On the Stability of Quadratic Forms based Model Predictive Control of Constrained PWA Systems

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***Abstract*— In this paper we investigate the stability of discrete-time PWA systems in closed-loop with quadratic cost based Model Predictive Controllers (MPC) and we derive *a priori* sufficient conditions for Lyapunov asymptotic stability. We prove that Lyapunov stability can be achieved for the closed-loop system even though the considered Lyapunov function and the system dynamics may be discontinuous. The stabilization conditions are derived using a terminal cost and constraint set method. An $\mathcal{S}$-procedure technique is employed to reduce conservativeness of the stabilization conditions and a linear matrix inequalities set-up is developed in order to calculate the terminal cost. A new algorithm for computing *piecewise polyhedral* positively invariant sets for PWA systems is also presented. In this manner, the on-line optimization problem associated with MPC leads to a mixed integer quadratic programming problem, which can be solved by standard optimization tools.**

***Index Terms*— Hybrid systems, Lyapunov stability, Piecewise affine systems, Model predictive control.**

I. INTRODUCTION

Hybrid systems provide a unified framework for modeling complex processes that include both continuous and discrete dynamics. Several modeling formalisms have been developed for describing hybrid systems, such as Mixed Logical Dynamical (MLD) systems [1] or Piecewise Affine (PWA) systems [2], and several control strategies have been proposed for relevant classes of hybrid systems. PWA systems in particular have become popular due to their accessible mathematical description on one hand, and their ability to model a broad class of hybrid systems [3] on the other. Many of the control schemes for hybrid systems are based on Model Predictive Control (MPC), e.g., as the ones in [1], [4], [5], [6]. The implementation of MPC for hybrid systems faces two difficult problems: how to reduce the on-line computational complexity and, how to guarantee closed-loop stability. In this paper we focus on the latter problem and we aim at deriving sufficient conditions that guarantee asymptotic stability in the Lyapunov sense [7] for hybrid MPC based on quadratic performance indices. Note that many of the hybrid MPC schemes, such as [1], [4], [5], [6], only guarantee attractivity, although Lyapunov stability is a desirable property from a practical point of view as well. This is due to the fact that if attractivity alone is ensured, then in principle, an arbitrarily small perturbation from the equilibrium may cause the state of the closed-loop system to drift far away by a fixed distance before converging back to the origin.

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For PWA systems in closed-loop with hybrid MPC based on quadratic costs, the stabilization conditions translate into Linear Matrix Inequalities (LMI), as shown in [6], [8]. A terminal cost and constraint set method [9] has been used in [6] to guarantee attractivity for PWA systems in closed-loop with MPC controllers. The terminal weight is calculated using semi-definite programming and the terminal state is constrained to a *polyhedral* positively invariant set. Another option to guarantee attractivity for hybrid MPC based on quadratic costs is to impose a *terminal equality constraint*, as done in [1]. However, this method has the disadvantage that the predicted state must be brought to the origin in finite time. This requires that the PWA system is controllable, while stabilizability should be sufficient in general. Moreover, a longer prediction horizon may be needed for ensuring feasibility of the MPC optimization problem, which increases the computational complexity. Controllers with reduced complexity are proposed for this case in [8], although convergence can only be established by an *a posteriori* analysis.

In this paper we extend the work of [6], [8] based on a terminal cost and constraint set method [9]. We derive *a priori* sufficient conditions for asymptotic stability (including next to attractivity, also Lyapunov stability) of hybrid MPC with costs expressed as quadratic forms. We show that Lyapunov stability can be achieved even though the MPC value function and the system dynamics may be discontinuous. We employ an S-procedure technique [10] to reduce the conservativeness of the stabilization conditions with respect to [6], [8] (the example illustrates the improvements) and we develop an LMI set-up in order to calculate the terminal cost. A new algorithm for calculating piecewise polyhedral positively invariant sets (needed as the terminal set) for PWA systems is also developed. As a consequence, the MPC optimization problem leads to an Mixed Integer Quadratic Programming (MIQP) problem, which is a standard problem in hybrid MPC [1].

II. PRELIMINARIES

Let $\mathbb{R}$, $\mathbb{R}_+$, $\mathbb{Z}$ and $\mathbb{N}$ denote the field of real numbers, the set of non-negative reals, the set of integer numbers and the set of non-negative integers, respectively. Given a set

$\mathcal{S} \subseteq \mathbb{R}^n$, we denote by $\partial\mathcal{S}$ the boundary of $\mathcal{S}$, by $\mathrm{int}(\mathcal{S})$ its interior, and by $\mathrm{cl}(\mathcal{S})$ its closure. Consider the time-invariant discrete-time autonomous nonlinear system described by

$$x_{k+1} = G(x_k), \tag{1}$$

where $G : \mathbb{R}^n \to \mathbb{R}^n$ is an arbitrary nonlinear function.

Definition II.1 Let $0 \leq \lambda \leq 1$ be given. A set $\mathcal{P} \subseteq \mathbb{R}^n$ is called a λ-*contractive set* for system (1) if for all $x \in \mathcal{P}$ it holds that $G(x) \in \lambda\mathcal{P}$. For $\lambda = 1$ a λ-contractive set is called a *positively invariant set*.

