An Efficient Algorithm for Computing the State Feedback Optimal Control Law for Discrete Time Hybrid Systems

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Abstract

In this paper we propose an efficient algorithm for computing the solution to the finite time optimal control problem for discrete time linear hybrid systems with a quadratic performance criterion. The algorithm is based on a dynamic programming recursion and a multiparametric quadratic programming solver.

1 Introduction

Different methods for the analysis and design of controllers for hybrid systems have emerged over the last few years [16, 9, 13, 4]. Among them, the class of optimal controllers is one of the most studied. Most of the literature deals with optimal control of continuous-time hybrid systems and is focused on the study of necessary conditions for a trajectory to be optimal [17, 15], and on the computation of optimal or sub-optimal solutions by means of Dynamic Programming or the Maximum Principle [10, 8]. Although some techniques for determining feedback control laws seem to be very promising, many suffer from the "curse of dimensionality" arising from the *discretization* of the state space necessary in order to solve the corresponding Hamilton-Jacobi-Bellman or Euler-Lagrange differential equations.

In this paper we study how to compute the solution to optimal control problems for linear discrete time hybrid systems. Interesting mathematical phenomena occurring in hybrid systems such as Zeno behaviors [12] do not exist in discrete time. The advantage of the discrete time formulation is, however, that one can *characterize* and *compute* the optimal control law *without gridding* the state space. In [1] we proposed a procedure for synthesizing piecewise affine optimal controllers for discrete time linear hybrid systems. It is based on multiparametric programming and determines the state-feedback solution to finite-time optimal control problems with performance criteria based on linear (1 or ∞) norms. Sometimes the use of linear norms has practical disadvantages. A satisfactory performance may be only achieved with long time-horizons, with a consequent increase of complexity. Also, generally, the closed-loop performance does not depend smoothly on the weights used in the performance index, i.e., slight changes of the weights can lead to very different closed-loop trajectories, so that the tuning of the controller becomes difficult.

In his plenary presentation [14] at the European Control Conference David Mayne presented an intuitively appealing characterization of the state-feedback solution to optimal control problems for linear hybrid systems with performance criteria based on quadratic and linear norms. The detailed exposition [2] by the authors follows a similar line of argumentation and shows that the state-feedback solution to the finite time optimal control problem is a time-varying piecewise affine feedback control law, possibly defined over non convex regions. The proposed procedures are constructive, but based on the enumeration of all the possible switching sequences of the hybrid system, the number of which grows exponentially with the time horizon. Although the computation is performed off-line (the on-line complexity is the one associated with the evaluation of the piecewise (PWA) control law), more efficient methods than enumeration are desirable. In this paper we present an algorithm to efficiently compute the state-feedback optimal control law. The algorithm is based on a dynamic programming recursion and a multiparametric quadratic solver [5].

The infinite horizon optimal controller can be approximated by implementing in a receding horizon fashion a finite-time optimal control law. The implementation, as a consequence of the results presented here on finite-time optimal control, requires only the evaluation of a piecewise affine function. This opens up the route to use receding horizon techniques to control hybrid systems characterized by fast sampling and relatively small size.

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2 Definitions

We give the following definitions:

Definition 1 (Polyhedron) *A set $\Theta \subseteq \mathbb{R}^s$ presented in the form $\Theta = \{x \mid Hx \leq k\}$ for some $H \in \mathbb{R}^{m\times s}$, $k \in \mathbb{R}^m$ is called polyhedron.*

Definition 2 *A collection of sets $R_1, \ldots, R_N$ is a* partition *of a set Θ if* (i) $\bigcup_{i=1}^N R_i = \Theta$, (ii) $R_i \cap R_j = \emptyset$, $\forall i \neq j$. *Moreover $R_1, \ldots, R_N$ is a* polyhedral partition *of a polyhedral set Θ if $R_1, \ldots, R_N$ is a partition of Θ and the $\bar{R}_i$'s are polyhedral sets, where $\bar{R}_i$ denotes the closure of the set R_i.*

Definition 3 *A function $h : \Theta \to \mathbb{R}^k$, where $\Theta \subseteq \mathbb{R}^s$, is* piecewise affine (PWA) *if there exists a partition $\mathcal{R}_1,\ldots,\mathcal{R}_N$ of Θ and $h(\theta) = H^i\theta + k^i$, $\forall \theta \in \mathcal{R}_i$, $i = 1, \ldots, N$.*

Definition 4 *A function $h : \Theta \to \mathbb{R}^k$, where $\Theta \subseteq \mathbb{R}^s$, is* PWA on polyhedrons (PPWA) *if there exists a polyhedral partition $\mathcal{R}_1,\ldots,\mathcal{R}_N$ of Θ and $h(\theta) = H^i\theta + k^i$, $\forall \theta \in R_i$, $i = 1, \ldots, N$.*

Piecewise quadratic functions (PWQ) and piecewise quadratic functions on polyhedra (PPWQ) are defined analogously.

