Parallel Investments in Multiple Call and Put Options for the Tracking of Desired Profit Profiles

Mogens Graf Plessen and Alberto Bemporad

***Abstract*— A hierarchical algorithm is presented for optimally and automatically combining various option investments to cost-efficiently realize a desired profit-vs.-underlying-price profile (profit profile). The algorithm assumes that a user-defined reference shape is defined and a set of plain vanilla options in which long and short investment positions can be taken are given. Within the presented framework, the desired profit profile can be of arbitrary piecewise-affine (PWA) shape. Depending on future underlying price predictions, it typically represents a bearish or bullish market outlook, or displays bi-modal shape for conditional market outlooks. The method provides a tool for portfolio optimization that is flexible enough to trade off different user-preferences such as exploiting on conditional market outlooks, realizing leverage, and most notably guaranteeing predictable worst-case losses for risk-minimization. The framework can easily be extended to account for different derivative contracts such as exotic options.**

I. INTRODUCTION

For a financial institution the *writing* of an option consists of determining option parameters (such as strike, expiration date, etc.) and an initial price the customer must pay to buy the option. See [1] for background on options, futures and other derivatives. Part of the initial wealth paid by the customer is used to create a portfolio of underlying assets, whose composition is changed periodically during the option life so that at expiration the value of the portfolio is as close as possible to the payoff-value to be paid to the customer, then referred to as *dynamic option hedging*. See [2] and [3] for dynamic option hedging approaches based on stochastic optimization. The focus of this paper is to develop a method for how to cost-efficiently take parallel investements in (potentially) multiple call and put options for the tracking and realization of desired profit profiles.

The motivation for this paper is the usage of the proposed tool in portfolio optimization. Standard Markowitz portfolio selection as in [4] trades-off the mean and variance of the return. For the influence of linear and fixed transaction costs in the Markowitz framework, see [5], where additionally *shortfall risk constraints* are discussed, preserving convexity of the portfolio optimization problem, making, however, the assumption of a jointly Gaussian distribution of asset returns, and not guaranteeing a bound on the worst-case loss. In practice, observed returns frequently reveal "fat tails", i.e., higher probabilities for high price fluctuations. This motivates the contributors of this paper to look at portfolio optimization in terms of desired *profit profiles*, most notably guaranteeing predictable worst-case losses and profiting upon various market evolutions. Here, we do not discuss wealth dynamics that render a closed-loop control system and are characteristic for portfolio optimization. Instead, we present a method to realize an investment in a desired *profit profile* as an alternative to buying a specific stock or the purchase of a single option type. To realize such investments, at every portfolio rebalancing instant, static (i.e., independent between different sampling times) optimization problems are solved exploiting a given set of plain vanilla options.

While there exist well-known option strategies combining multiple options (such as the *bull call spread*, the *iron condor* and the like), see [6], [7], [8], [9], [10], [11], the novel contribution of this paper is to present a general *optimization-based* method for the cost-efficient and automated realization of an *aritrarily-shaped* PWA desired profit profile given a database of available option investments. Within the context of portfolio optimization, the presented tool allows one to concentrate on price predictions of the underlying asset and the design of desired profit profiles.

This paper is organized as follows. In Section II we discuss the four types of derivative contracts ultimately used for the realization of the proposed algorithm, and introduce notation. The designated problem at hand is formulated in Section III, whereby our solution approach is presented in Section IV. The results of numerical experiments on real-world data are stated in Section V, before concluding with Section VI.

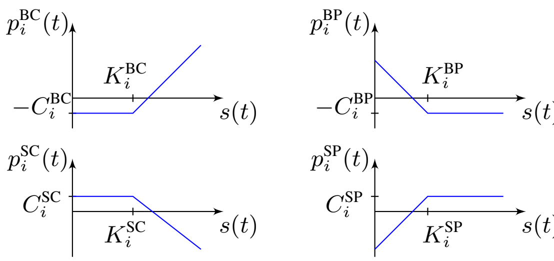
II. CALL AND PUT OPTIONS

A. Employing options as derivatives

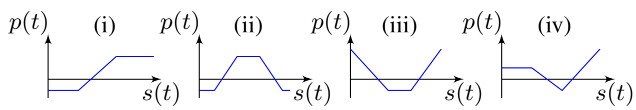
In financial terms, *securities* refer to tradable financial assets such as stocks, bonds and options. A *derivative* is a security where the value of the derivative depends explicitly on the value of another so-called *underlying* security. The derivatives of interest in this paper are *options* whose underlying can be bought or sold, such as, e.g., a stock. Derivatives are standardizedly traded on *option exchanges*, e.g., on the Chicago Board Options Exchange (CBOE), or *over-the-counter* (OTC) for tailored contracts between investment parties.

There are two main types of vanilla options: a *call/put option* gives the holder the right to buy/sell the underlying asset at a given *expiration date* T in the future for a predetermined *strike price*. *American*-style or simply *American* options allow to *exercise* (i.e., to buy/sell the underlying asset) at any time before and including the expiration date. In contrast, a *European*-style or simply *European* option allows the exercise right only on the expiration date. More exercise styles exist. The party who agreed to *buy* or *sell* an option is said to be *long* or *short*, respectively. We refer to *uncovered* options if the seller of an option does *not* holds a position in the underlying. The opposite are *covered* options.

