Optimal and Receding Horizon Drift Counteraction Control: Linear Programming Approaches

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Abstract— **This paper presents new approaches based on linear programming (LP) and mixed-integer linear programming (MILP) to solve the optimal exit-time control problem, that is to maximize the first exit-time of a system from a prescribed set. For linear discrete-time systems with known disturbance inputs, we show that an optimal solution can be obtained by solving an MILP and suboptimal solutions are obtained via LP. For both the MILP and LP, an iterative scheme is introduced that improves robustness and computation time. In addition, feedback control strategies are formulated using model predictive control (MPC) techniques. Two numerical examples of a linearized van der Pol oscillator and of spacecraft attitude control demonstrate the efficiency of the proposed approaches in solving optimal exit-time control problems.**

I. Introduction

Let the state and control input vector at time instant $t \in \mathbb{Z}_{\geq 0}$ be $x_t \in \mathbb{R}^n$ and $u_t \in \mathbb{R}^p$, respectively, and $d_t \in \mathbb{R}^n$ denote a disturbance. Let the set of admissible control sequences $\{u_t\}$ be defined by

$$U_{\text{seq}} = \{\{u_t\} : u_t \in U_t \text{ for all } t\}, \quad (1)$$

where

$$U_t = \{u \in \mathbb{R}^p : C_{\text{c},t} u \leq b_{\text{c},t}\}. \quad (2)$$

We consider the following deterministic optimal control problem of finding an admissible sequence of control inputs $\{u_t\}$ that maximizes the first exit-time from a given set

$$\begin{aligned} &\max_{\{u_t\} \in U_{\text{seq}}} \tau(x_0, \{u_t\}) \\ &\text{subject to } x_{t+1} = A_t x_t + B_t u_t + d_t,\ x_0 \in G_0, \end{aligned} \quad (3)$$

where the first exit-time is defined as

$$\begin{aligned} \tau(x_0, \{u_t\}) = \inf\{t \in \mathbb{Z}_+ : x_t \notin G_t \,|\, x_0 \in G_0, \\ \{u_t\} \in U_{\text{seq}}\}, \end{aligned} \quad (4)$$

and

$$G_t = \{x \in \mathbb{R}^n : C_{\text{s},t} x \leq b_{\text{s},t}\} \quad (5)$$

is the set one wants the state vector to remain inside.

The optimal exit-time control problem (3) arises in many engineering applications, for example, underactuated systems or systems with finite resources (fuel, energy, component life). Since the control action may be viewed as a way of counteracting drift from disturbances or system dynamics in order to satisfy prescribed constraints for as long as possible, we refer to such problems as drift counteraction optimal control (DCOC) problems.

Problems similar to (3) were studied for continuous-time systems in [1], [2], [3], [4], [5] and references therein. Most of the previous research, however, considered minimizing non-negative or discounted cost/reward functions instead of explicitly maximizing the first exit-time. Furthermore, the continuous-time formulation of the problem requires solving the Hamilton-Jacobi-Bellman (HJB) equation, which is a first-order partial differential equation (PDE) in this case. Explicit solutions to the HJB exist only for a few special problems. On the other hand, the discrete-time formulation of the problem appears computationally more tractable compared to solving a PDE numerically [6].

Problems of the form (3) were solved by obtaining control policies using dynamic programming techniques in [6], [7], [8], [9]. While these approaches are quite general and can address the case of nonlinear systems, they suffer from the classical *curse of dimensionality* of dynamic programming, as the computational complexity increases exponentially with the number of states. In this paper, we propose new approaches based on mathematical programming to efficiently solve problem (3). We show that (3) is equivalent to a *mixed integer linear program* (MILP). A general framework for modeling and control using mixed integer programming was outlined in [10] and MILP has been used for control problems such as trajectory planning with obstacle avoidance for spacecraft [11], minimum-time control of ground vehicles [12], or for control of microgrids [13].

However, MILP is in the class of NP-complete problems and the worst-case computation time increases exponentially with the number of integer variables [10], [14], [15]. In addition, employing MILP to solve (3) is challenging because the optimal exit-time is unknown, which makes it difficult to choose an appropriate time horizon for the MILP formulation. We address this with an iterative scheme based on MILP that efficiently solves (3) by reducing the number of integer variables while adapting the time horizon. Furthermore, we efficiently obtain good-quality suboptimal solutions by solving a similar problem without integer variables using standard *linear programming* (LP). Besides, we propose to use the MILP and LP formulations in a model predictive control (MPC) / receding horizon control (RHC) scheme for improved robustness due to introducing state feedback.

The paper is structured as follows. The MILP and LP formulations are discussed in Section II. The iterative procedure for robustly solving the respective problem is outlined in Section III. The MPC implementation is discussed in

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Section IV. Section V provides two numerical examples of a linearized van der Pol oscillator and of attitude control of an underactuated spacecraft with two reaction wheels (RWs). A conclusion is given in Section VI.