Definition 5 *A function $q : \Theta \to \mathbb{R}$, where $\Theta \subseteq \mathbb{R}^s$, is a* multiple quadratic function *of multiplicity $d \in \mathbb{N}^+$ if $q(\theta) = \min\{q^1(\theta) \triangleq \theta' Q^1\theta + l^1\theta + c^1, \ldots, q^d(\theta) \triangleq \theta' Q^d\theta + l^d\theta + c^d\}$ and Θ is a convex polyhedron.*

Definition 6 *A function $q : \Theta \to \mathbb{R}$, where $\Theta \subseteq \mathbb{R}^s$, is a* multiple PWQ on polyhedrons *(multiple PPWQ) if there exists a polyhedral partition $\mathcal{R}_1,\ldots,\mathcal{R}_N$ of Θ and $q(\theta) = \min\{q_i^1\theta \triangleq \theta' Q_i^1\theta + l_i^1\theta + c_i^1, \ldots, q_i^{d_i} \triangleq \theta' Q_i^{d_i}\theta + l_i^{d_i}\theta + c_i^{d_i}\}$, $\forall \theta \in \mathcal{R}_i$, $i = 1, \ldots, N$. We define d_i to be the multiplicity of the function q in the polyhedron $\mathcal{R}_i$ and $d = \sum_{i=1}^N d_i$ to be the multiplicity of the function q. (Note that Θ in not necessary convex.)*

3 Hybrid Systems

Several modeling frameworks have been introduced for discrete time hybrid systems. Among them, *piecewise affine* (PWA) systems [16] are defined by partitioning the state space into polyhedral regions, and associating with each region a different linear state-update equation

$$\begin{aligned} x(t+1) &= A_i x(t) + B_i u(t) + f_i \\ &\quad \text{if } \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} \in \mathcal{P}_i \end{aligned} \tag{1}$$

where $x \in \mathbb{R}^{n_c} \times \{0,1\}^{n_\ell}$, $u \in \mathbb{R}^{m_c} \times \{0,1\}^{m_\ell}$, $\{\mathcal{P}_i\}_{i=1}^s$ is a polyhedral partition of the sets of state+input space $\mathbb{R}^{n+m}$, $n \triangleq n_c + n_\ell$, $m \triangleq m_c + m_\ell$.

PWA systems can model a large number of physical processes, such as systems with static nonlinearities, and can approximate nonlinear dynamics via multiple linearizations at different operating points.

Furthermore, we mention here linear complementarity (LC) systems and extended linear complementarity (ELC) systems, max-min-plus-scaling (MMPS) systems, and mixed logical dynamical (MLD) systems, which are equivalent in their discrete time version [11, 3]. Thus, the theoretical properties and tools can be easily transferred from one class to another. In particular, the optimal control synthesis technique developed in this paper for PWA systems can be immediately adopted for any of the former classes of hybrid systems.

4 Finite-Time Constrained Optimal Control: Problem Formulation

Consider the PWA system (1) subject to input and state constraints

$$E_c x(t) + L_c u(t) \leq M_c \tag{2}$$

for $t \geq 0$, and denote by constrained PWA system (CPWA) the restriction of the PWA system (1) over the set of states and inputs defined by (2),

$$\begin{aligned} x(t+1) &= A_i x(t) + B_i u(t) + f_i \\ &\quad \text{if } \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} \in \tilde{\mathcal{P}}_i \end{aligned} \tag{3}$$

where $\{\tilde{\mathcal{P}}_i\}_{i=1}^s$ is the new polyhedral partition of the sets of state+input space $\mathbb{R}^{n+m}$ obtained by intersecting the polyhedrons $\mathcal{P}_i$ in (1) with the polyhedron described by (2). Let $\tilde{\mathcal{P}} \triangleq \cup_{i=1}^s \{\tilde{\mathcal{P}}_i\}$. In the following we will substitute the CPWA system equations (3) with the shorter form

$$x(k+1) = \tilde{f}_{PWA}(x(k), u(k)) \tag{4}$$

where $\tilde{f}_{PWA} : \tilde{\mathcal{P}} \mapsto \mathbb{R}^n$ and $\tilde{f}_{PWA}(x, u) = A_i x + B_i u + f_i$ if $\left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right] \in \tilde{\mathcal{P}}_i$, $i = 1, \ldots, s$.