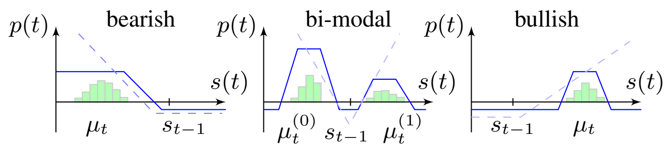
MGP and AB are with IMT School for Advanced Studies Lucca, Piazza S. Francesco 19, 55100 Lucca, Italy, {mogens.plessen, alberto.bemporad}@imtlucca.it



(a) Illustration of four option investment types, see (1).



(b) Examples of option combinations: (i) *bull call spread*, (ii) *iron condor*, (iii) *long strangle*, (iv) *call backspread*.



(c) Three market outlook scenarios: bearish, bi-modal and bullish. For each scenario, two exemplary profit profiles are displayed (blue solid and dashed lines). The (scaled) PMFs of expected underlying prices are displayed as green bars.

Fig. 1. Illustration of various profit profiles, see Section II.

B. Four general option investment types

Let the profit equations of four general option investment types be defined as

$$\begin{aligned} p_i^{\text{BC}}(t) &= \max\{s(t) - K_i^{\text{BC}}, 0\} - C_i^{\text{BC}},\ i = 1, \ldots, N^{\text{BC}}, \\ p_i^{\text{BP}}(t) &= \max\{K_i^{\text{BP}} - s(t), 0\} - C_i^{\text{BP}},\ i = 1, \ldots, N^{\text{BP}}, \\ p_i^{\text{SC}}(t) &= C_i^{\text{SC}} - \max\{s(t) - K_i^{\text{SC}}, 0\},\ i = 1, \ldots, N^{\text{SC}}, \\ p_i^{\text{SP}}(t) &= C_i^{\text{SP}} - \max\{K_i^{\text{SP}} - s(t), 0\},\ i = 1, \ldots, N^{\text{SP}}, \end{aligned} \tag{1}$$

whereby the time index is indicated by $t \in \mathbb{Z}_+$, associated with sampling time T_s such that time instances can be described as tT_s, whereby T_s may be, for example, a trading period of one month. Superscripts BC, BP, SC and SP denote "buy call option" (take a long position), "buy put", "sell call" and "sell put". For visualization of (1), see Figure 1(a). Let $y_i \in \{0, 1, 2, 3\}$ denote one of the four types. Profit from an investment is indicated by $p(t)$, e.g., $p_i^{\text{BC}}(t)$ denotes the profit at time t when holding a long position in the i-th of a set of N^{BC} call options for a specific underlying of price $s(t)$. Strike prices and costs of the option are defined by $K_i > 0$ and $C_i > 0$, respectively. We interchangeably use $s(t)$ and s_t when explicitly referring to time indices. Holding multiple option positions simultaneously then results in overall profit $p(t) = \sum_{i=1}^{N^{\text{BC}}} n_i^{\text{BC}} p_i^{\text{BC}} + \sum_{i=1}^{N^{\text{BP}}} n_i^{\text{BP}} p_i^{\text{BP}} + \sum_{i=1}^{N^{\text{SC}}} n_i^{\text{SC}} p_i^{\text{SC}} + \sum_{i=1}^{N^{\text{SP}}} n_i^{\text{SP}} p_i^{\text{SP}}$, whereby $n_i^{\text{BC}}, n_i^{\text{BP}}, n_i^{\text{SC}}, n_i^{\text{SP}} \in \mathbb{Z}_+$ denote the integer-valued number of options held and we omitted time-indices t for brevity.

C. Underlying price probability mass function

At time $t-1$, predictions about the future underlying price s_t can be made. We therefore describe s_t as a discrete random variable (DRV) with a (discrete) probability mass function (PMF) $f_{s_t}(s) \geq 0,\ \forall s \in \mathcal{S}_{s_t}$, which may in general be multi-modal. Here, our focus is on uni- and bi-modal distributions. The interest in uni-modal PMFs is natural due to bearish or bullish market outlooks. For the interest in bi-modal PMFs, consider the situation in which an investor is expecting a strong movement of the underlying price dependent on an earning report to be announced soon, but is uncertain about the movement direction. Predictions of the underlying price typically serve as prerequisite for the generation of a desired reference profit profile. Note, however, that the proposed algorithm allows tracking of *arbitrary* PWA profit profiles. Thus, estimated PMFs are not limiting trackable profit profiles, but merely can help in the design thereof. Assuming a uni-modal distribution, we may just consider the expected underlying price, here abbreviated by $\mu_t = \sum_{s \in \mathcal{S}_{s_t}} s f_{s_t}(s)$. For the bi-modal case, as an alternative to the complete PMF $f_{s_t}(s)$, we may just consider the conditional PMF denoted by $f_{s_t|z_t}(s|z) \geq 0$ with binary variable $z \in \{0, 1\}$ indicating one of two possible event outcomes, i.e., causing a decline ($z = 0$) or a rise ($z = 1$) in the underlying price. Naturally, it holds $f_{s_t}(s) = \sum_{z \in \{0,1\}} f_{s_t|z_t}(s|z) f_{z_t}(z)$. We abbreviate $\mu_t^{(z)} = \sum_{s \in \mathcal{S}_{s_t}} s f_{s_t|z_t}(s|z),\ \forall z \in \{0, 1\}$. For visualization, see Figure 1(c).