II. MILP and LP Formulations

Throughout the paper we make the following assumptions about the first exit-time and the sets G_t and U_t, which guarantee existence of a bounded solution to (3) [16].

Assumption 1: There exists $\bar{T} > 0$ such that $\tau(x, \{u_t\}) \leq \bar{T}$ for all $x \in G_0$ and $\{u_t\} \in U_{\mathrm{seq}}$.

Assumption 2: G_t and U_t are compact sets (polytopes) for all t.

A. MILP Formulation

Consider the MILP formulation

$$\begin{aligned}&\min_{\{x_t\},\{u_t\},\{\delta_t\}} \sum_{t=1}^{N} \delta_t\\&\text{subject to}\\&x_{t+1} = A_t x_t + B_t u_t + d_t\\&\delta_{t-1} \leq \delta_t,\ \delta_t \in \{0,1\} \subset \mathbb{Z}\\&C_{\mathrm{s},t} x_t \leq b_{\mathrm{s},t} + \mathbf{1} M \delta_t\\&u_t \in U_t,\end{aligned} \tag{6}$$

given $x_0 \in G_0$, where U_t is defined in (2), $N \in \mathbb{Z}_+$, $M \in \mathbb{R}_+$, $\mathbf{1}$ denotes the n-dimensional vector of ones, and $\delta_0 = 0$ due to $x_0 \in G_0$. The binary variable δ_t is an indicator variable for the condition $x_t \notin G_t$. If M is sufficiently large, a solution to the MILP exists as stated in Lemma 1.

Lemma 1: Assume $M \in \mathbb{R}_+$ exists such that $C_{\mathrm{s},t} x_t \leq b_{\mathrm{s},t} + \mathbf{1}M$ for all $t = 0, 1, ..., N$ and all $\{x_t\}$ satisfying (6) for any control sequence $\{u_t\} \in U_{\mathrm{seq}}$. Then a solution to (6) exists.

Proof: Since M is sufficiently large by assumption, $\delta_t \equiv 1$ is feasible for all $\{u_t\} \in U_{\mathrm{seq}}$ and $\{x_t\}$ satisfying $x_{t+1} = A_t x_t + B_t u_t + d_t$, $x_0 \in G_0$. Since the number of possible δ_t sequences is finite and a feasible solution exists for at least one of them, the solution existence to (6) follows. ∎

We can now state conditions under which solutions to MILP (6) are equivalent to solutions of (3).

Theorem 1: Suppose Assumptions 1 and 2 hold. Furthermore, suppose $N \geq \tau(x_0, \{u_t\})$ for all $\{u_t\} \in U_{\mathrm{seq}}$, $x_0 \in G_0$, and M is sufficiently large as in Lemma 1. Then, solutions to the MILP (6) and the original problem (3) are equivalent, i.e., a solution to the MILP is also a solution to (3) and vice versa.

Proof: A solution to problem (3) exists due to Assumptions 1 and 2 [16]. Suppose $\{u_t^*\}$ is a solution to (3) with corresponding state trajectory $\{x_t^*\}$ according to the system dynamics stated in (3). Then,

$$\tau(x_0, \{u_t^*\}) \geq \tau(x_0, \{u_t'\}), \tag{7}$$

for any $\{u_t'\} \in U_{\mathrm{seq}}$ with corresponding state trajectory $\{x_t'\}$ according to the system dynamics in (3), where $x_0 = x_0^* = x_0'$. Now (4), (5), the constraints in (6), and $N \geq \tau(x_0, \{u_t\})$ for all $\{u_t\} \in U_{\mathrm{seq}}$ imply that $\delta_t^* = 1$ for $t = \tau(x_0, \{u_t^*\}), ..., N$ and $\delta_t' = 1$ for $t = \tau(x_0, \{u_t'\}), ..., N$, where $\{\delta_t^*\}$ is the solution to (6) with $\{x_t\} = \{x_t^*\}$ and $\{u_t\} = \{u_t^*\}$ fixed, and $\{\delta_t'\}$ is the solution to (6) with $\{x_t\} = \{x_t'\}$ and $\{u_t\} = \{u_t'\}$ fixed. Consequently, $\delta_t^* = 0$ for $t < \tau(x_0, \{u_t^*\})$ and $\delta_t' = 0$ for $t < \tau(x_0, \{u_t'\})$. This and (7) imply that

$$\begin{aligned}\sum_{t=1}^{N} \delta_t^* &= N - \tau(x_0, \{u_t^*\}) + 1\\&\leq N - \tau(x_0, \{u_t'\}) + 1 = \sum_{t=1}^{N} \delta_t',\end{aligned} \tag{8}$$