Define the following cost function

$$J(U_0^{T-1}, x(0)) \triangleq \|Px(T)\|_2^2 + \sum_{k=0}^{T-1} \|Qx(k)\|_2^2 + \|Ru(k)\|_2^2 \tag{5}$$

and consider the finite-time constrained optimal control problem (FTCOC)

$$J^*(x(0)) \triangleq \min_{U_0^{T-1}} J(U_0^{T-1}, x(0)) \tag{6}$$

$$\text{s.t.} \quad \begin{cases} x(t+1) = \tilde{f}_{PWA}(x(t), u(t)) \\ x(T) \in \mathcal{X}^f \end{cases} \tag{7}$$

where the column vector $U_0^{T-1} \triangleq [u'(0), \dots, u(T-1)']' \in \mathbb{R}^{mT}$, is the optimization vector, T is the time horizon and $\mathcal{X}^f$ is the terminal region. In (5), $\|Qx\|_2^2 = x'Qx$ and $R = R' \succ 0,\ Q = Q', P = P' \succeq 0$. We denote by $\mathcal{X}^0 \subseteq \mathbb{R}^n$ the set of initial states $x(0)$ for which the optimal control problem (5)-(7) is feasible. Similarly $\mathcal{X}^k$ denotes the set of feasible states $x(k),\ k = 1, \dots, N$ at time k for the optimal control problem (5)-(7). See [7] for more details on the difference between minimization formulation (5) and mathematically correct infimum formulation.

In the following we recall the main property enjoyed by the solution of problem (5)-(7).

Theorem 1 *The solution to the optimal control problem (5)-(7) is a PWA state feedback control law of the form*

$$u^*(x(k)) = F_i^k x(k) + G_i^k \quad \textit{if } x(k) \in \mathcal{R}_i^k \tag{8}$$

where $\mathcal{R}_i^k,\ i = 1, \dots, N_i$ is a partition of the set $\mathcal{X}^k$ of feasible states $x(k)$ and the closure $\bar{\mathcal{R}}_i^k$ of the sets $\mathcal{R}_k^i$ has the following form:

$$\bar{\mathcal{R}}_i^k \triangleq \left\{x \mid x(k)'L(j)_i^k x(k) + M(j)_i^k x(k) \le N(j)_i^k\right\} \tag{9}$$

where $j = 1, \dots, n_i^k,\ k = 0, \dots, T-1$.

Proof: The piecewise linearity of the solution was first mentioned by Sontag in [16]. In [14] Mayne sketched a proof. More details can be found in [6, 2]. □

In general the optimizer $u^*(x(k))$ and the value function $J^*(x(k))$ are discontinuous, $\mathcal{X}^k$ may be non convex, disconnected and partitioned into convex and non convex sets $\mathcal{R}_i^k,\ i = 1, \dots, N_i$.

Despite the fact that the proof of Theorem 1 is constructive, it is based on the enumeration of all the possible switching sequences of the hybrid system, the number of which grows exponentially with the time horizon. Although the computation is performed off line (the on-line complexity is the one associated with the evaluation of the PWA control law (8)), more efficient methods than enumeration are desirable. In [6] it was also shown that under more restrictive assumptions on the objective function $J(U_0^{T-1}, x(0))$ and on the PWA system (3) the optimal control law (8) can assume a PPWA form. We will not impose such restrictions on optimal control problem (5)-(7) here. In the next section we propose an algorithm that efficiently compute the optimal control law (8).

5 Preliminaries and Basic Steps

Denote with $f_{PPWA}(x)$ and $f_{PPWQ}(x)$ a generic PPWA and PPWQ function of x, respectively.