III. High-level algorithm

There exist option strategies, see e.g. [6], that combine (superimpose) multiple options to generate *profit profiles*, see Figure 1(b) for illustration. Depending on the market outlook, specific selections are preferable. Our proposed high-level algorithm for profit profile generation and realization is summarized in Algorithm 1. Let us discuss the first three substeps in Sections III-A to III-C. Step 4 is treated in all of Section IV.

Algorithm 1 Profit profile realization @$t-1$

1: **Input**: underlying price s_{t-1}, and database $\mathcal{D}$ @$t-1$.
2: **Predict future underlying price**: given past financial time-series until $t-1$, predict at least μ_t in case of a bearish, bullish or neutral market outlook, or $\mu_t^{(0)}$ and $\mu_t^{(1)}$ for a conditional market outlook; ideally, predict arbitrarily accurate the corresponding underlying price PMFs, see Section II-C.
3: **Generate desired profit profile**: design a desired PWA $p^{\text{ref}}(s)$ according (3) by
- deciding upon a desired *shape*, e.g., according to Table I.
- constructing $p^{\text{ref}}(s)$ considering Section III-C for slope, plateau levels, and kink points selections.
4: **Solve optimization problem**: solve (11) for $n^\star \in \mathbb{Z}_+^{N_n \times 1}$ according to Section IV.
5: **Wait until next rebalancing time**: initiate/terminate option investment positions according to $n^\star$.

A. Step 1

A database of available option investment positions can be summarized as

$$\mathcal{D}=\begin{bmatrix}\{K_i^{\text{BC}}\}_{i=1}^{N^{\text{BC}}} & \{C_i^{\text{BC}}\}_{i=1}^{N^{\text{BC}}} & \{y_i^{\text{BC}}\}_{i=1}^{N^{\text{BC}}}\\ \{K_i^{\text{BP}}\}_{i=1}^{N^{\text{BP}}} & \{C_i^{\text{BP}}\}_{i=1}^{N^{\text{BP}}} & \{y_i^{\text{BP}}\}_{i=1}^{N^{\text{BP}}}\\ \{K_i^{\text{SC}}\}_{i=1}^{N^{\text{SC}}} & \{C_i^{\text{SC}}\}_{i=1}^{N^{\text{SC}}} & \{y_i^{\text{SC}}\}_{i=1}^{N^{\text{SC}}}\\ \{K_i^{\text{SP}}\}_{i=1}^{N^{\text{SP}}} & \{C_i^{\text{SP}}\}_{i=1}^{N^{\text{SP}}} & \{y_i^{\text{SP}}\}_{i=1}^{N^{\text{SP}}}\end{bmatrix}\in\mathbb{R}_+^{N_n\times 3}, \tag{2}$$

with $N_n=(N^{\text{BC}}+N^{\text{BP}}+N^{\text{SC}}+N^{\text{SP}})$, where we abbreviate $\mathcal{D}=\begin{bmatrix}\mathcal{K}^{\mathcal{D}} & \mathcal{C}^{\mathcal{D}} & \mathcal{Y}^{\mathcal{D}}\end{bmatrix}$, and where the last column indicates one of the four types of options. Note that elements of database $\mathcal{D}$ must be synchronized according to expiration dates, which becomes relevant when mixing European and American exercise style options.

B. Step 2

The *prediction* of future underlying prices is crucial when taking any financial investment decisions. Predictions may be based on financial accountancy or technical chart analysis. For a method based on support vector machines, see [12]. There exists a plethora of approaches for *financial times-series prediction*. They are here not not our focus. We concentrate on profit profile designs and their optimization-based realizations by means of option combinations.

C. Step 3

We denote a desired PWA reference profit profile by

$$p^{\text{ref}}(s)=a_j s+b_j,\ \ s\in[K_j,K_{j+1}],\ \forall K_j\in\mathcal{K}^{\text{ref}}, \tag{3}$$

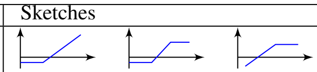
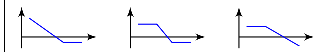
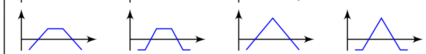
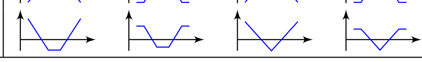
for all $j=0,1,\dots,N^{\text{ref}}$, whereby the underlying price segments are defined by the set $\mathcal{K}^{\text{ref}}$. Any arbitrary PWA function is admissable. The design can be regarded as *engineering art* and is subject to user-preferences, see Figure 1(c). Various slope rates can be achieved by adjusting the number of options sold or bought.