for all $(\{x_t'\}, \{u_t'\}, \{\delta_t'\})$ that satisfy the constraints of the MILP. Therefore, $(\{x_t^*\}, \{u_t^*\})$ together with $\{\delta_t^*\}$ solve the MILP as well. For the second part of the proof we need to show that a solution to the MILP is also a solution of (3), where a solution to the MILP exists by Lemma 1. Suppose that $(\{x_t^{\mathrm{MILP}}\}, \{u_t^{\mathrm{MILP}}\}, \{\delta_t^{\mathrm{MILP}}\})$ solves the MILP, i.e.,

$$\sum_{t=1}^{N} \delta_t^{\mathrm{MILP}} \leq \sum_{t=1}^{N} \delta_t', \tag{9}$$

for all $(\{x_t'\}, \{u_t'\}, \{\delta_t'\})$ that satisfy the constraints in (6) and $x_0 = x_0^{\mathrm{MILP}} = x_0'$. For any given admissible $\{u_t'\}$, let $\{\bar{\delta}_t'\}$ be such that $\bar{\delta}_t' = 0$ iff $t < \tau(x_0, \{u_t'\})$, which is always feasible with respect to (6) due to M being sufficiently large by assumption. Hence,

$$\tau(x_0, \{u_t'\}) = 1 + \sum_{t=1}^{N} (1 - \bar{\delta}_t') = N + 1 - \sum_{t=1}^{N} \bar{\delta}_t'. \tag{10}$$

Then, by (9),

$$\begin{aligned}\tau(x_0, &\{u_t^{\mathrm{MILP}}\}) = \min\{t : \delta_t^{\mathrm{MILP}} = 1\} = N + 1\\&- \sum_{t=1}^{N} \delta_t^{\mathrm{MILP}} \geq N + 1 - \sum_{t=1}^{N} \bar{\delta}_t' = \tau(x_0, \{u_t'\}),\end{aligned} \tag{11}$$

for all $\{u_t'\} \in U_{\mathrm{seq}}$. Consequently, $\{u_t^{\mathrm{MILP}}\}$ solves (3). ∎

$$\begin{aligned}&\min_{\{x_t\},\{u_t\},\{\delta_{\tau_{\mathrm{lb}}},...,\delta_N\}} \sum_{t=\tau_{\mathrm{lb}}}^{N} \delta_t\\&\text{subject to}\\&x_{t+1} = A_t x_t + B_t u_t + d_t\\&\delta_{t-1} \leq \delta_t,\ \delta_t \in \{0,1\} \subset \mathbb{Z},\ t = \tau_{\mathrm{lb}}, ..., N\\&C_{\mathrm{s},t} x_t \leq b_{\mathrm{s},t},\ t = 1, ..., \tau_{\mathrm{lb}} - 1\\&C_{\mathrm{s},t} x_t \leq b_{\mathrm{s},t} + \mathbf{1} M \delta_t,\ t = \tau_{\mathrm{lb}}, ..., N\\&u_t \in U_t.\end{aligned} \tag{12}$$

In practice, the complexity of MILP (6) can be reduced if a lower bound $\tau_{\mathrm{lb}} \in \mathbb{Z}_+$ for the optimal exit-time is known, i.e., $\tau(x_0, \{u_t^*\}) \geq \tau_{\mathrm{lb}}$, where $\{u_t^*\}$ is a solution of (3). In this case we can set $\delta_t = 0$ for $t = 1, ..., \tau_{\mathrm{lb}} - 1$ and get MILP (12), where $x_0 \in G_0$. Compared to (6), MILP (12) reduces the number of binary variables to optimize from N

to $N-\tau_{\mathrm{lb}}+1$. Note that τ_{lb} can be chosen as corresponding to exit-time under any given admissible control law.

B. LP Formulation

Instead of using MILP (12) to solve (3), relaxing (12) and obtaining a suboptimal solution based on an LP formulation may provide a better balance between computation time and performance of the solution. The MILP is relaxed by replacing the binary variables δ_t with non-negative continuous variables ε_t, yielding the following LP

$$
\begin{aligned}
&\min_{\{x_t\},\{u_t\},\{\varepsilon_{\tau_{\mathrm{lb}}},\dots,\varepsilon_N\}} \sum_{t=\tau_{\mathrm{lb}}}^{N} q_t\varepsilon_t \\
&\text{subject to} \\
&x_{t+1} = A_t x_t + B_t u_t + d_t \\
&0 \le \varepsilon_{t-1} \le \varepsilon_t \\
&C_{\mathrm{s},t} x_t \le b_{\mathrm{s},t},\ t = 1,\dots,\tau_{\mathrm{lb}}-1 \\
&C_{\mathrm{s},t} x_t \le b_{\mathrm{s},t} + \mathbf{1}\varepsilon_t,\ t = \tau_{\mathrm{lb}},\dots,N \\
&u_t \in U_t,
\end{aligned}
\tag{13}
$$

