MPC controller based on piecewise affine costs only, see the example in Bemporad and Muñoz de la Peña (2009).

Consider the multiobjective MPC controller h_{mo}^{QP} based on the vector objective function $J = [J_0\ J_1]'$ with $N = 2$, and let the target weight vector be $\alpha_d(t) = \left[\frac{1}{1+0.02\|x(t)\|_2^2} \ \ \frac{0.02\|x(t)\|_2^2}{1+0.02\|x(t)\|_2^2} \right]'$. Also consider the (single-objective) controller $h_1^{LP}(x(t)) = E_0U^*(x(t), [0\ \ 1]')$, where $U^*(x, \alpha)$ is the solution of (5) with $J = [J_0\ \ J_1]'$ and $N = 2$, $U^*(t)$ is the optimal input trajectory of the proposed multiobjective scheme for J, and $E_0 = [I\ 0\ \dots\ 0]$ is such that $E_0U = u_0$, and consider the (single-objective) MPC law h_0^{QP} based on the cost function J_0 and subject to the same set of constraints of h_1^{LP}. Controller h_0^{QP} also guarantees that the closed-loop system is asymptotically stable and has the same feasibility region as the one of h_1^{LP}. However, h_0^{QP} provides a slower closed-loop convergence to the origin than h_1^{LP} but more robustness with respect to measurement noise. The target weight vector $\alpha_d(t)$ was chosen to give priority to h_1^{LP} when the state is far from the origin, and to h_0^{QP} near the origin. Note that the goal of this example is not to prove that any of the controllers outperforms the others, but to demonstrate how the target weight vector can be used to smoothly move between two different performance objectives, capturing the best closed-loop properties of both without destabilizing the system.

A set of simulations was carried out starting from different states inside the feasibility region of the controllers (note that this region is equal for all of them). In the simulations we consider the presence of random measurement noise $w(t)$, that is $u(t) = h(x(t) + w(t))$, with $\|w(t)\|_\infty \leq 0.5$ and $w(t) = 0$ for all $t \leq 30$. In order to measure the robustness of the closed-loop system with respect to measurement noise the following values of signal-to-noise (SNR) ratios are measured:

$$SNR_u = \frac{\sum_{t=31}^{f} \|u(t)\|_2}{\sum_{t=31}^{f} \|w(t)\|_2}, \qquad SNR_x = \frac{\sum_{t=31}^{f} \|x(t)\|_2}{\sum_{k=31}^{f} \|w(t)\|_2}, \tag{24}$$

where f is the total number of simulated time steps. Convergence performance is evaluated according to the time t_r needed for $\|x(t)\|_2$ to go below 10% of its initial value. The average results over 100 simulations with $N = 2$ and $f = 100$ are shown in the following table.

Controller	SNR_u	SNR_x	t_r
h_0^{QP}	0.4807	0.6313	7.11
h_1^{LP}	1.3141	0.9196	3.11
h_{mo}^{QP}	0.5090	0.6352	4.08

The results show that the multiobjective controller h_{mo}^{QP} provides SNR ratios similar to h_0^{QP}, with a time t_r similar to the one provided by h_1^{LP}.

In the upper plots of Fig. 1, the state and input trajectories of a simulation with initial state $x(0) = [7.5\ 7.5]'$ are shown. It can be seen that both h_1^{LP} and the proposed controller h_{mo}^{QP} converge faster to the origin than h_0^{QP}. In the lower plot are shown the second components of the target weight vector $\alpha_d(t)$ and of the optimal weight vector $\alpha^*(t)$ solving the online convex programming problem (23).

The explicit solution of (20) was obtained by applying the multiparametric LCP solver of Jones and Morari (2006) to (22), with respect to the same three parameters (the two-dimensional state x

Fig. 1. Upper three plots: closed-loop trajectories for $h_0^{QP}(x)$ (dashed line), $h_1^{LP}(x)$ (solid thin line), and $h_{mo}^{QP}(x)$ (solid line). Lowest plot: trajectories of $\alpha_{d2}(t)$ (solid line) and $\alpha_2^*(t)$ (dashed line).

