Pergamon

0005-1098(94)00109-X

Automatica, Vol. 30, No. 12, pp. 2013–2015, 1994
Elsevier Science Ltd
Printed in Great Britain
0005-1098/94 $7.00 + 0.00

Technical Communique

On the Stabilizing Property of SIORHC*†

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Key Words—Predictive control; receding horizon control; LQ control; dynamic programming.

Abstract—It is shown that the stabilizing property of SIORHC (stabilizing I/O receding horizon control) holds for general stabilizable discrete-time linear plants irrespective of the condition that the plant has no poles at the origin.

1. *Introduction*

A PREDICTIVE CONTROLLER with a guaranteed stabilizing property has been presented recently in Mosca *et al.* (1990), Clarke and Scattolini (1991) and Mosca and Zhang (1992) under the acronym SIORHC (stabilizing I/O receding horizon control). In short SIORHC consists, at each sample time, of:

(i) finding a finite sequence of future input increments which minimizes a quadratic criterion defined over a finite prediction horizon subject to the constraint that the plant output matches a desired constant setpoint beyond the prediction horizon; and

(ii) applying the first sample of the optimal sequence to the plant according to the receding horizon control strategy. For a related predictive controller see also the SGPC (stable generalized predictive controller) of Kouvaritakis *et al.* (1992). Recently, Rawlings and Muske (1991) have proposed a related approach which is based on a finite control horizon but, in contrast to SIORHC, uses an infinite prediction horizon. It was shown that the resulting receding horizon controller guarantees closed-loop stability for any linear stabilizable plant providing that the control horizon is at least equal to the number of unstable poles of the plant. In Mosca *et al.* (1990), Clarke and Scattolini (1991) and Mosca and Zhang (1992) it has been shown that SIORHC stabilizes any stabilizable discrete-time linear plant under the only limitative assumption that the plant transfer function has no poles at the origin of the complex plane. Note that the above limitation only ensures the validity of the stabilizing property but is irrelevant for the existence of the SIORHC law which is well-defined irrespective of the presence of poles at the origin.

The aim of the present note is to remove the above-mentioned limitation by adopting two alternative methods of stability proof different from the one used in Mosca *et al.* (1990), Clarke and Scattolini (1991) and Mosca and Zhang (1992). Of these two proofs, the first, based on the monotonicity property of the Riccati equation relevant to SIORHC and on the so-called fake algebraic Riccati equation (FARE) argument (see Bitmead *et al.*, 1990), enlightens the connection between SIORHC and the standard linear quadratic output regulation (LQOR) problem. The second, which is based on an argument that seems to be used first by Keerthi and Gilbert (1988) in receding horizon control problems, though nonconstructive and subject to problem feasibility, can also cover nonlinear plants with hard constraints. For a discussion on the stability issue in the predictive control of continuous-time plants also see the survey paper of Mayne and Polak (1993).

* Received 11 April 1994; received in final form 8 July 1994; recommended for publication by Editor P. Dorato.

† Work partially supported by MURST and CNR.

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2. *Stabilizing I/O receding horizon control*

For the motivations and solution of the SIORHC problem, the reader is referred to Mosca and Zhang (1992). Here, just for convenience, we briefly recall the problem statement. Consider the discrete-time SISO plant

$$A(d)\Delta(d)y(t) = B(d)\,\delta u(t), \tag{1}$$

where d denotes the unit delay operator, $A(d) = 1 + a_1 d + \cdots + a_{n_a} d^{n_a}$, $B(d) = b_1 d + \cdots + b_{n_b} d^{n_b}$, $|a_{n_a}| + |b_{n_b}| > 0$, $\Delta(d) := 1 - d$ and $\delta u(t) := u(t) - u(t-1)$ denotes the input increment. Let $\{r(k)\}$ be the output reference to be tracked and $\varepsilon(k) := y(k) - r(k)$ the corresponding tracking error. Consider the problem of finding, whenever it exists, a sequence of $T \geq 1$ input increments $\widehat{\delta u}_{[t,t+T)} := \{\widehat{\delta u}(t), \ldots, \widehat{\delta u}[t+T-1)\}$ which solves the following constrained optimization problem