A polyhedron is a convex set obtained as the intersection of a finite number of open and/or closed half-spaces. Moreover, a convex and compact set in $\mathbb{R}^n$ that contains the origin in its interior is called a C-set. A *piecewise polyhedral set* is a finite union of polyhedral sets. The 2-norm of a vector $x \in \mathbb{R}^n$ is defined as:

$$\|x\|_2 \triangleq \sqrt{|x_1|^2 + \ldots + |x_n|^2},$$

where x_i, $i = 1, \ldots, n$ is the i-th component of x. For a positive definite matrix Z, $\lambda_{\min}(Z)$ and $\lambda_{\max}(Z)$ denote the smallest and the largest eigenvalue of Z, respectively.

III. PROBLEM STATEMENT

Consider the time-invariant discrete-time PWA system described by equations of the form [2]

$$x_{k+1} = A_j x_k + B_j u_k + f_j \quad \text{when} \quad x_k \in \Omega_j, \tag{2}$$

where $x_k \in \mathbb{X} \subseteq \mathbb{R}^n$ is the state and $u_k \in \mathbb{U} \subseteq \mathbb{R}^m$ is the control input at the discrete-time instant $k \geq 0$. $A_j \in \mathbb{R}^{n\times n}$, $B_j \in \mathbb{R}^{n\times m}$, $f_j \in \mathbb{R}^n$, $j \in \mathcal{S}$ with $\mathcal{S} := \{1, 2, \ldots, s\}$ a *finite set* of indices and s denoting the number of discrete modes. Here, $f_j \in \mathbb{R}^n$ denotes a fixed offset vector for all $j \in \mathcal{S}$. The sets $\mathbb{X}$ and $\mathbb{U}$ are assumed to be polyhedral C-sets. The collection $\{\Omega_j \mid j \in \mathcal{S}\}$ defines a partition of $\mathbb{X}$, meaning that $\cup_{j\in\mathcal{S}}\Omega_j = \mathbb{X}$ and $\Omega_i \cap \Omega_j = \emptyset$ for $i \neq j$. Each Ω_j is assumed to be a polyhedron (not necessarily closed). Let $\mathcal{S}_0 := \{j \in \mathcal{S} \mid 0 \in \mathrm{cl}(\Omega_j)\}$ and let $\mathcal{S}_1 := \{j \in \mathcal{S} \mid 0 \notin \mathrm{cl}(\Omega_j)\}$, so that $\mathcal{S} = \mathcal{S}_0 \cup \mathcal{S}_1$. We assume that the origin is an equilibrium state for (2) with $u = 0$ and we require that

$$f_j = 0 \text{ for all } j \in \mathcal{S}_0. \tag{3}$$

The class of hybrid systems described by (2)-(3) contains PWA systems which *may be discontinuous over the boundaries* and which are Piecewise Linear (PWL), instead of PWA, in the state space region $\cup_{j\in\mathcal{S}_0}\Omega_j$. For a fixed $N \in \mathbb{N}$, $N \geq 1$, let $\mathbf{x}_k(x_k, \mathbf{u}_k) := (x_{k+1}, \ldots, x_{k+N})$ denote a state sequence generated by system (2) from initial state x_k and by applying the input sequence $\mathbf{u}_k := (u_k, \ldots, u_{k+N-1}) \in \mathbb{U}^N$. Furthermore, let $\mathcal{X}_T \subseteq \mathbb{X}$ denote a desired target set that contains the origin.

Definition III.1 The class of *admissible input sequences* defined with respect to $\mathcal{X}_T$ and state $x_k \in \mathbb{X}$ is $\mathcal{U}_N(x_k) := \{\mathbf{u}_k \in \mathbb{U}^N \mid \mathbf{x}_k(x_k, \mathbf{u}_k) \in \mathbb{X}^N,\ x_{k+N} \in \mathcal{X}_T\}$.

Consider the following constrained optimization problem.

Problem III.2 At time $k \geq 0$ let $x_k \in \mathbb{X}$, the target set $\mathcal{X}_T \subseteq \mathbb{X}$ and $N \geq 1$ be given. Minimize the cost function

$$J(x_k, \mathbf{u}_k) \triangleq x_{k+N}^\top P(x_{k+N}) x_{k+N} + \sum_{i=0}^{N-1} x_{k+i}^\top Q x_{k+i} + u_{k+i}^\top R u_{k+i} \tag{4}$$

over all input sequences $\mathbf{u}_k \in \mathcal{U}_N(x_k)$, where $P(x_{k+N}) = P_j$ when $x_{k+N} \in \mathcal{X}_T \cap \Omega_j$ and (x_{k+i}, u_{k+i}) satisfy (2) for $i = 0, \ldots, N-1$.