Let us discuss a heuristic design of $p^{\text{ref}}(s)$. For the realization of different market outlooks and risk/reward demands, we summarize typical $p(s)$-schemes in Table I. We remark that some profiles exhibit *plateau levels*, defined by constant p for consecutive s, and with $\min(p(s))>-\infty$ and $\max(p(s))<+\infty$. In case the designated $p(s)$ profile is selected to have limited downside risk, i.e., $\min(p(s))>-\infty$, we scale $p^{\text{ref}}(s)$ such that $\min\left(p^{\text{ref}}(s)\right)=0$ and introduce an optimization *slack variable* responsible for constant cost-efficient offset as later discussed in Section IV. For simplicity, we may assign $\frac{dp^{\text{ref}}(s)}{ds}\in\{0,1\}$, i.e., permit only two possible slope rates. Naturally, other slope rates are possible, too.

For uni-modal market outlooks (bullish, bearish or neutral), we define $p^{\text{ref}}(s)\geq\tilde{p}_t,\ \forall s\in[\alpha\mu_t,\mu_t+(1-\alpha)\mu_t]$ with parameters $\alpha\in(0,1]$ and $m_t\geq 0$ such that $\tilde{p}_t=m_t(\alpha\mu_t-s_{t-1})$. Figure 2 illustrates a corresponding example in which we define $m_t=1$, $\mathcal{K}^{\text{ref}}=\{0,K_1,K_2,K_3,K_4\}=\{0,\alpha\mu_t-\tilde{p}_t,\alpha\mu_t,\mu_t+(1-\alpha)\mu_t,\mu_t+(1-\alpha)\mu_t+\tilde{p}_t\}$ and

$$p^{\text{ref}}(s)=\begin{cases}0, & 0\leq s\leq K_1 \text{ and } K_4\leq s<\infty,\\ s-K_1, & K_1\leq s\leq K_2,\\ \tilde{p}_t, & K_2\leq s\leq K_3,\\ \tilde{p}_t-s+K_3, & K_3\leq s\leq K_4.\end{cases}$$

TABLE I. Library of typical shape designs for $p(s)$ that may be used for the realization of different market outlooks and risk/reward demands. For a discussion of corresponding kink points, slopes and plateau levels selections, see Section III-C and Figure 2.

Outlook	Sketches
bullish	
bearish	
neutral	
conditional	

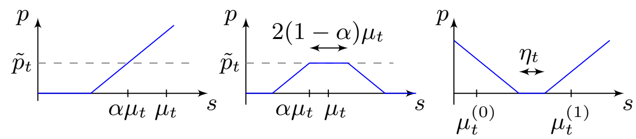


Fig. 2. Illustration of heuristic methods for the PWA generation of $p^{\text{ref}}(s)$ including plateau level and kink points selections. See Section III-C for the exemplary discussion of the center and right subplot. We have $\eta_t=\alpha^{(1)}\mu_t^{(1)}-(1+\alpha^{(0)})\mu_t^{(0)}$.

For conditional market outlooks, we select tuning parameters $\alpha^{(0)},\alpha^{(1)}\in(0,1]$. To give an example with respect to Figure 2, we define $\mathcal{K}^{\text{ref}}=\{0,K_1,K_2\}=\{0,(1+\alpha^{(0)})\mu_t^{(0)},\alpha^{(1)}\mu_t^{(1)}\}$ and

$$p^{\text{ref}}(s)=\begin{cases}K_1-s, & 0\leq s\leq K_1,\\ 0, & K_1\leq s\leq K_2,\\ s-K_2, & K_2\leq s<\infty.\end{cases} \tag{4}$$

It remains to discuss the selections of upper plateau levels for conditional market outlooks and desired limited upside rewards (for overall cost reduction). Saturating (4), we may, for example, select two different plateau levels $\tilde{p}_t^{(0)}=\alpha^{(0)}\mu_t^{(0)},\ \forall 0\leq s\leq\mu_t^{(0)}$ and $\tilde{p}_t^{(1)}=(1-\alpha^{(1)})\mu_t^{(1)},\ \forall s\geq\mu_t^{(1)}$, thereby trading-off different likelihoods of upside or downside market outlooks.

IV. Optimization Problem Formulation

A. Preparation

Vector set $\mathcal{K}^{\text{ref}}$ defines all *kink points* (s_t-coordinates at which $p^{\text{ref}}(s)$ is continuous but with discontinuous gradient) of $p^{\text{ref}}(s)$. Likewise, $\mathcal{K}^{\mathcal{D}}$ describes a similar set, see Section III-A. In a first step, we unite and sort them according ascending s_t-coordinate, thereby creating the vector set

$$\mathcal{K}=\text{sort}\left(\mathcal{K}^{\mathcal{D}}\cup\mathcal{K}^{\text{ref}}\right), \tag{5}$$

where we denote the number of elements by $N_{\mathcal{K}}$. Then, all of options of (1) (each PWA with *one* kink point) can be cast into general PWA form with $N_{\mathcal{K}}$ kink points then common to all, i.e.,

$$p_i(s)=\begin{cases}(f_{i,0}s+g_{i,0}+c_i)n_i, & s\in[0,K_1],\\ (f_{i,1}s+g_{i,1}+c_i)n_i, & s\in[K_1,K_2],\\ \vdots & \\ (f_{i,N_{\mathcal{K}}}s+g_{i,N_{\mathcal{K}}}+c_i)n_i, & s\in[K_{N_{\mathcal{K}}},\infty),\end{cases}$$