$$\min_{\delta u_{[t,t+T)}} \left\{ \sum_{k=0}^{T-1} \|\varepsilon(t+k)\|^2_{\Psi_y} + \|\delta u(t+k)\|^2_{\Psi_u} \right\}, \tag{2}$$

$$\text{subject to } \delta u_{t+T}^{t+T+n-2} = 0_{n-1} \quad \text{and} \quad y_{t+T}^{t+T+n-1} = \underline{r}(t+T), \tag{3}$$

where $n := \max\{n_a + 1, n_b\}$, $\Psi_y > 0$, $\Psi_u > O$, $\|x\|^2_{\Psi} := x'\Psi x$, prime denotes transpose, $y_{t+k}^{t} := [y(t) \cdots y(t+k)]'$ and $\underline{r}(k) := \underbrace{[r(k) \cdots r(k)]'}_{n \text{ times}}$. Then the SIORHC law is defined at each sample time t as

$$\delta u(t) = \widehat{\delta u}(t). \tag{4}$$

In Mosca and Zhang (1992) it has been shown that, if $A(d)\Delta(d)$ and $B(d)$ are coprime polynomials, the SIORHC problem (1)–(3) admits a unique solution and that, if in addition

$$n_b \leq n_a + 1, \tag{5}$$

SIORHC law (4) stabilizes plant (1) provided that $T \geq n$. Condition (5), which imposes restrictions on the plant that are both conceptually and practically disturbing, is due to the fact that the proofs given in Mosca *et al.* (1990), Clarke and Scattolini (1991) and Mosca and Zhang (1992) rely on the classical stability results of receding horizon control (Bitmead *et al.*, 1990) which are valid for state space representations with nonsingular state-transition matrices. Alternative proofs, not requiring such a limitative assumption, will be given in the next section.

3. *The stabilizing property of SIORHC*

Method of proof 1. By defining the state vector

$$x(t) := [(y_t^{t-n+1})'(\delta u_{t-1}^{t-n+1})']' \tag{6}$$

plant (1) can be cast into the state-space representation

$$\begin{aligned} x(k+1) &= \Phi x(k) + G\,\delta u(k) \\ y(k) &= Hx(k) \end{aligned} \tag{7}$$

for suitably defined matrices Φ, G and H. It is known that, if $A(d)\Delta(d)$ and $B(d)$ are coprime, the triplet $\Sigma = (\Phi, G, H)$ is completely reachable and reconstructible, hence detectable. For the subsequent stability analysis we assume, without loss of generality, that $r(k) \equiv 0$ and thus $\epsilon(k) \equiv y(k)$. For the sake of simplicity, in equations (2) and (3) we set $t = 0$ and

$x(0) = x$. Then, by equation (6), the terminal constraint (3) becomes

$$x(N) = 0_X \quad \text{with} \quad N := T + n - 1. \tag{8}$$

In the following we shall consider minimization of (2) subject to constraint (8) and the relative RHC law for a generic, possibly MIMO, completely reachable and detectable plant Σ of the form of equation (7). Let $\tilde{R}_\nu$ be the νth order reachability matrix of Σ

$$\tilde{R}_\nu := [\Phi^{\nu-1}G \cdots \Phi G \quad G],$$

ν being the reachability index of Σ. Then, $\tilde{R}_\nu$ has full row rank. Consequently, there are input increment sequences $\delta u_{[0,\nu)}$ which satisfy the zero terminal state constraint

$$0_X = x(\nu) = \Phi^\nu x + \tilde{R}_\nu \, \delta u^0_{\nu-1} \tag{9}$$

for any initial state x. For every $\delta u_{[0,\nu)}$ satisfying equation (9) we write

$$J(x, \delta u_{[0,\nu)} \mid x(\nu) = 0_X) := \sum_{k=0}^{\nu-1} \|y(k)\|^2_{\Psi_y} + \|\delta u(k)\|^2_{\Psi_u}, \tag{10}$$