Here, N denotes the prediction horizon and P_j, Q and R are assumed to be positive definite matrices. We call an initial state $x_k \in \mathbb{X}$ *feasible* if $\mathcal{U}_N(x_k) \neq \emptyset$. Similarly, Problem III.2 is said to be feasible (or *solvable*) for $x_k \in \mathbb{X}$ if $\mathcal{U}_N(x_k) \neq \emptyset$. Let $\mathcal{X}_f(N)$ denote the set of *feasible* initial states x_k with respect to Problem III.2 and let

$$V_{\text{MPC}} : \mathcal{X}_f(N) \to \mathbb{R}_+, \quad V_{\text{MPC}}(x_k) \triangleq \inf_{\mathbf{u}_k \in \mathcal{U}_N(x_k)} J(x_k, \mathbf{u}_k) \tag{5}$$

denote the value function corresponding to (4). Throughout the paper we assume that there exists an optimal sequence of controls calculated by solving Problem III.2 for state $x_k \in \mathcal{X}_f(N)$, i.e. $\mathbf{u}_k^* := (u_k^*, u_{k+1}^*, \ldots, u_{k+N-1}^*)$. Hence, the infimum in (5) is a minimum and $V_{\text{MPC}}(x_k) = J(x_k, \mathbf{u}_k^*)$. The following stability analysis is not affected by the possible non-uniqueness of the optimal control sequence, i.e. all results apply irrespective of which optimal sequence is selected. Let $\mathbf{x}_k^*(x_k, \mathbf{u}_k^*) := (x_{k+1}^*, \ldots, x_{k+N}^*)$ denote the state sequence generated by system (2) from initial state $x_k \in \mathcal{X}_f(N)$ and by applying the optimal sequence of controls $\mathbf{u}_k^*$. Let $\mathbf{u}_k^*(1)$ denote the first element of $\mathbf{u}_k^*$. According to the receding horizon strategy, the *MPC control law* is defined as

$$u_k^{\text{MPC}} = \mathbf{u}_k^*(1); \quad k \in \mathbb{N}. \tag{6}$$

A precise problem formulation can now be stated as follows.

Problem III.3 Let a desired set of initial states $\mathbb{X}_0 \subseteq \mathbb{X}$, system (2) and the matrices Q, R be given. Determine the terminal weights P_j, the terminal constraint set $\mathcal{X}_T$ and the prediction horizon N such that system (2) in closed-loop with the MPC control (6) is asymptotically stable in the Lyapunov sense and $\mathbb{X}_0 \subseteq \mathcal{X}_f(N)$.

Moreover, it is desirable that a solution to the above problem should be such that Problem III.2 leads to an MIQP problem, which can be solved by standard optimization tools [11].

Note that many of the hybrid MPC schemes only guarantee attractivity, e.g., see [1], [5], and not Lyapunov stability [7], which is an important property in practice. This is due to the fact that if attractivity alone is ensured, then in principle, an arbitrarily small perturbation from the equilibrium may cause the state of the closed-loop system to drift far away by a fixed distance before converging back to the origin.

IV. MAIN RESULTS

In this section we derive a priori sufficient conditions that guarantee asymptotic stability in the Lyapunov sense for the closed-loop hybrid system (2)-(6). Consider an auxiliary local PWL control law of the form

$$\tilde{u}_k \triangleq K_j x_k, \quad x_k \in \Omega_j, k \in \mathbb{N}, K_j \in \mathbb{R}^{m\times n}, j \in \mathcal{S}_0. \quad (7)$$

Let $\mathcal{X}_{\mathbb{U}} := \cup_{j\in\mathcal{S}_0}\{x \in \Omega_j \mid K_j x \in \mathbb{U}\}$ denote the safe set with respect to *state and input* constraints for this controller. Let $\mathcal{X}_T \subseteq \mathcal{X}_{\mathbb{U}} \subseteq \cup_{j\in\mathcal{S}_0}\Omega_j$ denote the terminal constraint set from Problem III.2. Let $\mathcal{Q}_{ji} := \{x \in \Omega_j \mid \exists u \in \mathbb{U} : A_j x + B_j u + f_j \in \Omega_i\}$, $(j,i) \in \mathcal{S}_0 \times \mathcal{S}_0$ and let $\mathcal{S}_{t0} := \{(j,i) \in \mathcal{S}_0 \times \mathcal{S}_0 \mid \mathcal{Q}_{ji} \neq \emptyset\}$. The set of pairs of indices $\mathcal{S}_{t0}$ can be easily determined off-line by solving s_0^2 linear programs, where s_0 is the number of elements of $\mathcal{S}_0$. Let $\mathbf{x}_k^*(x_k, \mathbf{u}_k^*) := (x_{k+1}^*, \ldots, x_{k+N}^*)$ denote the state sequence generated by system (2) from initial state $x_k \in \mathcal{X}_f(N)$ and by applying the optimal sequence of controls $\mathbf{u}_k^*$.

Theorem IV.1 [12] *Consider system* (2) *and suppose* $\mathcal{X}_T \subseteq \mathcal{X}_{\mathbb{U}}$ *is a closed positively invariant set for the closed-loop system* (2)-(7) *that contains the origin in its interior. Assume that there exists* $N \geq 1$ *such that* $\mathbb{X}_0 \subseteq \mathcal{X}_f(N)$ *and that*

$$\tilde{x}_{k+1}^\top P_i \tilde{x}_{k+1} - x_k^\top P_j x_k + x_k^\top Q x_k + \tilde{u}_k^\top R \tilde{u}_k \leq 0 \quad (8)$$

for all $x_k \in \mathcal{X}_T \cap \Omega_j$, $(j,i) \in \mathcal{S}_{t0}$, *where*

$$\begin{cases} \tilde{x}_{k+1} \triangleq A_j x_k + B_j \tilde{u}_k + f_j \\ \tilde{u}_k = K_j x_k \end{cases} \quad \text{when} \quad x_k \in \mathcal{X}_T \cap \Omega_j. \quad (9)$$