with $n_i \in \mathbb{Z}_+$ the number of option i for all $i = 1, 2, \ldots, N_n$. Combining all options by superposition, we obtain

$$p(s) = \begin{cases} \sum_{i=1}^{N_n} (f_{i,0}s + g_{i,0} + c_i)n_i, & s \in [0, K_1], \\ \sum_{i=1}^{N_n} (f_{i,1}s + g_{i,1} + c_i)n_i, & s \in [K_1, K_2], \\ \vdots \\ \sum_{i=1}^{N_n} (f_{i,N_\mathcal{K}}s + g_{i,N_\mathcal{K}} + c_i)n_i, & s \in [K_{N_\mathcal{K}}, \infty), \end{cases}$$
$$= (sf_j^T + g_j^T + c^T)n, \; s \in [K_j, K_{j+1}], \; \forall K_j \in \mathcal{K}, \tag{6}$$

for all $j = 0, 1, \ldots, N_\mathcal{K}$. Likewise, we cast $p^{\text{ref}}(s)$ from (3) into general PWA form with the same $N_\mathcal{K}$ kink points, then denoted by

$$p^{\text{bound}}(s) = a_j^{\text{bnd}} s + b_j^{\text{bnd}}, \; s \in [K_j, K_{j+1}], \; \forall K_j \in \mathcal{K}, \tag{7}$$

for all $j = 0, 1, \ldots, N_\mathcal{K}$, and further define

$$\tilde{b}_j = a_j^{\text{bnd}} K_j + b_j^{\text{bnd}}, \; \forall K_j \in \mathcal{K}, \; \forall j = 0, 1, \ldots, N_\mathcal{K}. \tag{8}$$

Proposition 1: To ensure that (7) serve as a lower bound on the desired profit profile $p(s)$ described in (6), it suffices to just evaluate at the kink points and constrain

$$(K_j f_j^T + g_j^T + c^T)n \geq \tilde{b}_j, \tag{9}$$
$$f_{N_\mathcal{K}}^T n \geq a_{N_\mathcal{K}}^{\text{bnd}}, \tag{10}$$

for all $K_j \in \mathcal{K}$ and $j = 0, 1, \ldots, N_\mathcal{K}$.

Proof: Let us abbreviate $z_j = (K_j f_j^T + g_j^T + c^T)n$ for all $K_j \in \mathcal{K}$ and $j = 0, 1, \ldots, N_\mathcal{K}$. Because of all $N_\mathcal{K} + 1$ segments being PWA, w.l.o.g. we can consider any of the segments. Then, we note that any $z = p(s)$ for $s \in [K_j, K_{j+1}]$ can be described as a linear combination $z = \gamma z_j + (1-\gamma)z_{j+1}$ for $\gamma \in [0, 1]$. Assume now $z_j \geq \tilde{b}_j$, $z_{j+1} \geq \tilde{b}_{j+1}$ and Proposition 1 as stated is wrong. The proof is then by contradiction. Similarly as above, we can write $\tilde{b} = \gamma \tilde{b}_j + (1-\gamma)\tilde{b}_{j+1}$. According to our assumption there exists a $\gamma \in [0, 1]$ such that $\gamma z_j + (1-\gamma)z_{j+1} < \gamma \tilde{b}_j + (1-\gamma)\tilde{b}_{j+1}$. This can be rewritten as $\gamma(z_j - \tilde{b}_j) + (1-\gamma)(z_{j+1} - \tilde{b}_{j+1}) < 0$, which is a contradiction since all of the left-hand side is positive. Ultimately, as a constraint on the slope, (10) is introduced as an alternative to account for the kink point $s \to \infty$. This concludes the proof. ∎

With respect to the discussion in Section III-A, we remark for the sign of costs (when options are purchased) or premiums (when options are sold) that

$$c_i = \begin{cases} +C_i^\mathcal{D}, & \text{if } i \in \{1, 2, \ldots, N^{\text{BC}} + N^{\text{BP}}\}, \\ -C_i^\mathcal{D}, & \text{if } i \in \{N^{\text{BC}} + N^{\text{BP}} + 1, \ldots, N_n\}. \end{cases}$$

Thus, to summarize, we construct a vector set (i.e., a grid) $\mathcal{K}$ of kink points according to (5), before organizing $c \in \mathbb{R}^{N_n \times 1}$, $f_j \in \mathbb{R}^{N_n \times 1}$, $g_j \in \mathbb{R}^{N_n \times 1}$, $\tilde{b}_j \in \mathbb{R}$ according to (6) and (8) for all $j = 0, 1, \ldots, N_\mathcal{K}$, and $a_{N_\mathcal{K}}^{\text{bnd}} \in \mathbb{R}$.

