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# $\mathcal{L}_2$ Anti-Windup Via Receding Horizon Optimal Control

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## Abstract

The nonlinear $\mathcal{L}_2$ anti-windup framework introduced by Teel and Kapoor (1997) reduces the anti-windup synthesis problem to a state feedback synthesis problem for linear systems with input saturation and input matched $\mathcal{L}_2$ disturbances. In this paper, such a state feedback is synthesized using receding horizon optimal control techniques, and its equivalent piecewise affine closed-form is computed using the techniques of Bemporad et al. (2002). The properties of the resulting anti-windup compensation scheme are analyzed in the paper, and its performance is investigated through a simulation example.

## 1 Introduction

Actuator saturation is an ubiquitous nonlinearity in control systems. One of the control problems that arises when actuator saturation is present is the anti-windup synthesis problem. In this problem, a linear compensator is prespecified and the overall control system must:
1) preserve the given linear closed-loop behavior when the signals in the control loop are small enough not to activate actuator saturation, and
2) avoid undesired (and unpredictable) behavior in the presence of large signals, and guarantee that the linear performance degrades gracefully as signals in the control loop become large.

It was not until the end of the 1980's that the anti-windup problem started to receive significant attention from the scientific research community (see, *e.g.*, [8, 10] for surveys of early anti-windup techniques). In recent years, a broad number of anti-windup designs with formal stability (and, to a certain extent, performance) guarantees have been proposed, thus lifting the anti-windup setting from a merely experimental and industrial discipline into a theoretical research topic associated with precise mathematical formulations and formal stability/performance requirements (see, *e.g.*, [1, 5–7, 12, 13, 17, 19, 22]).

Based on the two requirements above, anti-windup can be seen as an "augmentation problem," where a linear control system (including a linear plant and a linear nominal controller) constitutes the fixed parameters of the anti-windup design, and the anti-windup compensator (or anti-windup augmentation) action is the free parameter to be designed in such a way as to meet those two requirements.

The nonlinear $\mathcal{L}_2$ anti-windup framework introduced in [19] reduced the anti-windup synthesis problem to a state feedback synthesis problem for linear systems with input saturation and input matched $\mathcal{L}_2$ disturbances. Optimal synthesis of these feedbacks is an open problem, but many suboptimal approaches were proposed recently (see, e.g., [11, 21]). In this paper we adopt the explicit Receding Horizon optimal Control (RHC) synthesis techniques of [2] within the $\mathcal{L}_2$ anti-windup framework.

Receding horizon optimal control, often referred to as "model predictive control" (MPC), is a widely used technique in the process industries for designing controllers that handle complex constrained multivariable problems [14, 15]. Here, at each sampling instant, an open-loop optimal control problem is solved over a finite horizon, starting at the current state. At the next sampling instant, the computation is repeated starting from the new state and over a shifted horizon, leading to a receding horizon policy. The solution relies on a linear dynamic model, respects all input and output constraints, and optimizes a quadratic performance index. The main historical drawback of MPC is its relatively formidable on-line computational effort, which has limited its applicability to relatively slow processes. Recently, [2] showed that the MPC control law is a piecewise affine function of the state, and that it can be computed off-line by employing techniques of multiparametric quadratic programming [2, 20], so that the on-line complexity of MPC (RHC) reduces to the evaluation of such a piecewise affine map.

Within the $\mathcal{L}_2$ anti-windup framework of this paper, we will show that RHC provides a piecewise affine state-feedback law which is geared toward minimizing a quadratic cost over a finite time interval. Despite the fact that RHC is not, in general, $\mathcal{L}_2$ optimal because of the receding horizon mechanism, it provides high performance anti-windup compensation, for instance when the windup problem is induced by large and abrupt changes of the reference to the control system, as we will show through an example in Section 4.

One may argue that using a combination of both RHC and anti-windup techniques for control of saturated systems is a mere redundancy, and that RHC alone would be enough. There are at least two main reasons against this argument. One is a very practical one: very often control engineers are not willing to replace an existing controller that has a successful history of usage (even if for small signals) with a new RHC controller, that requires a complete tuning and testing. A second argument is that even if RHC is put on top of the existing controller, most likely it will completely change the closed-loop response, as it does not attempt at recovering the nominal (unconstrained) performance after saturation occurs, and it even modifies the poles of the unconstrained closed-loop system, whose frequency response (and, therefore, noise rejection) properties may be very different from the nominal ones. On the other hand, unlike RHC, the attempt of a pure anti-windup scheme at recovering the unconstrained behavior is usually not driven by an optimality criterion, thus the combination of the two techniques provides an appealing solution to the anti-windup problem.