Then, the origin of the PWA system (2) *in closed-loop with the MPC control* (6) *is asymptotically stable in the Lyapunov sense in* $\mathcal{X}_f(N)$, *while satisfying the state and input constraints.*

Proof: Let $\Delta V_{\text{MPC}}(x_k) := V_{\text{MPC}}(x_{k+1}) - V_{\text{MPC}}(x_k)$. Consider the shifted sequence of controls $\mathbf{u}_{k+1} := (u_{k+1}^*, \ldots, u_{k+N-1}^*, \tilde{u}_{k+N})$. By optimality, we observe that for all $x_k \in \mathcal{X}_f(N)$

$$\begin{aligned} \Delta V_{\text{MPC}}(x_k) &\leq J(x_{k+1}, \mathbf{u}_{k+1}) - J(x_k, \mathbf{u}_k^*) = \\ &= -x_k^\top Q x_k - u_k^{*\top} R u_k^* + \tilde{x}_{k+N+1}^\top P_i \tilde{x}_{k+N+1} - \\ &- x_{k+N}^{*\top} P_j x_{k+N}^* + x_{k+N}^{*\top} Q x_{k+N}^* + \tilde{u}_{k+N}^\top R \tilde{u}_{k+N}. \quad (10) \end{aligned}$$

Since $x_{k+N}^* \in \mathcal{X}_T$, from the hypothesis (8) it follows that

$$\Delta V_{\text{MPC}}(x_k) \leq -x_k^\top Q x_k \leq \lambda_{\min}(Q)\|x_k\|_2^2. \quad (11)$$

Then, it follows that V_{MPC} has a negative definite forward difference [7]. From (4) it follows that

$$V_{\text{MPC}}(x_k) \geq x_k^\top Q x_k \geq \lambda_{\min}(Q)\|x_k\|_2^2, \quad (12)$$

for all $x_k \in \mathcal{X}_f(N)$. Hence, V_{MPC} is a *positive definite* and *radially unbounded* function [7].

Let $\tilde{\mathbf{x}}_k(x_k, \tilde{\mathbf{u}}_k) := (\tilde{x}_{k+1}, \ldots, \tilde{x}_{k+N})$ denote the state sequence generated by system (9) from initial state $x_k \in \mathcal{X}_T$. Since $\tilde{\mathbf{x}}_k \in \mathcal{X}_T^N$, (8) holds for all elements of the sequence $\tilde{\mathbf{x}}_k$ and by optimality it follows that (e.g., see [12] for details):

$$V_{\text{MPC}}(x_k) \leq \max_{j\in\mathcal{S}_0} x_k^\top P_j x_k \leq \max_{j\in\mathcal{S}_0} \lambda_{\max}(P_j)\|x_k\|_2^2, \quad (13)$$

for all $x_k \in \mathcal{X}_T$. Hence, V_{MPC} is a *decrescent* function [7] on $\mathcal{X}_T$ (note that $\mathcal{X}_T$ contains the origin in its interior).

In [9] it is proven that if V_{MPC} satisfies the conditions (11)-(12)-(13), and if V_{MPC} *is continuous*, then asymptotic stability in the Lyapunov sense is guaranteed. We prove in [12] that the conditions (11)-(12)-(13) are sufficient for asymptotic stability in the Lyapunov sense, even though the V_{MPC} *is discontinuous*. The reader is referred to [12] for details, due to space limitations.

Hence, from (11)-(12)-(13) and [12] it follows that the PWA system (2) in closed-loop with the MPC control (6) is asymptotically stable in the Lyapunov sense in $\mathcal{X}_f(N)$. ■

Remark IV.2 The results of [9] regarding stability of MPC rely on the fact that V_{MPC} is continuous (e.g., see Section 3.2 of [9]). Theorem IV.1 shows that Lyapunov stability can be achieved in quadratic forms based hybrid MPC, even though V_{MPC} may be discontinuous (with the exception of $x = 0$). Note that V_{MPC} is always continuous in $x = 0$, since by (13) we have that $\lim_{x\to\infty} V_{\text{MPC}} = V_{\text{MPC}}(0) = 0$.

A. Computation of the terminal weights and control gains

Under the assumption that the closed-loop system (9) admits a common quadratic or a Piecewise Quadratic (PWQ) Lyapunov function, a solution to inequality (8) can be found via semi-definite programming, as it has been shown in [6] (see also [8] for an alternative LMI set-up). In the sequel we employ an S-procedure technique with respect to inequality (8) in order to reduce the conservativeness of the stabilization conditions (as done in [10]), i.e. we consider the matrix inequality

$$\begin{aligned} &P_j - (A_j + B_j K_j)^\top P_i (A_j + B_j K_j) - Q \\ &\quad - K_j^\top R K_j - E_{ji}^\top U_{ji} E_{ji} > 0 \text{ for all } (j,i) \in \mathcal{S}_{t0} \quad (14) \end{aligned}$$

in the unknowns (P_j, K_j, U_{ji}), where the matrices P_j are the terminal weights employed in cost (4), the matrices U_{ji} have all entries non-negative and the matrices E_{ji} define the cones $\mathcal{C}_{ji}$, which are such that $\mathcal{C}_{ji} := \{x \in \mathbb{R}^n \mid E_{ji} x \geq 0\}$ and $\mathcal{Q}_{ji} \subseteq \mathcal{C}_{ji}$ for all $(j,i) \in \mathcal{S}_{t0}$. Note that if (P_j, K_j, U_{ji}) with $P_j > 0$ and U_{ji} with all entries non-negative for all $(j,i) \in \mathcal{S}_{t0}$ satisfy (14), then it follows that