given $x_0 \in G_0$, where $\varepsilon_t \in \mathbb{R}_{\ge 0}$, and $q_t \in \mathbb{R}_+$ are weights. As in MILP (12), $\tau_{\mathrm{lb}} \in \mathbb{Z}_+$ is a lower bound on the optimal exit-time and $\varepsilon_t = 0$ for $t = 1,\dots,\tau_{\mathrm{lb}}-1$. The solution to (13) is only guaranteed to be optimal with respect to (3) when the time horizon is $N = \tau(x_0,\{u_t^*\})-1$, where $\{u_t^*\}$ is a solution to (3). In contrast to the MILP formulation, $N \ge \tau(x_0,\{u_t^*\})$ does not guarantee an optimal solution with respect to (3). Furthermore, note that (13) does not require the upper bound M used in (12). On the other hand, if such an M is known, under the additional constraint $\varepsilon_t \le M$ and for $q_t \equiv 1/M$, (13) corresponds to the LP relaxation of (12) by setting $\delta_t = \varepsilon_t/M$, $0 \le \delta_t \le 1$.

III. Iterative Solution Procedure

In Theorem 1 we assume that the time horizon satisfies $N \ge \tau(x_0,\{u_t\})$ for all admissible control sequences $\{u_t\}$. This condition can be further reformulated as $N \ge \tau(x_0,\{u_t^*\})-1$, where $\{u_t^*\}$ is a solution to (3). In fact, it is straightforward to show that solutions to the MILP can only be optimal with respect to (3) if N satisfies this condition. However, the optimal exit-time is a priori unknown and, consequently, it is not possible to choose N such that $N \ge \tau(x_0,\{u_t^*\})-1$ is guaranteed to hold. Moreover, choosing N very large, i.e., $N \gg \tau(x_0,\{u_t^*\})$, is prohibitive because it increases the number of integer variables, which in turn increases (possibly exponentially) the computation time.

A similar problem arises when solving LP (13). While the solution to the LP is not guaranteed to be optimal with respect to (3) for $N \neq \tau(x_0,\{u_t^*\})-1$, the best solutions appear to be obtained when $N = \tau(x_0,\{u_t^*\})-1+\gamma$ for some small $\gamma \in \mathbb{Z}_{\ge 0}$. Therefore, we propose an algorithm that iteratively adapts N while reducing the number of decision variables δ_t or ε_t, respectively. The algorithm for the LP is stated first (Section III-A) because its solution is used to initialize the algorithm for the MILP (Section III-B).

A. LP-Based Iterative Procedure

The LP-based Algorithm 1 is as follows. In Step 1, the lower bound τ_{lb} is initialized using the zero-control solution. Besides, the time horizon N is initialized by adding a constant $\alpha^{\mathrm{LP}} \in \mathbb{Z}_+$ to τ_{lb} at Step 2. Then LP (13) is solved. If the solution does not exit G_t for the current time horizon, the solution is used as a new lower bound (Step 6) and the time horizon N is increased by α^{LP} (Step 2). The procedure is repeated until the solution exits G_t. The number of decision variables for each LP in Algorithm 1 is $N(n+p)+\alpha^{\mathrm{LP}}+1$, where n and p are the dimensions of the state and control input vector, respectively.

Algorithm 1 Obtain suboptimal solution to (3) based on LP

1: $\tau_{\mathrm{lb}} \leftarrow \tau(x_0,\{0,0,\dots,0\})$
2: $N \leftarrow \tau_{\mathrm{lb}} + \alpha^{\mathrm{LP}},\ \alpha^{\mathrm{LP}} \in \mathbb{Z}_+$
3: $\{x_t^{\mathrm{LP}}\},\{u_t^{\mathrm{LP}}\},\{\varepsilon_{\tau_{\mathrm{lb}}}^{\mathrm{LP}},\dots,\varepsilon_N^{\mathrm{LP}}\} \leftarrow$ solution of (13)
4: $\tau \leftarrow \max\{t \le N : \varepsilon_t^{\mathrm{LP}} = 0\}+1$
5: **if** $\varepsilon_N^{\mathrm{LP}} = 0$ **then**
6: $\quad\tau_{\mathrm{lb}} \leftarrow \tau$; go to Step 2
7: **end if**

B. MILP-Based Iterative Procedure

Algorithm 2 Obtain solution to (3) based on MILP

1: $\tau_{\mathrm{lb}} \leftarrow$ output of Algorithm 1
2: $N \leftarrow \tau_{\mathrm{lb}} + \alpha^{\mathrm{MILP}}$
3: $\{x_t^{\mathrm{MILP}}\},\{u_t^{\mathrm{MILP}}\},\{\delta_{\tau_{\mathrm{lb}}}^{\mathrm{MILP}},\dots,\delta_N^{\mathrm{MILP}}\} \leftarrow$ sol. of (12)
4: $\tau \leftarrow \max\{t \le N : \delta_t^{\mathrm{MILP}} = 0\}+1$
5: **if** $\delta_N^{\mathrm{MILP}} = 0$ **then**
6: $\quad\tau_{\mathrm{lb}} \leftarrow \tau$; go to Step 2
7: **end if**

Algorithm 2 outlines the iterative procedure based on MILP, which obtains an optimal solution with respect to (3). The LP-based Algorithm 1 is used to initialize the lower bound τ_{lb} in Step 1. The time horizon N is initialized in Step 2 by adding a constant integer α^{MILP} to τ_{lb} as in Algorithm 1. Then the MILP (12) is solved and the time horizon and lower bound are adapted until the solution exits the set G_t. Note that this procedure can be very effective for solving MILP because the number of binary variables at each iteration is $\alpha^{\mathrm{MILP}}+1$, where α^{MILP} can be specified by the user.