Multiparametric Quadratic Programming: The following quadratic program

$$\begin{array}{rll} V(x) = \frac{1}{2}x'Yx + & \min\limits_u & \frac{1}{2}u'Hu + x'FU \\ & \text{subj. to} & Gu \le W + Ex \end{array} \tag{10}$$

can be solved for all x by using an Multiparametric Quadratic Programming solver (mp-QP) described in [5]. The solution to the parametric program (10) is a PPWA law $u^*(x) = f_{PPWA}(x)$ and the value function is PPWQ, $V(x) = f_{PPWQ}(x)$.

Procedure Intersect and Compare: Consider the PWA map $\zeta(x)$

$$\zeta:\ x \mapsto F_i x + G_i \text{ if } x \in \mathcal{R}_i\ i = 1, \dots, N_{\mathcal{R}} \tag{11}$$

where $\mathcal{R}_i,\ i = 1, \dots, N_{\mathcal{R}}$ are sets of the $x-$space. If there exist $l, m \in \{1, \dots, N_{\mathcal{R}}\}$ such that for $x \in \mathcal{R}_l \cap \mathcal{R}_m$, $F_l x + G_l \neq F_m x + G_m$ the map $\zeta(x)$ (11) is not single valued.

Definition 7 *Given a PWA map (11) we define the function $f_{PWA}(x) = \zeta_o(x)$ as the* ordered region *single-valued function associated with (11) when $\zeta_o(x) = F_j x + G_j,\ j \in \{1, \dots, N_{\mathcal{R}}\} | x \in \mathcal{R}_j$ and $\forall i < j\ x \notin \mathcal{R}_i$.*

Note that given a PWA map (11) the corresponding *ordered region* single-valued function changes if the order used to store the regions $\mathcal{R}_i$ and the corresponding affine gains changes.

In the following we assume that the sets $\mathcal{R}_i^k$ in the optimal solution (8) can overlap. When we refer to the PWA function $u^*(x(k))$ in (8) we will implicitly mean the *ordered region single-valued function* associated with the mapping (8).

Theorem 2 *Let $J_1^* : \mathcal{R}_1 \to \mathbb{R}$ and $J_2^* : \mathcal{R}_2 \to \mathbb{R}$ be two quadratic functions, $J_1^*(x) \triangleq x'L_1x + M_1x + N_1$ and $J_2^*(x) \triangleq x'L_2x + M_2x + N_2$, where $\mathcal{R}_1$ and $\mathcal{R}_2$ are convex polyhedron and $J_i^*(x) = +\infty$ if $x \notin \mathcal{R}_i$. Consider the nontrivial case $\mathcal{R}_1 \cap \mathcal{R}_2 \triangleq \mathcal{R}_3 \neq \varnothing$ and the expressions*

$$J^*(x) = \min\{J_1^*(x), J_2^*(x)\} \tag{12}$$

$$u^*(x) = \begin{cases} u_1^*(x) & \textit{if } J_1^*(x) \le J_2^*(x) \\ u_2^*(x) & \textit{if } J_1^*(x) > J_2^*(x) \end{cases} \tag{13}$$

Define $\lambda(x) \triangleq x'(L_2 - L_1)x + (M_2 - M_1)x + (N_2 - N_1)$. Then, corresponding to the three following cases

1. $J_1^*(x) \leq J_2^*(x)\ \forall x \in \mathcal{R}_3$
2. $J_1^*(x) \geq J_2^*(x)\ \forall x \in \mathcal{R}_3$
3. $\exists x_1, x_2 \in \mathcal{R}_3\ | J_1^*(x_1) < J_2^*(x_1)\ \&\ J_1^*(x_2) > J_2^*(x_2)$

the expressions (12) and (13) can be written equivalently as:

1.

$$J^*(x) = \begin{cases} J_1^*(x) & \text{if } x \in \mathcal{R}_1 \\ J_2^*(x) & \text{if } x \in \mathcal{R}_2 \end{cases} \tag{14}$$

$$u^*(x) = \begin{cases} u_1^*(x) & \text{if } x \in \mathcal{R}_1 \\ u_2^*(x) & \text{if } x \in \mathcal{R}_2 \end{cases} \tag{15}$$

2. *as in (14) and (15) by switching the indices* 1 *and* 2
3.