$$\begin{aligned} &x^\top (P_j - (A_j + B_j K_j)^\top P_i (A_j + B_j K_j) - Q \\ &\quad - K_j^\top R K_j) x \geq x^\top (E_{ji}^\top U_{ji} E_{ji}) x \geq 0 \quad (15) \end{aligned}$$

whenever $x \in \mathcal{Q}_{ji} \subseteq \mathcal{C}_{ji}$, $(j,i) \in \mathcal{S}_{t0}$. Hence, (8) is satisfied and conservativeness is reduced when comparing to the corresponding nonlinear matrix inequality, i.e.

$$P_j - (A_j + B_j K_j)^\top P_i (A_j + B_j K_j) - Q - K_j^\top R K_j > 0.$$

Next, we develop a method for finding a solution to the matrix inequality (14). This method is based on solving a sequence of LMIs that is obtained by fixing a suitable basis of the state space and successively selecting tuning parameters. Consider an eigenvalue decomposition of the terminal weight matrices from cost (4), i.e. $P_j = V_j \Sigma_j V_j^\top$, $j \in \mathcal{S}_0$ where $\Sigma_j = \mathrm{diag}(\sigma_{1j}, \ldots, \sigma_{nj})$, $\sigma_{1j} \geq \ldots \geq \sigma_{nj}$ and $V_j^\top = V_j^{-1}$. In the sequel we assume that the orthonormal matrices V_j are known and let $\Gamma_j := \mathrm{diag}(\gamma_{1j}, \ldots, \gamma_{nj})$, $j \in \mathcal{S}_0$ denote an arbitrary diagonal matrix. Consider now the following LMI:

$$\Delta_{ji} > 0, \quad (j,i) \in \mathcal{S}_{t0}, \tag{16}$$

with

$$\Delta_{ji} := \begin{pmatrix} V_j \Sigma_j V_j^\top - Q - E_{ji}^\top U_{ji} E_{ji} & * & * \\ V_i^\top (A_j + B_j K_j) & \Gamma_i & 0 \\ K_j & 0 & R^{-1} \end{pmatrix},$$

in the unknowns $(\sigma_{1j}, \ldots, \sigma_{nj}), (\gamma_{1i}, \ldots, \gamma_{ni}), K_j, U_{ji}$, $(j,i) \in \mathcal{S}_{t0}$. In addition to (16) we require that the linear scalar inequalities

$$\sigma_{1j} \geq \ldots \geq \sigma_{nj} > 0, \quad \gamma_{nj} \geq \ldots \geq \gamma_{1j} > 0, \tag{17a}$$

$$\frac{1}{\epsilon_{lj}} - \sigma_{lj} \geq 0, \quad \epsilon_{lj} - \gamma_{lj} \geq 0, \quad l = 1, \ldots, n, \tag{17b}$$

with ϵ_{lj} fixed constants (tuning factors) in $(0,1]$, are satisfied for all $j \in \mathcal{S}_0$ and that

$$U_{ji} \quad \text{has all entries non-negative,} \quad \forall (j,i) \in \mathcal{S}_{t0}. \tag{18}$$

Note that the tuning factors $\epsilon_{lj} \in (0,1]$ are fixed in (17) and that condition (18) can be easily written as an LMI. Hence, the conditions (16)-(17)-(18) are in the LMI form.

Theorem IV.3 *Choose the orthonormal matrices V_j and the tuning factors $\epsilon_{lj} \in (0,1]$, $l = 1, \ldots, n$, $j \in \mathcal{S}_0$ such that the LMI (16)-(17)-(18) is feasible. Let $(\sigma_{1j}, \ldots, \sigma_{nj})$, $(\gamma_{1i}, \ldots, \gamma_{ni})$, K_j, U_{ji}, $(j,i) \in \mathcal{S}_{t0}$ be a solution. Then (P_j, K_j, U_{ji}) with $P_j = V_j \,\mathrm{diag}(\sigma_{1j}, \ldots, \sigma_{nj}) V_j^\top > 0$ is a solution of the matrix inequality (14).*

The proof of Theorem IV.3 is given in the Appendix. Solving the LMI (16)-(17)-(18) hinges on the fact that the orthonormal matrices V_j and the scaling factors ϵ_{lj} must be chosen a priori. This is not a problem with respect to the tuning factors, which can be chosen arbitrarily small. However, when it comes to fixing the matrices V_j, it is interesting to find out how they should be chosen such that by varying $\sigma_{1j}, \ldots, \sigma_{nj}$ a sufficiently wide range of P_j matrices is covered. An answer to this question can be obtained for the two dimensional case, where all orthonormal matrices can be parameterized according to

$$V_j := \begin{pmatrix} -\sin\theta_j & \cos\theta_j \\ \cos\theta_j & \sin\theta_j \end{pmatrix}, \tag{19}$$

where $0 \leq \theta_j \leq \pi$. In this way, multiple solutions of the LMI (16)-(17)-(18) can be obtained by varying θ_j. A similar explicit form of V_j can be specified also in the three dimensional case, by using two angles, i.e., θ_{1j} and θ_{2j}. However, these expressions get more complicated in higher dimensional spaces.