RHC strategies have been previously employed successfully in the context of anti-windup schemes in the Ref-

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erence Governor (RG) proposed in [1, 6], a device that smooths out the reference signal to a given nominal closed-loop system whenever this is needed to enforce the fulfillment of the constraints. Among other things, one important difference between RG and the scheme proposed in this paper is that RG only recovers the nominal trajectory after a finite time, typically when the system has reached its steady-state, while here we attempt at recovering it also during the transient. Moreover, RG only employed modifications to the nominal scheme acting at the reference input, while our approach admits modifications acting within the feedback loop, thus possessing extra degrees of freedom for stabilization and performance improvement. Finally, arbitrarily large external disturbances are allowed in our approach, while they are not in RG schemes.

The main technical challenges in proving that the control laws from [2] provide a suboptimal solution to the nonlinear $\mathcal{L}_2$ anti-windup problem revolve around combining the discrete-time RHC algorithm of [2] with the continuous-time controller/plant interconnection that is prone to windup in the presence of input saturation. When using a discrete-time anti-windup algorithm, the dynamics that characterize the mismatch between the nominal closed-loop behavior and the saturated closed-loop behavior is a sampled-data nonlinear system whose $\mathcal{L}_2$ stability needs to be established. Results on $\mathcal{L}_2$ (in fact, $\mathcal{L}_p$) stability for *linear* sampled-data systems can be found in [3] for the time-invariant case, and in [9] for the time-varying case. To the best of our knowledge, the literature does not contain any results that are general enough to cover the problem we encounter here. For this reason, we will also spend some time developing results sufficient for establishing the $\mathcal{L}_2$ stability of the sampled-data system that arises in our anti-windup problem.

### 1.1 Notation

We use $\mathbb{Z}_{\geq k}$ (respectively, $\mathbb{R}_{\geq\alpha}$) to denote the set of integers (respectively, reals) greater than or equal to the integer $k$ (respectively, to the real $\alpha$). For a function $v : \mathbb{R}_{\geq 0} \to \mathbb{R}^m$, we define

$$||v(\cdot)||_{\mathcal{L}_2} := \left(\int_0^\infty |v(\tau)|^2 d\tau\right)^{1/2},$$

and for a function $\xi : \mathbb{Z}_{\geq 0} \to \mathbb{R}^n$, we define

$$||\xi(\cdot)||_{\ell_2} := \left(\sum_{k\geq 0} |\xi(k)|^2\right)^{1/2}.$$

When $||v(\cdot)||_{\mathcal{L}_2} < \infty$, respectively $||\xi(\cdot)||_{\ell_2} < \infty$, we say that $v(\cdot) \in \mathcal{L}_2$, respectively, $\xi(\cdot) \in \ell_2$. Given a matrix $Q$, $Q'$ denotes the transpose of $Q$ and, if $Q$ is square, $Q \succ 0$ means that $Q$ is positive definite, while $\lambda_{min}(Q)$, $\lambda_{MAX}(Q)$ denote the eigenvalues of $Q$ whose modulus is minimum and maximum, respectively.

## 2 $\mathcal{L}_2$ Anti-Windup

Consider a linear plant whose state-space representation is

$$\begin{array}{rcl} \dot{x} & = & A\,x + B\,u + B_d\,d \\ z & = & C_z\,x + D_z\,u + D_{dz}\,d \\ y & = & C_y\,x + D_y\,u + D_{dy}\,d, \end{array} \quad (1)$$

where $x \in \mathbb{R}^n$ is the state, $y \in \mathbb{R}^{n_y}$ is the measured output, $z \in \mathbb{R}^{n_z}$ is the performance output, $u \in \mathbb{R}^{n_u}$ is the control input and $d$ is a disturbance input.