B. Formulation

For the cost-efficient realization of the desired profit profile, we propose the following optimization problem:

$$\max_{n,l,\sigma} \; \lambda_0 c^T n - \lambda_1 \|n\|_1 + \lambda_2 l - \lambda_3 \sigma - \lambda_4 v^T n \tag{11a}$$
$$\text{s.t.} \; \left(K_j f_j^T + g_j^T + c^T\right) n \geq \tilde{b}_j - \sigma, \tag{11b}$$
$$f_{N_\mathcal{K}}^T n \geq a_{N_\mathcal{K}}^{\text{bnd}}, \tag{11c}$$
$$\left(K_j f_j^T + g_j^T + c^T\right) n \geq l, \tag{11d}$$
$$n \geq 0, \; l \geq l_{\text{min}}, \; \sigma \geq 0, \tag{11e}$$
$$n \in \mathbb{Z}_+^{N_n \times 1}, \; l \in \mathbb{R}_+, \; \sigma \in \mathbb{R}_+, \tag{11f}$$
$$\forall K_j \in \mathcal{K}, \; \forall j = 0, 1, \ldots, N_\mathcal{K}, \tag{11g}$$

where $\lambda_0, \lambda_1, \ldots, \lambda_4 \in \mathbb{R}_+$ denote penalty weights that can easily trade-off or omit (by setting the corresponding $\lambda = 0$) different objectives. The objective function (11a) is composed of five components. The first component denotes the maximization of accumulated fixed costs/gains for purchasing/selling of available options. The second component is introduced to encourage sparsity in the integer-valued decision vector $n \in \mathbb{Z}_+^{N_n \times 1}$ with n_i according to Sections IV-A. The third component results from the introduction of *slack variable* $l \in \mathbb{R}_+$ to minimize the maximal profit profile loss (min-max problem). The fourth component penalizes another slack variable, $\sigma \in \mathbb{R}_+$, introduced for softening the constraint on the lower bound on the desired profit profile (*soft constraint*), see (11b). The fifth component indicates costs incurred when having to cover the selling of options (e.g., purchase and transaction costs for buying the underlying as a prerequisite for selling a covered call option). A sixth component such as $\sum_{i=1}^{N_\mu} \lambda_{4+i} \left(K_{\mu_i} f_{\mu_i}^T + g_{\mu_i} + c_{\mu_i}\right) n$ with $K_{\mu_i} \in \{K_j : K_j \in \mathcal{K}, \; K_j = \mu_i \text{ modal peaks}\}$ may additionally be added to maximize profit for expected underlying prices (that possibly may be multi-modal). The first inequality (11b) describes the aforementioned *soft constraint* on the lower bound on the desired profit profile. It is introduced since a *reasonbale* hard lower bound on the profit profile is a priori unknown since depending on $\mathcal{D}$. The second constraint (11c) stems from (10). The third constraint (11d) results from the aforementioned introduction of slack variable l to minimize the maximal profit profile loss. In (11e), $l_{\text{min}} \in \mathbb{R}_+$ is defined to enforce a potential hard threshold on the maximal admissable profit profile loss. Dimensions of all optimization variables are stated in (11f). The coverage of all of the desired underlying price range segments is indicated by (11g). The solution vector of (11) shall be denoted by $n^\star$ and the corresponding profit profile by $p^\star(s)$.

C. Solution

For the solution of the mixed-integer optimization problem (11), we employ the domain-specific language CVXPY for optimization embedded in Python [13]. Note that when relaxing $n \in \mathbb{Z}_+^{N_n \times 1}$ to be real-valued, (11) is a convex problem; in fact, a linear program with an additionally added ℓ_1-norm in the objective function. All numerical experiments throughout this paper were conducted on a laptop running Ubuntu 14.04 equipped with an Intel Core i7 CPU @2.80GHz×8, 15.6GB of memory, and using Python 2.7.

V. Numerical examples

We consider real-world option price data, drawn from the CBOE at http://www.cboe.com/delayedquote/quotetable.aspx on August 24, 2016. As underlying, we selected Alphabet Inc. Class C (Nasdaq symbol GOOG), which was quoted with a stock price of 772.4\$ at the time of the data retrieval. The expiration date was selected to be December 16, 2016. For that datum, we retrieved the maximum amount of option data available, i.e., a total of 65 different strike prices, valued between 440\$ and 1020\$, for both call and put options. To give two examples for a call and put option with strike price 440\$, respectively: `GOOG1616L440-E` and `GOOG1616X440-E`. For C_i^{BC} and C_i^{BP}, and for C_i^{SC} and C_i^{SP}, we considered the ask and bid prices of call and put options on time of data retrieval, respectively. Thus, in total we retrieved $N_n = 260$ unique option investment opportunities as illustrated in Figure 3.