B. Computation of the terminal constraint set

In the sequel we develop a method for calculating a terminal constraint set $\mathcal{X}_T \subseteq \mathcal{X}_{\mathbb{U}}$ that satisfies the hypothesis of Theorem IV.1 and solves Problem III.3.

Consider system (9) with the feedback gains calculated as in Section IV-A. From the hypothesis of Theorem IV.1 it follows that

$$x^\top (A_j + B_j K_j)^\top P_i (A_j + B_j K_j) x - \\ - x^\top P_j x \leq -\lambda_{\min}(Q) \|x\|_2^2 < 0 \tag{20}$$

for all $x \in \mathcal{X}_T \setminus \{0\}$, $(j,i) \in \mathcal{S}_{t0}$. Then, it can be proven along the lines of the proof of Theorem IV.1 that the possibly discontinuous function $V(x) := x^\top P_j x$ when $x \in \Omega_j$, $j \in \mathcal{S}_0$ is a local PWQ Lyapunov function for the closed-loop system (9). Let

$$\mathcal{E} := \cup_{j \in \mathcal{S}_0} \mathcal{E}_j \quad \text{with} \quad \mathcal{E}_j := \{x \in \mathcal{X}_{\mathbb{U}} \cap \Omega_j \mid V(x) \leq c\},$$

where $c > 0$, $j \in \mathcal{S}_0$, be a (piecewise ellipsoidal) sublevel set of V. From (20) it follows that there exists $\alpha \in (0,1)$ such that the set $\mathcal{E}$ is α-contractive.

Theorem IV.4 *Consider system (9) and assume that it admits a PWQ Lyapunov function $V(x) = x^\top P_j x$ when $x \in \Omega_j$, $j \in \mathcal{S}_0$. Let $\mathcal{E} \subseteq \mathcal{X}_{\mathbb{U}}$ be a sublevel set of V and let $\alpha \in (0,1)$ be such that $\mathcal{E}$ is α-contractive. Now assume that there exist polyhedral sets $\mathcal{P}_j$ that satisfy $\alpha\mathcal{E}_j \subseteq \mathcal{P}_j \subseteq \mathcal{E}_j$ for all $j \in \mathcal{S}_0$. Then the piecewise polyhedral set $\mathcal{P} := \cup_{j \in \mathcal{S}_0} \mathcal{P}_j$ is a positively invariant set for system (9) and $\mathcal{P} \subseteq \mathcal{X}_{\mathbb{U}}$.*

Proof: From $\alpha\mathcal{E}_j \subseteq \mathcal{P}_j \subseteq \mathcal{E}_j$ for all $j \in \mathcal{S}_0$ we have that $\alpha\mathcal{E} \subseteq \mathcal{P} \subseteq \mathcal{E}$. Thus, $\mathcal{P} \subseteq \mathcal{X}_{\mathbb{U}}$. Let $x \in \mathcal{P}$. Hence, there exists $j \in \mathcal{S}_0$ such that $x \in \mathcal{P}_j \subseteq \Omega_j$. Take $\gamma_j > 1$ such that $\gamma_j x \in \partial\mathcal{E}_j$. Then, it follows that $A_j^{cl}(\gamma_j x) \in \alpha\mathcal{E}$. Then, because of positive homogeneity of PWL dynamics, it follows that $A_j^{cl} x \in \frac{\alpha}{\gamma_j}\mathcal{E} \subseteq \alpha\mathcal{E}$. Since $\alpha\mathcal{E} \subseteq \mathcal{P}$, $\mathcal{P}$ is a positively invariant set for system (9). ■

The approach of Theorem IV.4 amounts to solving the problem of fitting a polyhedron in between two closed ellipsoidal sets where one is contained in the interior of the other. A possible way to solve this problem has been recently developed in [13] in the context of DC programming (difference of convex functions). Here, a polyhedral set is constructed by treating the ellipsoidal sets as sublevel sets of suitable quadratic functions, and by exploiting upper and lower piecewise affine bounds on such functions. Giving additional structure to the algorithm of [13] such that it generates a polyhedron with a finite number of facets for each region Ω_j, a piecewise polyhedral positively invariant set is obtained for system (9). This set can be used as the terminal constraint set in Problem III.2.

Note that this method yields a terminal set which is a union of at most s_0 polyhedral sets. Another option to obtain the terminal constraint set is to employ the algorithm developed in [14]. This algorithm computes the maximal positively invariant set for a PWA system, but this set might be a union of more than s_0 sets. If this is the case, then one has to introduce additional Boolean variables in order to formulate Problem III.2 as an MIQP problem.

C. How to determine the prediction horizon

In the case of the quadratic forms cost (4), Problem III.2 with the terminal constraint set calculated as in Theorem IV.4 leads to an MIQP problem. The minimum value of N needed to ensure feasibility of this problem for a desired set of initial conditions $\mathbb{X}_0 \subseteq \mathbb{X}$ (i.e. the minimum N for which $\mathbb{X}_0 \subseteq \mathcal{X}_f(N)$) can be calculated using the *expcon* function of the Hybrid Toolbox [11]. The function *expcon* computes the explicit MPC control law and returns the feasible state-space region $\mathcal{X}_f(N)$. Thus, one can check if $\mathbb{X}_0 \subseteq \mathcal{X}_f(N)$ for a fixed N.