with $\Psi_y = \Psi'_y > 0$ and $\Psi_u = \Psi'_u > 0$. Defining

$$V_\nu(x \mid x(\nu) = 0_X) := \min_{\delta u_{[0,\nu)}} J(x, \delta u_{[0,\nu)} \mid x(\nu) = 0_X) \tag{11}$$

in Chisci and Mosca (1993) it is constructively shown that, irrespective of the possible singularity of Φ, we can construct a matrix $P(\nu)$ such that

$$V_\nu(x \mid x(\nu) = 0_X) = x'P(\nu)x \tag{12}$$

$$P(\nu) = P'(\nu) \geq 0. \tag{13}$$

Consider next the zero terminal state regulation over the interval $[0, \nu]$. Taking into account equation (12), we have

$$\begin{aligned} V_{\nu+1}(x \mid x(\nu+1) = 0_X) &= \min_{\delta u(0)} \{\|y(0)\|^2_{\Psi_y} + \|\delta u(0)\|^2_{\Psi_u} \\ &\quad + V(x(1) \mid x(\nu) = 0_X)\} \\ &= \min_{\delta u(0)} \{\|y(0)\|^2_{\Psi_y} + \|\delta u(0)\|^2_{\Psi_u} \\ &\quad + x'(1)P(\nu)x(1)\}, \end{aligned} \tag{14}$$

where

$$x(1) = \Phi x + G\,\delta u(0).$$

Equation (14) is the same as the dynamic programming step in the standard LQOR problem and yields

$$\delta u(0) = -[\Psi_u + G'P(\nu)G]^{-1}G'P(\nu)\Phi x := Fx \tag{15}$$

$$V_{\nu+1}(x \mid x(\nu+1) = 0_X) = x'P(\nu+1)x \tag{16}$$

$$\begin{aligned} P(\nu+1) &= \Phi'P(\nu)\Phi - \Phi'P(\nu)G(\Psi_u + G'P(\nu)G)^{-1} \\ &\quad \times G'P(\nu)\Phi + H'\Psi_y H. \end{aligned} \tag{17}$$

Further,

$$P(\nu+1) \leq P(\nu). \tag{18}$$

In fact,

$$\begin{aligned} x'P(\nu+1)x &= \min_{\delta u_{[0,\nu]}} J(x, \delta u_{[0,\nu]} \mid x(\nu+1) = 0_X) \\ &\leq J(x, \widehat{\delta u}_{[0,\nu)} \otimes 0_U \mid x(\nu+1) = 0_X) \\ &= J(x, \widehat{\delta u}_{[0,\nu)} \mid x(\nu) = 0_X) = x'P(\nu)x. \end{aligned}$$

Here $\widehat{\delta u}_{[0,\nu)}$ denotes the optimal input increment sequence over $[0, \nu)$ and $\otimes$ concatenation. By monotonicity of the RDE (Bitmead *et al.*, 1990), equation (18) yields

$$P(k+1) \leq P(k), \quad \forall k \geq \nu. \tag{19}$$

Then, by the FARE argument (Bitmead *et al.*, 1990) we conclude that the RHC law $\delta u(t) = Fx(t)$ yields an asymptotically stable closed-loop system whenever $N \geq \nu + 1$, irrespective of the possible singularity of Φ. For single input plants the above fact can be extended to $N \geq \nu$. We observe that, relative to the state definition (6), $\nu = 2n - 1$ so that $N \geq \nu$ is equivalent to $T \geq n$. Thus, coming back to the original SIORHC problem (1)–(4), we can state the following stability result.

Theorem 1. Consider the SIORHC problem (1)–(4) with $\Psi_y > 0$ and $\Psi_u > 0$. Let $A(d)\Delta(d)$ and $B(d)$ in equation (1) be coprime polynomials. Then, irrespective of equation (5), the SIORHC law exists uniquely and stabilizes plant (1) whenever $T \geq n = \max\{n_a + 1, n_b\}$.