For all four experiments reported and according to (11), we set $(\lambda_0, \lambda_1, \lambda_2, \lambda_3, \lambda_4) = (100, 1, 1000, 100000, 0)$, $l_{\text{min}} = -\infty$ and $m_t = 3$. The experiments differ by the selection of α, μ_t (or $\mu_t^{(0)}$ and $\mu_t^{(1)}$ for the conditional case) and desired reference profit profiles $p^{\text{ref}}(s)$. All results are visualized in Figures 4, 5, 6 and 7, and quantitatively summarized in Table II. The fixed cost (if negative) or fixed premium (if positive) incurred at time $t-1$ is indicated by $c^T n^\star$. We define percentage returns by $r^{\text{stock}}(\mu_t) = \frac{\mu_t - s_{t-1}}{s_{t-1}} 100$ for a stock investment, and, for the option investments, $r(\mu_t) = \frac{p(\mu_t)}{-c^T n^\star} 100$ if $c^T n^\star < 0$ (i.e., an initial expenditure was required) and $r(\mu_t) = \infty$ if $c^T n^\star \geq 0$, i.e., an initial premium was received. All zero-crossings of $p^\star(s)$ are indicated by s_{BE} (*break-even* points). For the conditional (bi-modal) case, if appropriate, two quantities are stated. For simplicity, we here assumed permission to also conduct the selling of uncovered options. This assumption is justified when assuming a sufficient cash position to cover potential losses, but is to be revised when trading recursively in the context of self-financing portfolio optimization which is subject of ongoing work. With respect to database storage, we ordered options according to ascending strike prices, i.e., option identifier $i = 0, 1, 2, 3$ (see, e.g., the third subplot of Figure 4) correspond to BC,SC,BP,SP for the lowest possible strike price of 440\$.

Several observations can be made. First, the terms in (11a) associated with λ_1 and λ_3 are a *must*, i.e., for ensuring sparsity in the solution vector and enabling sufficient freedom in appropriately offsetting the desired reference profit profile, respectively. With λ_0 and λ_2 fine-tuning can be achieved in accordance with Section IV-B. Second, the smaller desired s-ranges for which $p^\star(s) \geq 0$ are, the smaller the worst case loss (i.e., the higher $\min(p^\star(s))$). These ranges can be controlled by selection of α. For example, selecting $\alpha = 0.98$ (instead of $\alpha = 0.95$) in the third experiment narrows the iron condor, but results in $\min p^\star(s) = -8.7$ and $\max p^\star(s) = 16.3$, see Table II for contrast. Thus, as accurate as possible *predictions* of underlying price evolutions are (as expected) preferable for cost minimization. Third, using CVXPY [13], remarkable differences could be observed when solving (11) as the original mixed-integer problem or

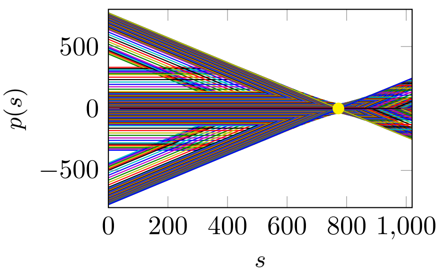


Fig. 3. Illustration of $p(s)$ for all $N_n = 260$ option contracts available at the CBOE on August 24, 2016, with underlying Alphabet Inc. Class C (Nasdaq symbol GOOG), which then was quoted at 772.4\$ as visualized by the yellow ball.

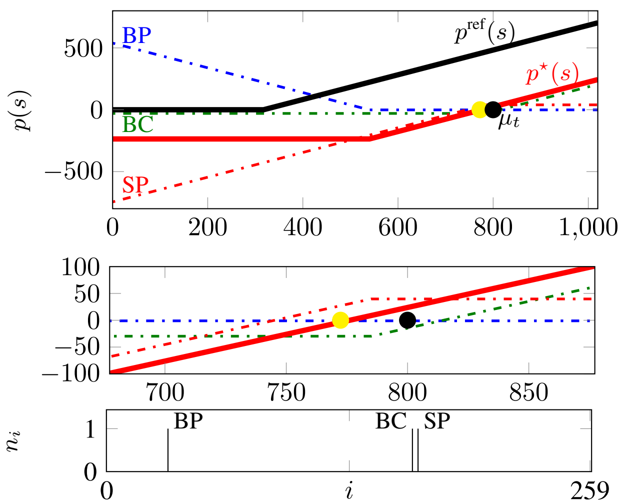


Fig. 4. Results of experiment 1. We selected $\mu_t = 800$ and $\alpha = 0.5$. The second subplot is a zoom-in of the first one. The third subplot indicates the number n_i and type (BC, SC, BP, SP) of options contracts sold or bought.

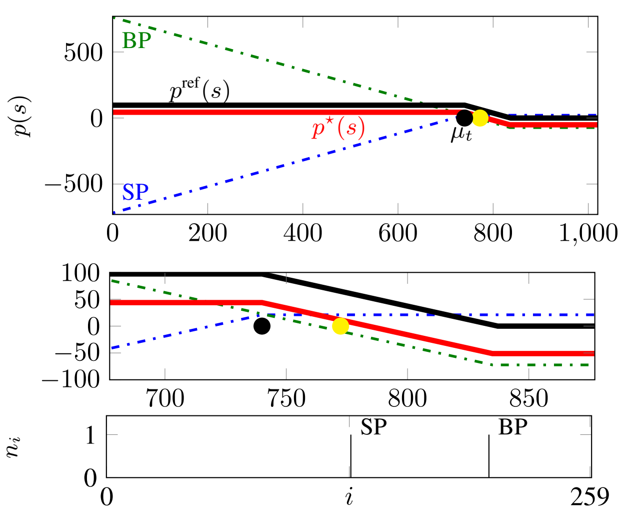


Fig. 5. Results of experiment 2. We selected $\mu_t = 740$ and $\alpha = 1$. Note that the desired shape is here realized optimally with two *put* options (typical for a *bear put spread*) rather than with two *call* options (*bear call spread*) which can result in the same shape, however, here at higher initial expenditure costs.