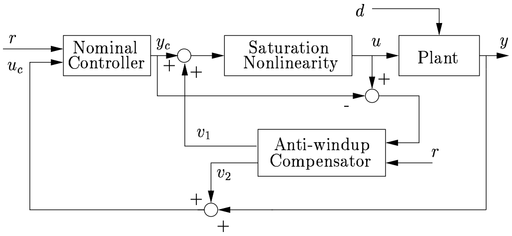


**Figure 1:** The $\mathcal{L}_2$ anti-windup scheme.

Assume that a linear controller has been designed for system (1) following a linear design technique[1] and that its state-space representation is

$$\begin{array}{rcl} \dot{x}_c & = & A_c\,x_c + B_{cu}\,u_c + B_{cr}\,r \\ y_c & = & C_c\,x_c + D_{cu}\,u_c + D_{cr}\,r, \end{array} \quad (2)$$

where $x_c \in \mathbb{R}^{n_c}$ is the controller state, $u_c \in \mathbb{R}^{n_y}$ and $y_c \in \mathbb{R}^{n_u}$ are the controller input and output, respectively, and $r \in \mathbb{R}^{n_z}$ is the reference input.

By referring to Figure 1, the dynamical system (2) will be denoted as the *nominal controller* henceforth. The system corresponding to the feedback connection of the linear plant (1) with the nominal controller (2) via the nominal interconnection equations

$$u = y_c, \qquad u_c = y, \quad (3)$$

will be referred to as the *nominal closed-loop system*, and we will suppose it satisfies the following assumption:

**Assumption 1** *The nominal closed-loop system (1), (2), (3) is well-posed and internally stable.*

If saturation is present at the input $u$ of the linear plant (1), the nominal interconnection (3) is replaced by the *saturated interconnection*

$$u = \mathrm{sat}(y_c), \qquad u_c = y, \quad (4)$$

and the linearity of the closed-loop system is lost. In this paper, we will consider the decentralized saturation function, as detailed in the following assumption:

**Assumption 2** *The input nonlinearity* $\mathrm{sat}(\cdot) : \mathbb{R}^m \to \mathbb{R}^m$ *is the standard decentralized saturation function, namely*

$$\mathrm{sat}(u) := [\mathrm{sat}_i(u_i), \ldots, \mathrm{sat}_m(u_m)]^T,$$

*where*

$$\mathrm{sat}_i(u_i) := \begin{cases} u_i^+ & \text{if } u_i \geq u_i^+ \\ u_i^- & \text{if } u_i \leq u_i^- \\ u_i & \text{otherwise,} \end{cases}$$

*and where* $u_i^- < 0 < u_i^+$ *for all* $i = 1, \ldots m$.

The closed-loop system (1), (2), (4), which we will call the *saturated closed-loop system* henceforth, often exhibits unpredictable behavior and, typically, performance and stability loss. This phenomenon is often referred to in the literature as "windup". Roughly speaking, anti-windup

[1] In general, the controller (2) does not need to be linear for the anti-windup construction to be applicable. However, in this paper we will assume it to be linear, to keep the discussion simple.

schemes attempt, in the presence of saturation, at recovering the nominal (unconstrained) performance as much as possible.

In [19], the goals of anti-windup construction for saturated linear systems have been formalized in an $\mathcal{L}_2$ setting, where the objective is to ensure that the trajectories of the saturated closed-loop, with anti-windup compensation, converge in an $\mathcal{L}_2$ sense to the trajectories of the nominal closed-loop system and to, perhaps, minimize the $\mathcal{L}_2$ error. Moreover, in [19], a model-based solution to the $\mathcal{L}_2$ anti-windup problem is proposed that guarantees stability of the arising closed-loop system by means of additional dynamics. These dynamics modify the control scheme upon activation of the saturation. The compensation scheme, represented in Figure 1, can be generalized to the case where sampled-data feedbacks are used to synthesize the anti-windup compensator block. We propose here a sampled-data anti-windup compensation scheme that solves the $\mathcal{L}_2$ anti-windup problem for asymptotically stable linear plants (1). The particular sampled-data structure that we adopt is the one detailed in Figure 2. The structure contains two principal blocks: a continuous-time one, reproducing the dynamics of the plant, driven by the difference between the plant input and the controller output, and a discrete-time one, driven by sampled versions of the state of the previous block and of an averaged version of the reference input. Note that, in Figure 2, two time scales are represented: the continuous time $t$ and the sampled time $t_s(t)$, which are related as depicted in Figure 3. Note also that, for the sake of generality, $t = 0$ is not necessarily a sampling instant.