$$J^*(x) = \begin{cases} min\{J_1^*(x), J_2^*(x)\} & \text{if } x \in \mathcal{R}_3 \\ J_1^*(x) & \text{if } x \in \mathcal{R}_1 \\ J_2^*(x) & \text{if } x \in \mathcal{R}_2 \end{cases} \tag{16}$$

$$u^*(x) = \begin{cases} u_1^*(x) & \text{if } x \in \mathcal{R}_3\ \&\ \lambda(x) \geq 0 \\ u_2^*(x) & \text{if } x \in \mathcal{R}_3\ \&\ \lambda(x) \leq 0 \\ u_1^*(x) & \text{if } x \in \mathcal{R}_1 \\ u_2^*(x) & \text{if } x \in \mathcal{R}_2 \end{cases} \tag{17}$$

where (14), (15), (16), and (17) have to be considered as PWA and PPWQ functions in the ordered region *sense.*

Proof: Straightforward. □

The results of Theorem 2 allow one

- to avoid the storage of the intersections of two polyhedra in case 1 and 2
- to avoid the storage of possibly non convex regions $\mathcal{R}_3 \setminus \mathcal{R}_1$ and $\mathcal{R}_3 \setminus \mathcal{R}_2$
- to work with multiple quadratic functions instead of quadratic functions defined over non-convex and non-polyhedral regions.

The three point listed above will be the three basic ingredients for storing and simplifying the optimal control law (8). Next we will show how to compute it.

Remark 1 To distinguish between cases 1, 2 and 3 of Theorem 2 one needs to solve an indefinite quadratic program. In our approach if one fails to distinguish between the three cases (e.g. if one solves a relaxed problem instead of the indefinite quadratic program) then the form (17) corresponding to the third case, will be used. The only drawback is that the form (17) could be a non-minimal representation of the value function and could therefore increase the computational complexity of on-line the algorithm for computing the optimal control action (8).

Basic Parametric Programming: Consider the multiparametric program

$$J^*(x)) \triangleq \min_u \quad l(x,u) + q(f(x,u)) \tag{18}$$

$$\text{s.t.} \quad f(x,u) \in \mathcal{R} \tag{19}$$

where $\mathcal{R} \subseteq \mathbb{R}^n$, $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$, $q : \mathcal{R} \to \mathbb{R}$, and $l : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ is a quadratic function of x and u. We aim at determining the region $\mathcal{X}$ of variables x such that the program (18)–(19) is feasible and the optimum $J^*(x)$ is finite, and at finding the expression of the optimizer $u^*(x)$.

We establish following results (for proofs see [7])

1. *one to one problem*: $f(x,u)$ is linear in x and u, $q(x)$ is quadratic in x, and $\mathcal{R}$ is a convex polyhedron. Problem is solved with one mp-QP.
2. *one to one problem of multiplicity d*: $f(x,u)$ linear in x and u, $q(x)$ is a multiple quadratic function of x of multiplicity d. Problem is solved with d mp-QP.
3. *one to r problem*: $f(x,u)$ is linear in x and u, $q(x)$ is a PPWQ function of x defined over r polyhedral regions. Problem is solved with r mp-QP's
4. *one to r problem of multiplicity d*: $f(x,u)$ is linear in x and u and $q(x)$ is a multiple PPWQ function of x of multiplicity d, defined over r polyhedral regions. Problem is solved with rd mp-QP's.

If the function f is PPWA defined over s regions then we have a *s to X problem* where X can belong to any of the combination listed above, i.e., we have a *s to r problem of multiplicity d* if $f(x,u)$ is PPWA in x and u defined over s regions and $q(x)$ is a multiple PPWQ function of x of multiplicity d, defined over r polyhedral regions. Such a problem can be decomposed into *s one to r problem of multiplicity d*, and consequently it may be solved with srd mp-QP's.

6 Efficient Dynamic Program for the Computation of the Solution

The PWA solution (8) to the FTCOC (5)-(7) can be computed efficiently in the following way.