TABLE II. Summary of quantitative results of the experiments from Section V. All quantities are reported in units \$ with exception of $r(\mu_t)$ and $r^{\text{stock}}(\mu_t)$, and the computation time τ_{cvxpy} required for the solution of (11), which is measured in milliseconds.

	Expt. 1	Expt. 2	Expt. 3	Expt. 4
$c^T n^\star$	8.8	-51.2	11.4	-136.6
min $p(s)$	-236.2	-51.2	-13.6	-131.6
max $p(s)$	∞	43.8	11.4	∞
$p(\mu_t)$	23.8	43.8	11.4	63.4/33.4
$r(\mu_t)$	∞	85.5%	∞	46.4/24.5%
$\mu_t - s_{t-1}$	27.6	-32.4	-5.4	-102.4/87.6
$r^{\text{stock}}(\mu_t)$	3.6%	-4.2%	-0.7%	-13.3/11.3%
s_{BE}	776.2	783.8	718.6/821.4	701.7/843.3
τ_{cvxpy}	109ms	106ms	124ms	105ms

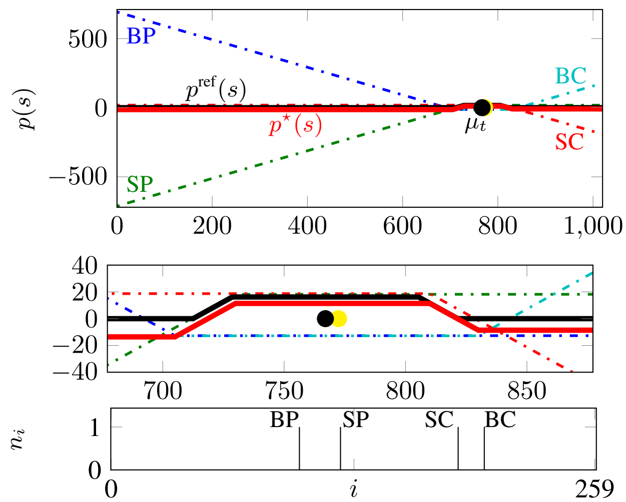


Fig. 6. Results of experiment 3. We selected $\mu_t = 767$ and $\alpha = 0.95$. Note that the optimal solution to (11) returned slightly different loss levels in case of strongly positive and strongly negative market evolutions (different plateau levels).

a (real-valued) relaxed version thereof admitting $n \in \mathbb{R}_+^{N_n \times 1}$ before then rounding the solution to the nearest integer. While the final results did only marginally (if at all) differ, computation times frequently lasted more than 12minutes vs. 100ms. All results reported in Table II stem from the real-valued relaxation and consequent integer-rounding solution. Ultimately, to point out a characteristic of the investment method via multiple options in parallel, consider experiment 1. While a stock purchase requires the initial expenditure of the stock price (plus transaction costs), an investment according to experiment 1 contrarily *generates* an initial *income*, here of $c^T n^\star = 8.8\$$ per option contract, which may immediately be used to undertake other investments.

VI. Conclusion

We proposed an optimization-based hierarchical algorithm for the cost-efficient and automated realization of desired profit profiles given a database of available option investments. Profit profiles can be designed arbitrarily as piecewise-affine, typically influenced by underlying price predictions, a bullish, bearish or conditional market outlook, and accounting for user-preferences such as a bound on maximum loss. Subject of ongoing research is the incorporation of the presented framework in the context of portfolio optimization, and the development of corresponding investment rebalancing strategies based on receding horizon control to react on predicted market evolutions.

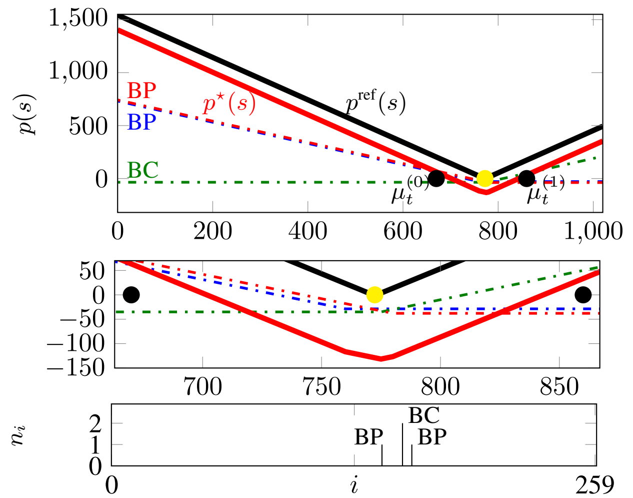


Fig. 7. Results of experiment 4. We selected $\mu_t^{(0)} = 670$, $\mu_t^{(1)} = 860$ and $\alpha = 0.999$. For interest, we now additionally changed the slope-rate from 1 to 2 in aim of stronger profit generation once the underlying price has passed the break-even points. The resulting profit profile is slightly asymmetrical with respect to $\mu_t^{(0)}$ and $\mu_t^{(1)}$.

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