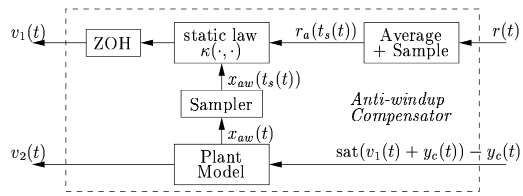


**Figure 2:** The sampled-data anti-windup compensator.

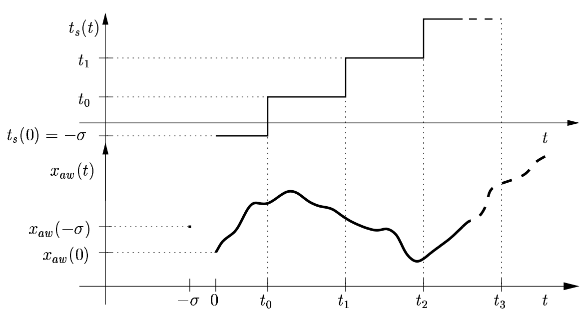


**Figure 3:** The solution to the sampled-data compensator (5) with indications of its initial conditions.

In formulas, the continuous-time block in Figure 2 is described by the equations (the dependence on $t$ has been omitted)

$$\begin{aligned} \dot{x}_{aw} &= A\,x_{aw} + B\left(\mathrm{sat}(y_c + v_1) - y_c\right) \\ v_2 &= -C_y\,x_{aw} - D_y\left(\mathrm{sat}(y_c + v_1) - y_c\right), \end{aligned} \tag{5a}$$

while the discrete-time block is represented by the static function (to be defined later)

$$v_1(t) = \kappa(x_{aw}(t_s(t)), r_a(t_s(t))), \tag{5b}$$

and these blocks are interconnected to the linear plant (1) and to the nominal controller (2) through the *anti-windup interconnection*

$$u_c = y + v_2, \qquad u = y_c + v_1. \tag{6}$$

The sampling instants $t_s(t)$ are related to the continuous time $t$ by

$$t_s(t) = \lfloor t + \widetilde{\sigma} \rfloor_T - \widetilde{\sigma}\ , \qquad \widetilde{\sigma} \in \mathbb{R} \tag{7}$$

where $T$ denotes the sampling period, $\lfloor s \rfloor_T := T\lfloor s/T \rfloor$, and $\lfloor q \rfloor := \max\{\chi \in \mathbb{Z},\ \chi \le q\}$, for all $q \in \mathbb{R}$. The value $\sigma := \widetilde{\sigma} - \lfloor \widetilde{\sigma} \rfloor_T = -t_s(0) = -t_{-1}$ is the time elapsed between the initial time $t = 0$ and the most recent sampling instant, denoted by $t_{-1}$. The signal $r_a(t_s(\cdot))$ is a piecewise constant average of the exogenous reference signal $r$ and is defined as

$$\begin{aligned} r_a(t_s(t)) &= \frac{1}{T}\int_{t_{k-1}}^{t_k} r(\tau)d\tau, \quad \forall t \in [t_k, t_{k+1}), \\ t_k &= t_s((k+1)T) = (k+1)T + t_s(0), \end{aligned} \tag{8}$$

for all $k \ge 0$, where the values $t_k$ represent the sampling instants associated with the sampled-data block. As shown in Figure 3, the information $x_{aw}(0)$, $x_{aw}(t_s(0))$ (where $t_s(0) = -\sigma$), $r_a(t_s(0))$, $r(t)$, $\forall t \in [t_{-1}, 0)$, is a minimal representation of the initial conditions of (5). In our simulations, we will use the initial conditions $(x_{aw}(0), x_{aw}(t_s(0)), r_a(t_s(0))) = (0, 0, 0)$, $r(t) = r(0)$, $\forall t \in [-\sigma, 0)$.

In the special case when the plant is asymptotically stable and the anti-windup compensator is a sampled-data system, the $\mathcal{L}_2$ anti-windup problem defined in [19] can be captured with the following two definitions:

**Definition 1** *A constant reference $r_\circ$ is said to be* feasible *if the response of the nominal closed-loop (1), (2), (3) to the input $(r, d) = (r_\circ, 0)$ is such that the steady-state value $y_{c,\infty}$ of the controller output satisfies*

$$y_{c,\infty} = \mathrm{sat}(y_{c,\infty}). \tag{9}$$

**Definition 2** *Let $z_\ell$ represent the performance output and $u_\ell$ represent the plant input for the nominal closed-loop system (1), (2), (3).*

*The performance response $z(\cdot)$ of the anti-windup closed-loop system (1), (2), (5), (6) satisfies:*

1. *if $x_{aw}(0) = 0$, $x_{aw}(t_s(0)) = 0$ and $u_\ell(\cdot) \equiv \mathrm{sat}(u_\ell(\cdot))$, then $z(\cdot) \equiv z_\ell(\cdot)$;*