IV. Receding Horizon Implementation

This section describes how the MILP and LP formulations can be used to implement feedback control in order to compensate for unmodeled effects online. For both the LP and MILP formulations, we introduce two different MPC schemes. The first MPC implementation (Algorithm 3) employs Algorithm 1 or 2 to solve the LP or MILP, respectively, see Step 3 of Algorithm 3. For the second MPC implementation (Algorithm 4), we assume there exists an upper bound $\tau_{\rm ub} > 0$ on the optimal exit-time, which is used to initialize the time horizon as stated at Step 2 of Algorithm 4. Then LP (13) or MILP (12), respectively, are solved directly without iteration (Step 5 of Algorithm 4) and the length of the receding time horizon is reduced by one (Step 7). Note that for both MPC schemes, the time-varying constraints and dynamics need to be shifted over the moving time horizon $t_{\rm sys}, ..., t_{\rm sys}+N$ since the LP and MILP formulations consider $t \in \{0, ..., N\}$. This is not shown in Algorithms 3 and 4.

For the numerical examples in this paper (Section V), the closed-loop simulation and the controller are based on the same model, i.e., there are no unmodeled effects. The control performance with unmodeled effects will be investigated in future work. We still use the proposed MPC schemes to investigate if the LP/MILP formulations and Algorithms 1 and 2 are suitable for MPC implementation.

Algorithm 3 MPC scheme based on Algorithm 1 or 2

1: $t_{\rm sys} \leftarrow 0$
2: $x_0 \leftarrow$ current state $x(t_{\rm sys})$
3: $\{u_t^{\rm LP}\} \leftarrow$ output of Algorithm 1
(or $\{u_t^{\rm MILP}\} \leftarrow$ output of Algorithm 2)
4: Apply $u_0^{\rm LP}$ (or $u_0^{\rm MILP}$) as input $u(t_{\rm sys})$ to the system
5: $t_{\rm sys} \leftarrow t_{\rm sys} + 1$; go to Step 2

Algorithm 4 MPC scheme based on (12) or (13)

1: $t_{\rm sys} \leftarrow 0$
2: $N \leftarrow \tau_{\rm ub}$
3: $x_0 \leftarrow$ current state $x(t_{\rm sys})$
4: $\tau_{\rm lb} \leftarrow \tau(x_0, \{0, 0, ..., 0\})$
5: $\{u_t^{\rm LP}\} \leftarrow$ solution of (13) (or $\{u_t^{\rm MILP}\} \leftarrow$ sol. of (12))
6: Apply $u_0^{\rm LP}$ (or $u_0^{\rm MILP}$) as input $u(t_{\rm sys})$ to the system
7: $N \leftarrow N - 1$
8: $t_{\rm sys} \leftarrow t_{\rm sys} + 1$; go to Step 3

V. Numerical Examples

Two numerical examples of a linearized van der Pol oscillator and of spacecraft attitude control with two RWs are considered. In both examples the objective is to maximize the time until specified constraints are violated for the first time according to (3). Based on the developed approaches in Sections II to IV, we investigate the performance of several controllers:

1) **MILP-**(12) / **LP-**(13): direct solution of MILP (12) / LP (13) with $\tau_{\rm lb} = \tau(x_0, \{0, 0, ..., 0\})$ (open-loop).
2) **LP-A1 / MILP-A2**: solution of LP or MILP using Algorithms 1 or 2, respectively (open-loop).
3) **MILP-MPC-A3 / MILP-MPC-A4**: MILP-based MPC implementation, Algorithm 3 or 4 (closed-loop).
4) **LP-MPC-A3 / LP-MPC-A4**: LP-based MPC implementation, Algorithm 3 or 4 (closed-loop).

The computation times reported herein are for a laptop with an i5-6300 processor and 8 GB RAM running MATLAB 2015a. We use the Hybrid Toolbox [17] (*lpsol* and *milpsol* functions with default settings) for solving LPs and MILPs. For each MILP, $M = 100$ and for LPs, we set $q_t \equiv 1$.