Consider the dynamic programming solution to the FTCOC (5)-(7)

$$J_j^*(x_j) \triangleq \min_{u_j} \quad \|Qx_j\|_2^2 + \|Ru_j\|_2^2 + J_{j+1}^*(x_{j+1})$$

$$\text{s.t.} \quad x_{j+1} \triangleq \tilde{f}_{PWA}(x_j, u_j) \in \mathcal{X}^{j+1} \tag{20}$$

for $j = T-1, \ldots, 0$, with boundary conditions

$$\mathcal{X}^T = \mathcal{X}^f \tag{21}$$

$$J_T^*(x) = \|Px\|_2^2 \tag{22}$$

where $\mathcal{X}^j$ is the set of all initial states for which problem (20) is feasible:

$$\mathcal{X}^j = \{x \in \mathbb{R}^n |\ \exists u,\ \tilde{f}_{PWA}(x,u) \in \mathcal{X}^{j+1}\} \qquad (23)$$

The dynamic program (22) can be solved backwards in time by using a multiparametric quadratic programming solver and the results of the previous Section. In the following we will assume assume that the constraints in (20) define a closed set. The procedure can be immediately extended in case of discontinuous PWA systems [6].

Consider the first step of the dynamic program (22)

$$J^*_{T-1}(x_{T-1}) \triangleq \min_{u_{T-1}} \|Qx_{T-1}\|_2^2 + \|Ru_{T-1}\|_2^2 + J^*_T(x_T)$$
$$\text{s.t. } x_T \triangleq \tilde{f}_{PWA}(x_{T-1}, u_{T-1}) \in \mathcal{X}^f \qquad (24)$$

The cost to go function $J^*_T(x)$ in (24) is quadratic, the terminal region $\mathcal{X}^f$ is a polyhedron and the constraints are piecewise affine. Problem (24) is a *s to one problem* that can be solved by solving s mp-QP's.

From the second step $j = T-2$ to the last one $j = 0$ the cost to go function $J^*_{j+1}(x)$ is a PPWQ with a certain multiplicity d_{j+1}, the terminal region $\mathcal{X}^{j+1}$ is a polyhedron (not necessary convex) and the constraints are piecewise affine. Therefore, problem (22) is a *s to N^r_{j+1} problem with multiplicity d_{j+1}* (where N^r_{j+1} is the number of polyhedra of the cost to go function J^*_{j+1}), that can be solved by solving $sN^r_{j+1}d_{j+1}$ mp-QP's. The resulting optimal solution will have the form (8) considered in the ordered region sense.

7 Examples

We reconsider here the problem of controlling a piecewise linear system to the origin reported in [4, Examples 5.1 and 6.1]. The problem was solved by expressing the PWL dynamics in MLD form and by using on-line mixed-integer quadratic optimization. We show here below that *exactly* the same behavior can be obtained by synthesizing an optimal control law according to the multiparametric technique developed in the previous sections.

Finite Time Optimal Control: Consider the problem of steering to a small region around the origin in three steps the piecewise affine system

$$\begin{aligned}
x(t+1) &= 0.8\begin{bmatrix}\cos\alpha(t) & -\sin\alpha(t)\\ \sin\alpha(t) & \cos\alpha(t)\end{bmatrix}x(t) + \begin{bmatrix}0\\1\end{bmatrix}u(t)\\
y(t) &= [1\ 0]x(t)\\
\alpha(t) &= \begin{cases}\frac{\pi}{3} & \text{if } [1\ 0]x(t) \geq 0\\ -\frac{\pi}{3} & \text{if } [1\ 0]x(t) < 0\end{cases}\\
x(t) &\in [-10, 10] \times [-10, 10]\\
u(t) &\in [-1, 1]
\end{aligned} \qquad (25)$$

while minimizing the cost function (5). The finite-time constrained optimal control problem (5)-(7) is solved with $N = 3$, $Q = [\begin{smallmatrix}1&0\\0&1\end{smallmatrix}]$, $R = 1$, $P = [\begin{smallmatrix}1&0\\0&1\end{smallmatrix}]$, and $\mathcal{X}^f = [-0.01\ 0.01] \times [-0.01\ 0.01]$. The solution was computed in less than 1 minute by using Matlab 5.3 on a Pentium II-500 MH. The polyhedral regions corresponding to the state feedback solution $u^*(x(k))$, $k = 0, \ldots, 2$ in (8) are depicted in Fig. 1. The resulting optimal trajectories for the initial state $x(0) = [-1\ 1]'$ are shown in Fig. 2.