The computational complexity of the on-line MPC optimization problem increases exponentially with both the length of the prediction horizon and the number of Boolean variables. Hence, one has to make a trade-off in choosing between a smaller terminal set, but which has a simple representation (e.g., a piecewise polyhedral set obtained as in Theorem IV.4 or a polyhedral set obtained as in [6]), and a larger terminal set, but possibly with a complex representation (e.g., as the set obtained in [14]).

V. EXAMPLE

Consider the following open-loop unstable system:

$$x_{k+1} = \begin{cases} A_1 x_k + B u_k & \text{if} \quad E_1 x_k > 0 \\ A_2 x_k + B u_k & \text{if} \quad E_2 x_k \geq 0 \\ A_3 x_k + B u_k & \text{if} \quad E_3 x_k > 0 \\ A_4 x_k + B u_k & \text{if} \quad E_4 x_k \geq 0 \end{cases} \tag{21}$$

subject to the constraints $x_k \in \mathbb{X} = [-10, 10] \times [-10, 10]$, $u_k \in \mathbb{U} = [-1, 1]$, where

$$A_1 = \begin{bmatrix} 0.5 & 0.61 \\ 0.9 & 1.345 \end{bmatrix}, A_2 = \begin{bmatrix} -0.92 & 0.644 \\ 0.758 & -0.71 \end{bmatrix}, B = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

$A_3 = A_1$ and $A_4 = A_2$. The state-space partition of the system is given by

$$E_1 = -E_3 = \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix}, E_2 = -E_4 = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}.$$

The tuning parameters of the MPC algorithm are $Q = 10^{-4} I_2$ and $R = 10^{-3}$. For system (21) the LMIs of [6], [8] turn out to be infeasible. With the S-procedure approach of subsection IV-A we have obtained the following solution by solving the LMI (16)-(17)-(18) for the tuning factors $\epsilon_{11} = 0.04, \epsilon_{21} = 0.3, \epsilon_{12} = 0.08, \epsilon_{22} = 1$ and for the orthonormal matrices V_1, V_2 defined as in (19) for $\theta_1 = 2.4$ and $\theta_2 = 0.9$:

$$\begin{aligned} &P_1 = \begin{bmatrix} 12.9707 & 10.9974 \\ 10.9974 & 14.9026 \end{bmatrix}, \ P_2 = \begin{bmatrix} 7.9915 & -5.5898 \\ -5.5898 & 5.3833 \end{bmatrix}, \\ &P_3 = P_1, \quad P_4 = P_2, \\ &K_1 = \begin{bmatrix} -0.7757 & -1.0299 \end{bmatrix}, \ K_2 = \begin{bmatrix} 0.6788 & -0.4302 \end{bmatrix}, \\ &K_3 = K_1, \quad K_4 = K_2, \\ &U_{11} = \begin{bmatrix} 0.4596 & 1.9626 \\ 1.9626 & 0.0198 \end{bmatrix}, \ U_{12} = \begin{bmatrix} 0.4545 & 2.0034 \\ 2.0034 & 0.0250 \end{bmatrix}, \\ &U_{21} = \begin{bmatrix} 0.0542 & 0.0841 \\ 0.0841 & 0.0506 \end{bmatrix}, \ U_{22} = \begin{bmatrix} 0.0599 & 0.0914 \\ 0.0914 & 0.0565 \end{bmatrix}, \\ &\sigma_{11} = 24.9765, \quad \sigma_{21} = 2.8969, \quad \sigma_{12} = 12.4273, \\ &\sigma_{22} = 0.9475, \quad \gamma_{11} = 0.0395, \quad \gamma_{21} = 0.2954, \\ &\gamma_{12} = 0.0791, \quad \gamma_{22} = 0.9675. \end{aligned} \tag{22}$$

A *piecewise polyhedral* positively invariant set has been computed for system (21) in closed-loop with (7) (with the feedbacks given in (22)) using the approach of Theorem IV.4 and the algorithm of [13] for the sublevel set $\mathcal{E}$ with $c = 14$, which satisfies $\mathcal{E} \subseteq \mathcal{X}_{\mathbb{U}}$. In this case $\mathcal{E}$ is α contractive for $\alpha = 0.9286$. The trajectories of the closed-loop system (21)-(7) (with K_j given in (22)) with the vertices of $\mathcal{X}_T$ as initial conditions are plotted in Figure 1 together with a plot of the safe set $\mathcal{X}_{\mathbb{U}}$. The simulation results illustrate the positive invariance of the terminal constraint set.

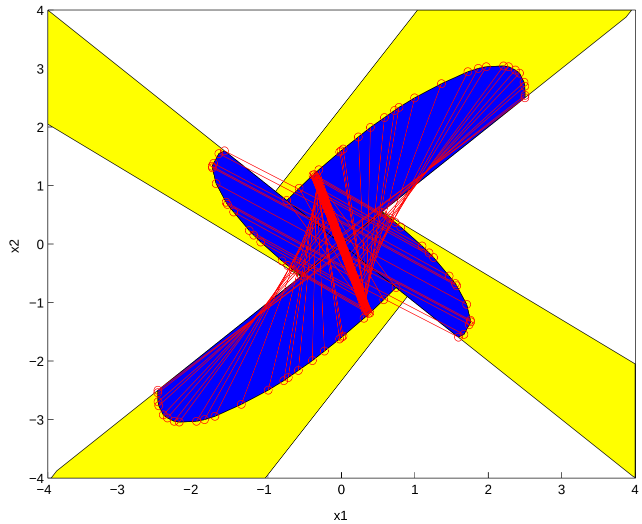

Fig. 1. State-feedback: State trajectories - red; $\mathcal{X}_T$ - blue polyhedra; $\mathcal{X}_{\mathbb{U}}$ - yellow and blue polyhedra.