A. Van der Pol Oscillator

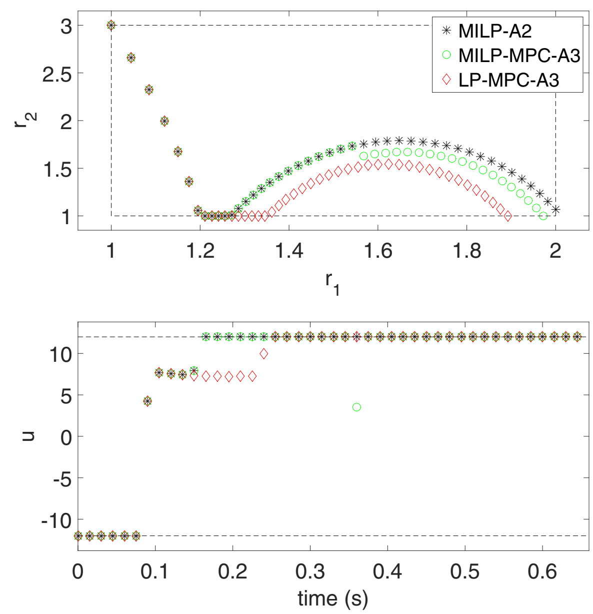

Fig. 1: Van der Pol oscillator example. Top plot: state trajectory in r_1-r_2 plane. Bottom plot: control u vs. time.

We consider $\ddot{r}_{\rm VdP} = (1 - r_{\rm VdP}^2)\dot{r}_{\rm VdP} - r_{\rm VdP} + u$ as the continuous-time nonlinear model. Let $x = (r_1, r_2)^{\rm T}$ be the state vector, where $r_1 = r_{\rm VdP}$ and $r_2 = \dot{r}_{\rm VdP}$. Through linearization about $x_{\rm lin} = (1.5, 2)^{\rm T}$ and using Euler's forward method with a sampling time $\Delta t = 0.015$ s, the discrete-time linear model with added sinusoidal disturbance is obtained as follows

$$\begin{bmatrix} r_{1,t+1} \\ r_{2,t+1} \end{bmatrix} = \begin{bmatrix} 1 & 0.015 \\ -0.105 & 0.9812 \end{bmatrix} \begin{bmatrix} r_{1,t} \\ r_{2,t} \end{bmatrix} + \begin{bmatrix} 0 \\ 0.015 \end{bmatrix} u_t + \begin{bmatrix} 0 \\ 0.05\sin(2\pi t \Delta t) \end{bmatrix}, \quad (14)$$

where the control input is $u_t \in [-12, 12]$. The state constraints for the DCOC problem are given by $G_t \equiv \{x : r_1 \in [1, 2], r_2 \in [1, 3]\}$. An initial $x_0 = (1, 3)^{\rm T}$ is assumed, for which the zero-control exit-time is 15. Table I shows the first exit-time τ and the required computation time for the open-loop controllers. The solution of the MILP-based iterative procedure is always optimal and the optimal exit-time for this example is 44. The LP-based controllers obtain a solution faster than with MILP. However, it is not guaranteed that the optimal exit-time is found. The direct solution of the LP, LP-(13), yields an optimal solution when N is close to the optimal exit-time ($N = 45$ and 55 in Table I). Furthermore, both LP-(13) and MILP-(12) cannot find an optimal solution if $N < 43$. In contrast, the iterative procedures (MILP-A2 and LP-A1) do not rely on guessing N sufficiently large.

Controller	Parameter	τ	Computation time (ms)
MILP-(12)	$N = 45$	44	7
	$N = 55$	44	10
	$N = 75$	44	14
MILP-A2 $\alpha^{\mathrm{LP}} = 25$	$\alpha^{\mathrm{MILP}} = 5$	44	10
	$\alpha^{\mathrm{MILP}} = 10$	44	11
	$\alpha^{\mathrm{MILP}} = 15$	44	11
LP-(13)	$N = 45$	44	3
	$N = 55$	44	4
	$N = 75$	41	7
LP-A1	$\alpha^{\mathrm{LP}} = 5$	44	7
	$\alpha^{\mathrm{LP}} = 15$	44	4
	$\alpha^{\mathrm{LP}} = 25$	43	7

TABLE I: Van der Pol oscillator example, open-loop control sequences with different parameters: First exit-time τ and computation time (worst-case over 100 samples).

The MPC implementations LP-MPC-A3 and MILP-MPC-A3 (with $\alpha^{\mathrm{LP}} = 25$ and $\alpha^{\mathrm{MILP}} = 5$) both obtain the optimal exit-time of 44. The average and worst-case computation times over all time instants are 3 ms and 7 ms, respectively, for LP-MPC-A3, and 4 ms and 10 ms, respectively, for MILP-MPC-A3. For LP-MPC-A4 and MILP-MPC-A4, an upper bound τ_{ub} on the optimal exit-time is obtained with Algorithm 2, where the initial state $x_0 \in G_0$ is added to the MILP as a decision variable, yielding 46 as the largest exit-time for $x_0 = (1, 1)^{\mathrm{T}}$. Then both LP-MPC-A4 and MILP-MPC-A4 achieve an exit-time of 44 with computation times of 1 ms (average) and 3 ms (worst-case) for LP-MPC-A4 and 2 ms (average) and 7 ms (worst-case) for MILP-MPC-A4. While the computation effort is slightly larger for the MILP-based controllers, the worst-case computation times of all feedback controllers are smaller than the sampling time $\Delta t = 15$ ms. Thus, it is feasible to recompute the control input at each time instant in real-time.