Method of proof 2. The second stability proof is based on a monotonicity property (Keerthi and Gilbert, 1988) and is similar to, yet different from, the one in equation (18). Its interest consists of the fact that it allows one to deal with nonlinear plants

$$\begin{aligned} x(k+1) &= \varphi(x(k), \delta u(k)) \\ y(k) &= \eta(x(k)) \end{aligned} \tag{20}$$

for which 0_X is an equilibrium point

$$\begin{aligned} 0_X &= \varphi(0_X, 0_U) \\ 0_Y &= \eta(0_X). \end{aligned} \tag{21}$$

Assume, as in the previous section, that $r(k) \equiv 0$. Also assume that, for plant (20) and (21), the problem of minimizing equation (2) subject to $x(t+N) = 0_X$ is uniquely solvable. Consider for a fixed N the Bellman function $V(t) := V_N(x(t) \mid x(t+N) = 0_X)$, the right-hand side being defined as in equation (11), along the trajectories of the controlled system. Let $\widehat{\delta u}_{[t,t+N)}$ be the optimal input increment sequence for the initial state $x(t)$. We see that $\widehat{\delta u}_{[t+1,t+N)} \otimes 0_m$ drives the plant state from $x(t+1) = \varphi(x(t), \widehat{\delta u}(t))$ to 0_X at time $t+N$ and hence by equation (21) also at time $t+1+N$. Then, by virtue of equation (21) we have

$$V(t) - V(t+1) \geq \|y(t)\|^2_{\Psi_y} + \|\delta u(t)\|^2_{\Psi_u}. \tag{22}$$

Therefore, $\{V(t)\}_{t=0}^{\infty}$ is a monotonically nonincreasing sequence. Hence, being $V(t)$ non-negative, as $t \to \infty$ it converges to V_∞, $0 \leq V_\infty \leq V(0)$. Consequently, summing the two sides of equation (22) from $t = 0$ to $t = \infty$, we get

$$\infty > V(0) - V(\infty) \geq \sum_{t=0}^{\infty} [\|y(t)\|^2_{\Psi_y} + \|\delta u(t)\|^2_{\Psi_u}]. \tag{23}$$

This, in turn, implies for $\Psi_y > 0$ and $\Psi_u > 0$

$$\lim_{t\to\infty} y(t) = 0_Y \quad \text{and} \quad \lim_{t\to\infty} \delta u(t) = 0_U. \tag{24}$$

Summing up, we have the following result.

Theorem 2. Suppose that the problem (2) and (3) with $\Psi_y > 0$, $\Psi_u > 0$ and equation (3) replaced by $x(t+N) = 0_X$, is uniquely solvable for the nonlinear plant (20) and (21). Then, RHC law (4) yields asymptotically vanishing I/O variables.

4. *Conclusive remarks*

(1) For linear controllable and detectable plants, Theorem 2 implies, at once, asymptotic stability of the controlled system.

(2) The method of proof of Theorem 2, though simple and general, does not unveil the strict connection between SIORHC and LQOR, nor the solvability conditions, issues which instead are explicitly focused on the constructive method of proof of Theorem 1.

(3) Also, the method of proof of Theorem 2 can be used to cover the case of weights $\Psi_y(k) > 0$ and $\Psi_u(k) > 0$, $k = 0, 1, \ldots, N-1$, in equation (2). In such a case the conclusions of Theorem 2 can be readily shown to hold true provided that

$$\Psi_y(k) \leq \Psi_y(k+1) \quad \text{and} \quad \Psi_u(k) \leq \Psi_u(k+1)$$

for $k = 0, 1, \ldots, N-2$.

(4) Theorem 2 is relevant for its far-reaching consequences on the stability of SIORHC applied to nonlinear plants once solvability is insured and complementary system-theoretic properties are added.

(5) The reader can verify that the presence of hard

constraints on input and state-dependent variables over the semi-infinite horizon $[t, \infty)$ is compatible with the method of proof of Theorem 2. This makes the results of Theorem 2 of paramount importance for practical applications where control problems with constraints are ubiquitous.

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