2. *if $d(\cdot) \in \mathcal{L}_2$ and there exists a feasible reference $r_\circ$ such that $r(\cdot) - r_\circ \in \mathcal{L}_2$, then*

$$(z_\ell - z)(\cdot) \in \mathcal{L}_2.$$

Definition 2 formalizes the two peculiar goals of anti-windup designs, formulated from an intuitive viewpoint in Section 1. As a matter of fact, item 1 imposes that whenever the initial conditions and external inputs are such that the arising trajectory would not exceed the saturation limits, then the anti-windup compensator must not enforce any modification to the linear closed-loop transfer function. Furthermore, item 2 formalizes the requirement that, whenever saturation is activated (thus making the desired linear trajectory unfeasible for the saturated plant), the performance output must at least converge (in an $\mathcal{L}_2$ sense) to the linear (equivalently, nominal) performance output. Evidently, the smaller the deviation $z_\ell - z$ between the nominal performance and the actual one, the better the anti-windup goal has been accomplished. For this reason, among all of the anti-windup compensators that solve this problem, we are interested in ones that are effective at making $\|(z_\ell - z)(\cdot)\|_{\mathcal{L}_2}$ small.

In this work, the synthesis of the static function $\kappa(\cdot,\cdot)$ in Figure 2 will be based on the discrete-time model

$$\xi(k+1) \;=\; A_d\xi(k) + B_d\Big(\mathrm{sat}(y_{c,\infty} + \nu(k)) - y_{c,\infty}\Big), \tag{10}$$

where $r_\circ$ is a generic feasible reference, the matrices $A_d$ and $B_d$ are defined as

$$A_d := e^{AT}, \quad B_d := e^{AT}\left(\int_0^T e^{-A\tau}d\tau\right)B\ , \tag{11}$$

and, by abuse of notation, $y_{c,\infty}(r_\circ)$ represents the steady-state input value associated with $(r,d) = (r_\circ, 0)$, according to Definition 1. The sampled-data feedback will have the form

$$\nu(k) = \kappa(\xi(k), r_\circ)\ , \tag{12}$$

and (in the next section) we will design a function $\kappa(\cdot,\cdot)$ that satisfies the following property:

**Property 1** *The function $\kappa(\cdot,\cdot)$ is such that*

1. *it is globally Lipschitz,*
2. $\mathrm{sat}(y_{c,\infty}(r_\circ) + \kappa(\xi, r_\circ)) \;=\; y_{c,\infty}(r_\circ) + \kappa(\xi, r_\circ)$, $\forall \xi \in \mathbb{R}^n$ , $r_\circ$ feasible
3. *the origin of*

$$\xi(k+1) \;=\; A_d\,\xi(k) + B_d\,\kappa(\xi(k), r_\circ), \tag{13}$$

*is globally exponentially stable.*

From a performance perspective, one great advantage residing in the $\mathcal{L}_2$ anti-windup compensation scheme is that, based on the linearity of the plant, the anti-windup performance can be measured in terms of the output

$$\begin{aligned} z_{aw} \;&:=\; C_z\,x_{aw} + D_z\Big(\mathrm{sat}(y_c + v_1) - y_c\Big), \\ &=\; z - z_\ell, \end{aligned} \tag{14}$$

which is shown in [19] to coincide with the difference between the nominal performance output (namely, the output $z_\ell$ corresponding to the unsaturated trajectory) and the actual performance output (namely, the output $z$ of the anti-windup closed-loop system). We can now state the following main result, whose proof is omitted due to space constraints.

**Theorem 1** *Under Assumptions 1 and 2, if the function $\kappa(\cdot,\cdot)$ satisfies Property 1, the anti-windup closed-loop system (1), (2), (5), (6) solves the $\mathcal{L}_2$ anti-windup problem of Definition 2.*

**Remark 1** We emphasize that Theorem 1 establishes a result for system (5) where the second argument of $\kappa(\cdot,\cdot)$ is time-varying, yet assumes properties for $\kappa(\cdot,\cdot)$ only when its second argument is a constant, feasible reference. $\circ$