Figure 1 shows the state and control trajectories for the open-loop controller MILP-A2 and the feedback controllers MILP-MPC-A3 and LP-MPC-A3. The state and control constraints are indicated by black dashed lines. The trajectories are different but each exits G_t after 44 steps. Thus, the optimal solution is not unique in this example.

B. Spacecraft Attitude Control

The second example is the attitude control of an underactuated spacecraft with body-fixed frame being a principal frame and principal axes denoted by 1, 2, and 3. The spacecraft is equipped with two RWs aligned with the 1- and 3-axis, respectively, where the moment of inertia of each wheel is $J_{\mathrm{w}} = 0.01\,\mathrm{kgm}^2$. The spacecraft principal moments of inertia are given by $J_1 = J_2 = 800\,\mathrm{kgm}^2$ and $J_3 = 300\,\mathrm{kgm}^2$. We assume that the spacecraft orientation is subject to drift caused by a constant external torque (e.g., from solar radiation pressure, where the orientation does not significantly change) with $M_1 = -1.2 \times 10^{-5}\,\mathrm{Nm}$, $M_2 = -10^{-5}\,\mathrm{Nm}$, and $M_3 = 0.9 \times 10^{-5}\,\mathrm{Nm}$.

The state vector is $x = (\phi, \theta, \psi, \omega_1, \omega_2, \omega_3, \omega_{\mathrm{w1}}, \omega_{\mathrm{w3}})^{\mathrm{T}}$, where ϕ, θ, and ψ are the 3-2-1 Euler angles describing the spacecraft orientation, ω_1, ω_2, and ω_3 are the spacecraft angular velocity vector projections on the principal axes, and ω_{w1} and ω_{w3} are the respective RW spin rates. The control input vector is $u = (\alpha_{\mathrm{w1}}, \alpha_{\mathrm{w3}})^{\mathrm{T}}$ comprising the angular accelerations of the two RWs, which are constrained by $\alpha_{\mathrm{w1}}, \alpha_{\mathrm{w3}} \in [-1, 1]\,\mathrm{rad/s}^2$. Note that since the spacecraft is acted on by an external torque, its angular momentum is not conserved and the reduced order equations, obtained by eliminating the angular velocities, cannot be used.

The objective is to maintain x within $G_t \equiv \{x : \phi, \theta \in [44.995, 45.005]\,\mathrm{deg},\ \psi \in [44.95, 45.05]\,\mathrm{deg},\ \omega_{\mathrm{w1}} \in [10, 200]\,\mathrm{rad/s},\ \omega_{\mathrm{w3}} \in [-200, -10]\,\mathrm{rad/s}\}$ for as long as possible. This set is defined by bounds on spacecraft attitude and RW spin rates (RWs must operate below maximum speeds and avoid zero crossing). The constraints on the orientation are very tight which corresponds to precise pointing requirements required for some missions such as Kepler [18].

The discrete-time linear model is derived from the nonlinear continuous-time model [19] by linearizing about $x_{\mathrm{lin}} = (0, 0, 0, 0, 0, 0, 190, -100)^{\mathrm{T}}$ and using Euler's forward method with a sampling time $\Delta t = 2$ s, yielding (for x and u in SI units) the following discrete-time equations

$$
\begin{aligned}
x_{t+1} = &\begin{bmatrix} 1 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0.003 & 0 & 0 & 0 \\ 0 & 0 & 0 & -0.003 & 1 & -0.005 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.013 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} x_t + \\
&\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ -2.5 \times 10^{-5} & 0 \\ 0 & 0 \\ 0 & -6.7 \times 10^{-5} \\ 2 & 0 \\ 0 & 2 \end{bmatrix} u_t + \begin{bmatrix} 0 \\ 0 \\ 0 \\ -3 \times 10^{-8} \\ -2.5 \times 10^{-8} \\ 6 \times 10^{-8} \\ 0 \\ 0 \end{bmatrix}.
\end{aligned} \tag{15}
$$