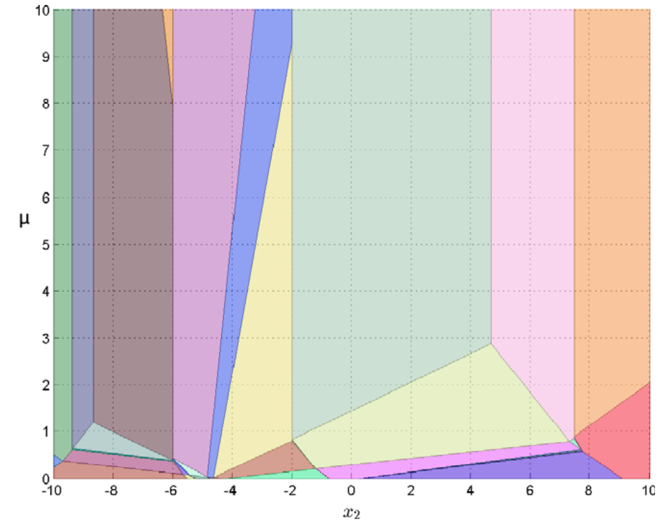

Fig. 3. Section of the LCP solution for controller h_{mo}^{QP} for $x_1 = 3$.

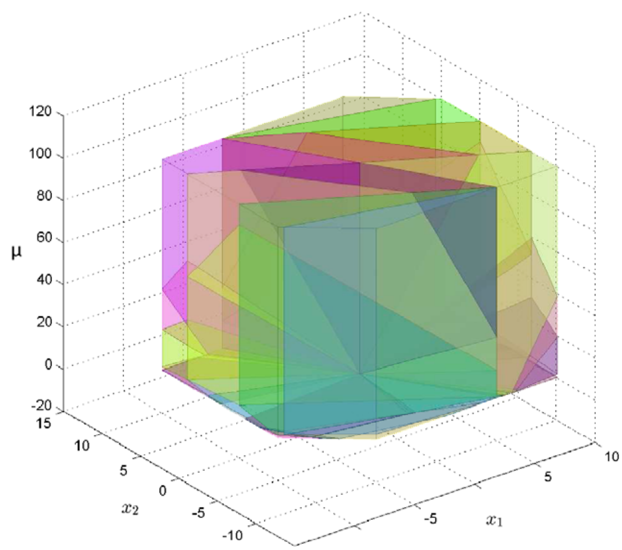

Fig. 2. Explicit LCP solution for controller h_{mo}^{QP} with respect to the parameter vector $[x' \ \mu]'$.

Fig. 4. Section of the LCP solution for controller h_{mo}^{QP} for $\mu = 0.5$.

and the scalar weight μ), as in Theorem 7. Fig. 2 shows the regions of the explicit piecewise affine solution $z^*(\mu, x)$ of Problem (20), where $n_r = 80$. Fig. 3 shows a close-up for $\mu \in [0, 10]$ of the section for $x_1 = 3$. Note that the facet-to-facet property does not hold for this class of non-strictly convex multiparametric quadratic programming problems. Note also that orthogonality does not hold anymore as in the piecewise-affine case. This can be easily justified by looking at the critical regions of the multiparametric problem (20), that are easily obtained from the associated KKT conditions. A section for $\mu = 0.5$ is shown in Fig. 4, where region sections correspond to the explicit solution of the mpQP problem defined by $\mu = 0.5$. Even in this case the facet to facet-to-facet property does not hold; see e.g. around the value $x = [-7 \ \ -7]'$ in Fig. 4.

6. Conclusions

This paper has proposed an MPC formulation based on multiple performance criteria that enjoys closed-loop stability properties. Compared to standard MPC formulations based on a single performance index, the multiobjective criterion allows one to take into account several, often irreconcilable, control specifications, such as high bandwidth (closed-loop promptness) far away from convergence, and low bandwidth (good noise rejection properties) near convergence. The corresponding optimization problem was solved as a multiparametric linear complementarity problem that provides the optimal Pareto solution as a piecewise affine function of the state vector and of the set of parameters that weight the different criteria in the equivalent scalarized problem. Thanks to such an explicit characterization of the solution, given a higher-level reference signal specified at each time step for the preferred weights, an optimal selection of the weights can be computed online by solving a simple convex programming problem, namely a linear programming problem when all objectives are convex piecewise affine functions, or a convex programming problem when one of the objectives is quadratic and the others are convex piecewise affine.

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