The state trajectory of system (21) with initial state $x_0 = [-5 \ -3.8]^\top$ and in closed-loop with the MPC control (6) calculated for $N = 4$ (obtained using the Hybrid Toolbox [11] as in subsection IV-C) is plotted in Figure 2. The MPC controller successfully stabilizes the open-loop unstable system (21) while satisfying the constraints.

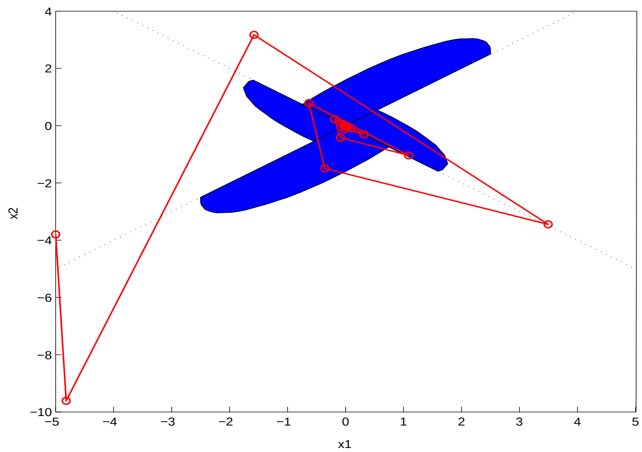

Fig. 2. MPC: State trajectory - red; $\mathcal{X}_T$ - blue polyhedra.

VI. CONCLUSIONS

In this paper we have derived sufficient *a priori* conditions for Lyapunov asymptotic stability of hybrid MPC based on quadratic costs. The stabilization conditions have been obtained using a terminal cost and constraint set method. We have shown that Laypunov stability can be achieved even if the considered Lyapunov function and the system dynamics are discontinuous. An S-procedure technique has been employed in order to reduce conservativeness with respect to earlier work [6], [8] and an LMI set-up has been developed for calculating the terminal cost. A new procedure for computing positively invariant sets for PWA systems has also been presented. As such, the MPC optimization problem leads to an MIQP problem, which can be solved by standard optimization tools.

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APPENDIX

A. Proof of Theorem IV.3

Since $(\sigma_{1j},\dots,\sigma_{nj}),(\gamma_{1i},\dots,\gamma_{ni}),K_j,U_{ji},\,(j,i)\in\mathcal{S}_{t0}$ satisfy the LMI (16)-(17)-(18) we can apply the Schur complement to (16), which yields

$$V_j\Sigma_jV_j^\top-(A_j+B_jK_j)^\top V_i\Gamma_i^{-1}V_i^\top(A_j+B_jK_j)\\ -Q-K_j^\top RK_j-E_{ji}^\top U_{ji}E_{ji}>0.$$

By adding and subtracting $(A_j+B_jK_j)^\top V_i\Sigma_iV_i^\top(A_j+B_jK_j)$ in the above inequality we obtain the equivalent

$$\begin{aligned}&V_j\Sigma_jV_j^\top-(A_j+B_jK_j)^\top V_i\Sigma_iV_i^\top(A_j+B_jK_j)-\\&-Q-K_j^\top RK_j-E_{ji}^\top U_{ji}E_{ji}>\\&>(A_j+B_jK_j)^\top V_i\Gamma_i^{-1}V_i^\top(A_j+B_jK_j)-\\&-(A_j+B_jK_j)^\top V_i\Sigma_iV_i^\top(A_j+B_jK_j).\end{aligned}\tag{23}$$

From (17b) we have that $1-\sigma_{lj}\gamma_{lj}\geq 0$ for all $l=1,\dots,n$ and all $j\in\mathcal{S}_0$. Then, the inequality

$$\Gamma_i^{-1}-\Sigma_i=\begin{pmatrix}\frac{1-\gamma_{1i}\sigma_{1i}}{\gamma_{1i}} & \dots & 0\\ \vdots & \ddots & \vdots\\ 0 & \dots & \frac{1-\gamma_{ni}\sigma_{ni}}{\gamma_{ni}}\end{pmatrix}\geq 0$$

holds for all $i\in\mathcal{S}_0$ and from (23) it follows that the inequality

$$V_j\Sigma_jV_j^\top-(A_j+B_jK_j)^\top V_i\Sigma_iV_i^\top(A_j+B_jK_j)\\ -Q-K_j^\top RK_j-E_{ji}^\top U_{ji}E_{ji}>0$$

is satisfied for all $(j,i)\in\mathcal{S}_{t0}$. The matrix inequality (14) is obtained by letting $P_j=V_j\Sigma_jV_j^\top>0$ for all $j\in\mathcal{S}_0$ in the above inequality.

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