We improve the numerical conditioning of each LP and MILP by normalizing the state vector according to $\hat{x} = O_{\mathrm{tf}} x + o_{\mathrm{tf}}$, where $O_{\mathrm{tf}} \in \mathbb{R}^{8 \times 8}$ and $o_{\mathrm{tf}} \in \mathbb{R}^8$ are such that G_t is transformed into $\hat{G}_t \equiv \{\hat{x} : \hat{\phi}, \hat{\theta}, \hat{\psi}, \hat{\omega}_{\mathrm{w1}}, \hat{\omega}_{\mathrm{w3}} \in [0, 1]\}$, $\omega_1, \omega_2 = -10^{-4}\,\mathrm{rad/s}$, $\omega_3 = -10^{-2}\,\mathrm{rad/s}$ correspond to $\hat{\omega}_1, \hat{\omega}_2, \hat{\omega}_3 = 0$, and $\omega_1, \omega_2 = 10^{-4}\,\mathrm{rad/s}$, $\omega_3 = 10^{-2}\,\mathrm{rad/s}$ correspond to $\hat{\omega}_1, \hat{\omega}_2, \hat{\omega}_3 = 1$.

The following results are for an initial $x_0 = x_{\mathrm{lin}}$ for which the zero-control exit-time is 54. Table II shows the first exit-time and computation time for different open-loop controllers. The results are similar to the van der Pol oscillator example (Section V-A). The optimal exit-time for this example is 172, see MILP-based controllers in Table II.

As done in the previous example, an upper bound of $\tau_{\mathrm{ub}} = 322$ is obtained for LP-MPC-A4 and MILP-MPC-A4 by adding the initial state $x_0 \in G_0$ to the decision variables of the MILP. However, compared to the optimal exit-time of 172, an upper bound of 322 is conservative and unnecessarily increases the number of variables for every LP and MILP. In fact, the MILP solver (MILP-MPC-A4) is not able to find a solution at $t = 11$, which is mainly due to the large number of binary variables, and the LP-based approach (LP-MPC-A4) violates the constraints after only 74 time steps.

In contrast, the feedback controllers based on Algorithm 3, LP-MPC-A3 and MILP-MPC-A3 (with $\alpha^{\mathrm{LP}} = 50$ and $\alpha^{\mathrm{MILP}} = 5$), do not rely on a tight upper bound on the optimal exit-time. Consequently, they are more robust and both controllers (LP-MPC-A3 and MILP-MPC-A3) achieve the optimal exit-time of 172. The respective worst-case computation times of 1.14 s (LP-MPC-A3) and 1.46 s (MILP-MPC-A3) are smaller than the sampling time ($\Delta t = 2$ s), which allows for real-time computation. While the solution of LP-MPC-A3 may not be optimal in general, computation times (0.33 s on average) are smaller compared to MILP-MPC-A3 (0.45 s on average). However, MILP-MPC-A3 should be used if optimality of the solution must be guaranteed.

Controller	Parameter	τ	Computation time (s)
MILP-(12)	$N = 175$	172	0.83
	$N = 200$	172	2.07
	$N = 225$	172	6.89
MILP-A2 $\alpha^{\mathrm{LP}} = 50$	$\alpha^{\mathrm{MILP}} = 5$	172	1.35
	$\alpha^{\mathrm{MILP}} = 10$	172	1.39
	$\alpha^{\mathrm{MILP}} = 15$	172	1.44
LP-(13)	$N = 175$	172	0.45
	$N = 200$	172	0.65
	$N = 225$	77	0.86
LP-A1	$\alpha^{\mathrm{LP}} = 25$	172	1.2
	$\alpha^{\mathrm{LP}} = 50$	172	1.02
	$\alpha^{\mathrm{LP}} = 75$	172	0.83

TABLE II: Spacecraft attitude control example, open-loop control sequences with different parameters: First exit-time τ and computation time (worst-case over 100 samples).

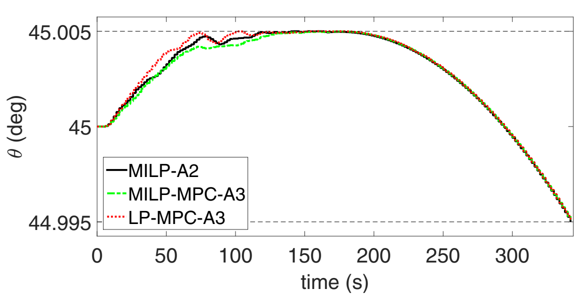

Fig. 2: Spacecraft attitude control example. Uncontrollable Euler angle θ vs. time.

Constraint violation occurs due to the uncontrollable Euler angle θ reaching its prescribed limit. Figure 2 shows θ over time for the open-loop controller MILP-A2 and the closed-loop controllers LP-MPC-A3 and MILP-MPC-A3, where the constraints are indicated by black dashed lines. As for the van der Pol oscillator example, the optimal solution is not unique here.

VI. Conclusion

We presented new approaches based on linear and mixed-integer linear programming and model predictive control (MPC) to solve optimal exit-time control problems with the objective of maximizing the time that prescribed constraints are satisfied. An iterative procedure was developed that efficiently obtains a solution. Two examples of a van der Pol oscillator and of spacecraft attitude control were treated.

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