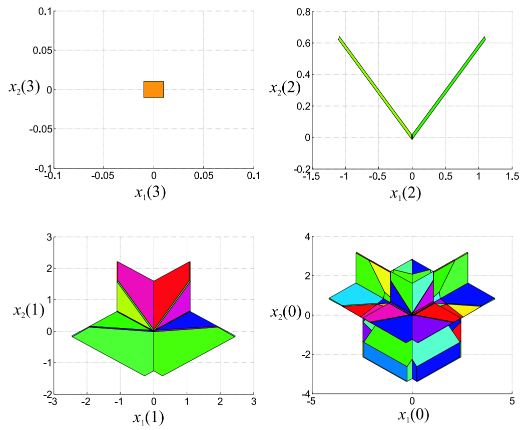

Figure 1: State space partition corresponding to the state-feedback finite time optimal control law $u^*(x(k))$ of system (25).

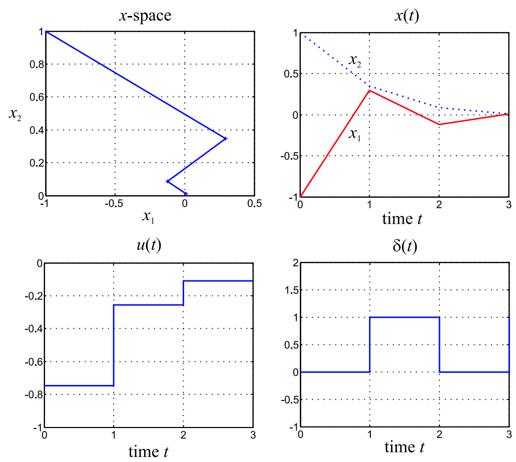

Figure 2: Finite time optimal control of system (25).

As explained in Section 5 the optimal control law is stored in a special data structure where:

1. The ordering of the regions is important.
2. The polyhedra can overlap.
3. The polyhedra can have an associated value function of multiplicity d larger than one. Thus, d quadratic functions have to be compared on-line in order to compute the optimal control action.

Receding Horizon Control: Consider the problem of regulating to the origin the piecewise affine system (25). The finite-time constrained optimal control (5)-(7) is solved with $N = 3$, $Q = [\begin{smallmatrix}1 & 0\\ 0 & 1\end{smallmatrix}]$, $R = 1$, $P = [\begin{smallmatrix}1 & 0\\ 0 & 1\end{smallmatrix}]$, $\mathcal{X}^f = \mathbb{R}^2$ and its state feedback solution (8) at time 0 $u^*(x(0)) = f_0^*(x(0))$ is implemented in a receding horizon fashion, i.e. $u(x(k)) = f_0^*(x(k))$. The state feedback control law consists of 48 polyhedral regions, and none of them has multiplicity higher than 2 (note that the enumeration of all possible switching sequences could lead to a multiplicity of 2^3 in all regions). In Fig. 3 we show the corresponding closed loop trajectories starting from the initial state $x(0) = [-1\ 1]'$.

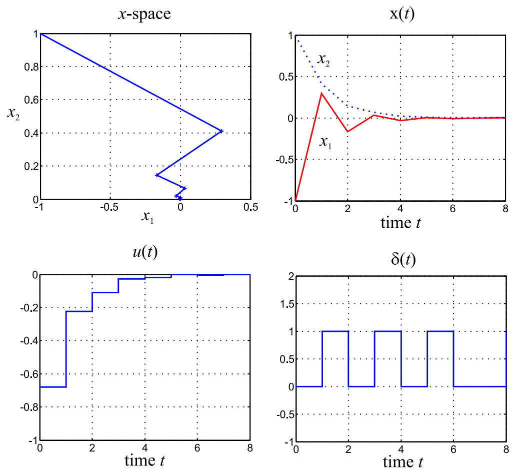

Figure 3: Receding horizon control of system (25)

8 Conclusions

For discrete-time linear hybrid systems, we have described an off-line procedure to synthesize optimal control laws based on the minimization of a quadratic performance index subject to linear constraints on inputs and states. The procedure is based on a combination of dynamic programming and multiparametric quadratic programming. Compared to the approach of [4], where the control law is implicitly defined as the result of a mixed-integer quadratic program which depends on the state vector, we have explicitly characterized the piecewise affine structure of the control law. This opens the use of hybrid quadratic optimal control to those applications where on-line optimization cannot be afforded because of limitations on the available CPU power and/or complexity of the control code.

Acknowledgements

This work was partially supported by the Swiss National Science Foundation and the European Project "Computation and Control".

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