In the next section, we will introduce an RHC-based design strategy for the synthesis of a feedback function $\kappa(\cdot,\cdot)$ that satisfies Property 1 (thus solving the $\mathcal{L}_2$ anti-windup problem by way of Theorem 1) while keeping small the $\ell_2$ norm of the discrete-time signal

$$\begin{aligned} \zeta_{aw}(k) \;&:=\; C_z\,\xi(k) + D_z\Big(\mathrm{sat}(y_{c,\infty} + \nu(k)) - y_{c,\infty}\Big) \\ &=\; C_z\,\xi(k) + D_z\nu(k), \end{aligned} \tag{15}$$

where by virtue of item 2 of Property 1 and (12), the equality in equation (15) necessarily holds, and where $\xi(k)$ is the trajectory of (10). This is especially effective at providing a good solution to the $\mathcal{L}_2$ anti-windup problem when the sampling period is small and $y_c(\cdot)$ is similar to an impulse function (for instance, as shown in the third plot of Figure 6, where the saturation limits are inactive for sufficiently long time intervals so that the nominal performance can be indeed recovered). When this is the case, $y_c(\cdot)$ can be thought of as inducing an initial condition $\xi(0) = x_{aw}(t_0)$ and thereafter satisfying $y_c(t) \approx y_{c,\infty}(r_\circ)$ so that the real problem is very close to the problem for which the RHC strategy was designed.

## 3 Anti-Windup Synthesis via Explicit RHC Techniques

The goal of this section is to synthesize a control law

$$\nu(k) = \kappa(\xi(k), r_\circ) =: \bar{\kappa}(\xi(k), y_{c,\infty}(r_\circ)) \tag{16}$$

that satisfies Property 1, and attempts at minimizing the $\ell_2$ norm of the output $\zeta_{aw}$ of the following system:

$$\begin{aligned} \xi(k+1) \;&=\; A_d\,\xi(k) + B_d\,\nu(k), \\ \zeta_{aw}(k) \;&=\; C_z\,\xi(k) + D_z\nu(k). \end{aligned} \tag{17}$$

Note that satisfying item 2 of Property 1 exactly corresponds to generating $\nu(k)$ so that

$$u^- \le y_{c,\infty}(r_\circ) + \nu(k) \le u^+\ . \tag{18}$$

For simplicity, we will use the notation $y_{c,\infty}$ to denote $y_{c,\infty}(r_\circ)$ hereafter. The design of $\kappa(\cdot,\cdot)$ is based on the result of a finite horizon optimization problem:

$$\bar{\kappa}(\xi, y_{c,\infty}) := \nu_0^*, \tag{19}$$

where, given a finite number of steps $N$, $\nu_0^*$ is the first element of the minimizer $\mathcal{V}^*$ of the following optimization problem:

$$J^*(\xi, y_{c,\infty}) := \tag{20a}$$

$$\min_{\mathcal{V}}\left\{ J(\mathcal{V}, \xi, y_{c,\infty}) = \eta_N' P \eta_N + \sum_{i=0}^{N}\left(|\zeta_i|^2 + \nu_i' R \nu_i\right)\right\}$$

$$\text{s.t.}\left\{\begin{array}{l} \eta_0 = \xi \\ u^- - y_{c,\infty} \le \nu_i \le u^+ - y_{c,\infty}, \\ \eta_{i+1} = A_d\eta_i + B_d\nu_i, \\ \zeta_i = C_z\eta_i + D_z\nu_i,\ i = 0,\ldots,N-1, \end{array}\right. \tag{20b}$$

where $R = R' \succ 0$, $P$ is the solution to the Lyapunov equation $P = A_d' P A_d + C_z' C_z \succeq 0$, and $|\cdot|$ denotes the standard Euclidean norm. Moreover, $\eta_0$ is the current state and $\eta_1, \dots, \eta_N$ are the predicted states for the future $N$ sampling instants; $\mathcal{V} := \{\nu_0,\ \nu_1,\ \dots,\ \nu_{N-1}\}$ denotes the set of free moves and $\mathcal{V}^* := \{\nu_0^*,\ \nu_1^*,\ \dots,\ \nu_{N-1}^*\}$ is the minimizer (the dependence on $\xi$ and $y_{c,\infty}$ is omitted for simplicity).

One great advantage in selecting (19)–(20) for the sampled-data feedback (16) resides in the fact that, based on the results of [2, 20], (19)–(20) can be computed analytically as the following globally Lipschitz and piecewise affine control law:

$$\bar{\kappa}(\xi, y_{c,\infty}) = F^i \xi + G^i y_{c,\infty} + a^i, \qquad (21)$$

if $H^i \xi + K^i y_{c,\infty} \leq b^i$, $i = 1, \dots, n_r$, where $F^i$, $G^i$, $a^i$, $H^i$, $K^i$ and $b^i$, $i = 1, \dots, n_r$, are matrices of suitable dimensions, whose values can be determined explicitly by following the construction in [2, 20]. This result, together with the suitability of the proposed control law as a candidate for the solution of the anti-windup problem described in the previous section, are formalized in the following lemmas (the proofs are omitted).

**Lemma 1** *The piecewise affine control law (21) is globally Lipschitz and coincides with the control law (19), defined by the optimization problem (20).*

**Lemma 2** *Let $A_d$ be a strictly Hurwitz matrix and $P$ be the solution to $P = A_d' P A_d + C_z' C_z$. Then, for all $r_\circ$ satisfying Definition 1, the RHC law defined by (19)–(20) globally exponentially stabilizes (17) while fulfilling the constraint (18) at all sampling steps $k$.*

Finally, based on Lemmas 1 and 2, we can prove that the piecewise affine function (21) satisfies Property 1 and, based on Theorem 1, solves the $\mathcal{L}_2$ anti-windup problem of Definition 2.

**Theorem 2** *Under Assumptions 1 and 2, if $A$ is Hurwitz, the anti-windup closed-loop system (1), (2), (5), (6) with the selection (21) solves the $\mathcal{L}_2$ anti-windup problem of Definition 2.*

**Proof:** By Theorem 1, it is sufficient to prove that equation (21) satisfies Property 1. Item 1 follows from Lemma 1 and from the fact that, by linearity, the map $r_\circ \mapsto y_{c,\infty}(r_\circ)$ is globally Lipschitz. To show items 2 and 3, first note that, since $A$ is Hurwitz, then by equation (11), $A_d$ is asymptotically stable. Hence, based on Lemma 2, items 2 and 3 follow. ∎

**Remark 2** In this paper we have chosen $P$ in (20a) as the solution to a Lyapunov equation. An alternative approach, as suggested in [2, 4, 16, 18], is to choose $P$ as the solution to the Riccati equation $P = (A_d + B_d K)' P (A_d + B_d K) + K' R K + C_z' C_z$, where $K = -(R + B_d' P B_d)^{-1} B - B_d' P A_d$ is the LQR gain. By choosing a sufficiently large finite number $N$ of free moves, which can be computed for any compact set as suggested in [2, 4], the receding horizon controller (19)–(20) solves the constrained linear quadratic regulation problem with output weight $I$, input weight $R$, and limits (18), i.e., we can achieve stability and $\ell_2$-optimality, but only semi-globally. In fact, for certain initial states outside the given compact set, not only optimality, but even convergence to the origin is not guaranteed. On the other hand, by setting $P$ as the solution to the Lyapunov equation we proved global stability. Moreover, even if $\ell_2$-optimality is not achieved, it can be arbitrarily approximated by increasing the number $N$ of free moves, although this usually increases the number $n_r$ of cells in the polyhedral partition of the piecewise affine control law (21). ∘

## 4 Simulation example

To illustrate the performance of the anti-windup construction proposed in Sections 2 and 3, consider a damped mass-spring system. Its equations of motion are given by

$$\begin{array}{rcl} \dot{x} & = & \begin{bmatrix} 0 & 1 \\ -k/m & -f/m \end{bmatrix} x + \begin{bmatrix} 0 \\ 1/m \end{bmatrix} (u + d) \\ z & = & y = \begin{bmatrix} 1 & 0 \end{bmatrix} x, \end{array} \qquad (22)$$

where $x := [q\ \dot{q}]^T$ represents position and speed of the body connected to the spring, $m$ is the mass of the body, $k$ is the elastic constant of the spring, $f$ is the damping coefficient, $u$ represents a force exerted on the mass and $d$ represents a disturbance at the plant's input. We choose the following values for the parameters: $m_0 = 0.1\ kg$, $k_0 = 1\ kg/s^2$, $f_0 = 0.005\ kg/s$.

Assume that $r \in \mathbb{R}$ is a reference input corresponding to the desired mass position and that the following linear controller

$$\begin{array}{l} y_c = C_{fb}(s)\big(C_{ff}(s)\, r - u_c\big), \\ C_{fb}(s) = 500\, \dfrac{(s+15)^2}{s\,(s+80)},\ C_{ff}(s) = \dfrac{5}{2s+5}, \end{array} \qquad (23)$$

has been determined with the aim of guaranteeing a fast response with zero steady-state error to step reference changes, robust to parametric uncertainties.

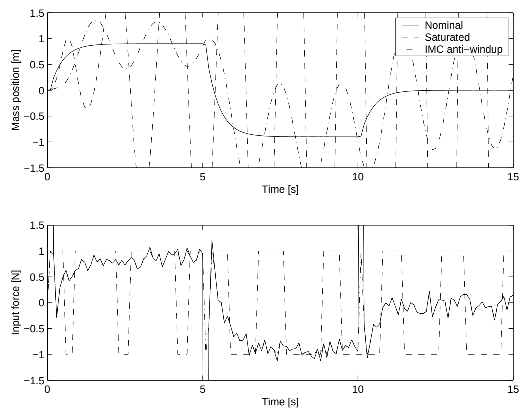


**Figure 4:** Time responses of the saturated (dashed), nominal or unsaturated (solid) and IMC anti-windup (dash-dotted) compensated systems to a double pulse reference.

For all of our simulations, the disturbance $d$ is chosen as band-limited white noise of power 0.0002 passed through a zero order holder with sampling instant 0.001 $s$. The response of the nominal closed-loop system (22), (23), (3), starting from the rest position and with the reference switching between $\pm 0.9$ meters every 5 seconds and going back to zero permanently after 15 seconds, is shown by the solid curve in Figure 4. If the force exerted at the plant's input $u$ is limited between $\pm 1\ Kg \cdot m/s^2$, the saturated response corresponds to the dotted curve in Figure 4, which converges to a limit cycle where the output persistently oscillates between positive and negative peaks $q_{PEAK} \approx 260\ m$. The windup effect shown by this saturated response is associated with a challenging compensation problem. Indeed, the following attempts for anti-windup design lead to unacceptable results:

1. *IMC/model-based anti-windup* leads to very large oscillations decaying at a very slow rate (corresponding to the slow modes of the open-loop plant dynamics) and is represented by the dash-dotted curve in Figure 4;
2. *optimal static linear anti-windup compensation* (proposed in [13]) is unfeasible for this particular example.

We synthesize the RHC control law $\bar{\kappa}(\xi, y_{c,\infty})$ by setting $R = 10^{-8}$, $N = 2$, $N_c = 2$, and $P$ solving the Lyapunov equation. The piecewise affine controller is computed in about 1.32 $s$ on a Pentium III 650Mhz running Matlab 5.3 by using the algorithm of [20]. The corresponding partition projected on the $\xi$ coordinates for $y_{c,\infty} = 0$ has 7 regions and is reported in Figure 5.

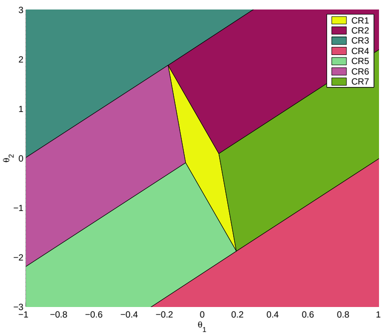


**Figure 5:** The partition associated with the RHC law, projected on the $\xi$ coordinates for $y_{c,\infty} = 0$.

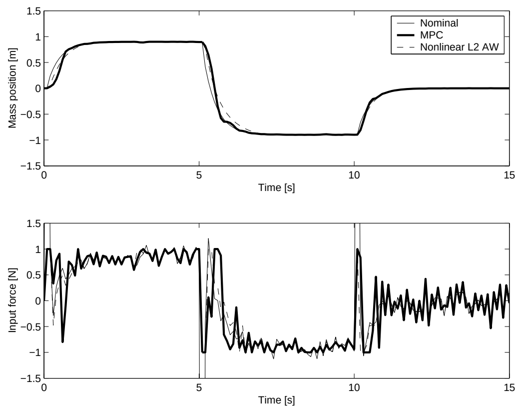


**Figure 6:** Time responses of nominal or unsaturated (solid), RHC-based anti-windup (bold) and nonlinear $\mathcal{L}_2$ anti-windup (dashed) compensated systems to a double pulse reference.

As already shown in [21], nonlinear $\mathcal{L}_2$ anti-windup provides a very effective solution to the windup problem associated with this mass-spring system. The RHC-based construction proposed in this paper, which attempts at approximating $\ell_2$-optimality, produces what can be viewed as better performance. This is shown in Figure 6 where it is seen that the bold curve converges to the nominal (solid) curve at a faster rate. To correctly simulate the sampled-data system, the quantity $\sigma$ indicated in Figure 3 has been selected as $\sigma = 0.08$. so that (according to a reasonable situation) the reference does not change exactly at the sampling instants.

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