Piecewise Linear Optimal Controllers for Hybrid Systems

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Abstract

In this paper we propose a procedure for synthesizing piecewise linear optimal controllers for discrete-time hybrid systems. A stabilizing controller is obtained by designing a *model predictive controller* (MPC), which is based on the minimization of a weighted ℓ_1/∞-norm of the tracking error and the input trajectories over a finite horizon. The control law is obtained by solving a *multiparametric mixed-integer linear program* (mp-MILP), which avoids solving mixed-integer programs on-line [6]. As the resulting control law is piecewise affine, on-line computation is drastically reduced to a simple linear function evaluation.

1 Introduction

Hybrid systems provide a unified framework for describing processes evolving according to continuous dynamics, discrete dynamics, and logic rules [2, 3, 13, 14]. The interest in hybrid systems is mainly motivated by the large variety of practical situations, for instance real-time systems, where physical processes interact with digital controllers. Several modeling formalisms have been developed to describe hybrid systems, among them the class of Mixed Logical Dynamical (MLD) systems introduced by Bemporad and Morari [6]. The MLD framework allows to model a broad class of systems arising in many applications: linear hybrid dynamical systems, hybrid automata, nonlinear dynamic systems where the nonlinearity can be approximated by a piecewise linear function, some classes of discrete event systems, linear systems with constraints, etc. Examples of real-world applications that can be naturally modeled within the MLD framework are reported in [5, 6, 7].

In [6], Bemporad and Morari propose *Model Predictive Control* (MPC) as a general approach to control hybrid systems. MPC has been widely adopted in industry to solve control problems of systems subject to input and output constraints. MPC is based on the so called *receding horizon* philosophy: a sequence of future control actions is chosen according to a prediction of the future evolution of the system and applied to the plant until new measurements are available. At that time, based on the new measurement, a new sequence is established which replaces the previous one. Each sequence is determined by means of an optimization procedure which takes into account two objectives: optimize the tracking performance, and protect the system from possible constraint violations. When the model of the system is a hybrid MLD model and the performance index is quadratic, the optimization problem is a Mixed-Integer Quadratic Programming (MIQP) problem. Similarly, ∞-norm/1-norm performance indices lead to Mixed-Integer Linear Programming (MILP) problems. By appropriately defining the concepts of equilibrium and stability for MLD systems, and by using Lyapunov arguments, it can be proven [6] that model predictive control laws stabilize MLD systems, as well as provide tracking of reference trajectories.

So far, the main drawback of such a control approach has been its intensive on-line computation requirement. Although efficient branch and bound algorithms exist to solve MIQP/MILP, these are known to be $\mathcal{NP}$-hard problems.

In this paper we propose a different approach where all computation is moved off line, by generalizing the result of [8] for linear systems to hybrid systems. The idea stems from observing that the linear part of the objective and the rhs of the constraints in the optimization problem depend linearly on the state vector $x(t)$, i.e. a vector of parameters. Then for ∞-norm/1-norm performance indices, the optimization problem can be treated as a Multiparametric MILP (mp-MILP). In this paper we consider an algorithm to efficiently solve mp-MILPs, and show that the solution is a piecewise linear function of the parameters. In other words, the proposed model predictive controller for hybrid MLD systems, besides being stabilizing and optimal with respect to an ∞-norm/1-norm performance design criterion, is also a piecewise linear controller. Therefore on-line computation reduces to a simple linear function evaluation, instead of an expensive mixed-integer linear program. The basic ideas of the approach are illustrated with an example of controller synthesis for a piecewise affine system.

2 Model Predictive Control of MLD Systems

Consider the *mixed logical dynamical* (MLD) system described by the relations

$$x(t+1) = Ax(t) + B_1u(t) + B_2\delta(t) + B_3z(t) \quad \text{(1a)}$$

$$y(t) = Cx(t) + D_1u(t) + D_2\delta(t) + D_3z(t) \quad \text{(1b)}$$

$$E_2\delta(t) + E_3z(t) \leq E_1u(t) + E_4x(t) + E_5 \quad \text{(1c)}$$

where $x \in \mathbb{R}^{n_c} \times \{0,1\}^{n_\ell}$ is a vector of continuous and binary states, $u \in \mathbb{R}^{m_c} \times \{0,1\}^{m_\ell}$ are the inputs, $y \in \mathbb{R}^{p_c} \times \{0,1\}^{p_\ell}$ the outputs, $\delta \in \{0,1\}^{r_\ell}$, $z \in \mathbb{R}^{r_c}$ represent auxiliary binary and continuous variables, respectively, which are introduced when transforming logic relations into mixed-integer linear inequalities [6], and A, B_1, B_2, B_3, C, D_1, D_2, D_3, E_1, ... ,E_5 are matrices of suitable dimensions. It is interesting from both a theoretical and practical point of view to ask whether or not an MLD system can be stabilized to an equilibrium state or can track a desired reference trajectory, possibly via feedback control. Despite the fact that the system is neither linear nor even smooth, we show in this section how model predictive control (MPC) provides successful tools to perform this task. As recalled above, the main idea of MPC is to use the *model* of the plant to *predict* the future evolution of the system, and based on this prediction to optimize a certain performance index under operating constraints in order to generate the *control* action. Only the first sample of the optimal sequence is actually applied to the plant at time t. At time $t+1$, a new sequence is evaluated to replace the previous one. This on-line "re-planning" provides the desired feedback control feature. Consider an equilibrium pair (x_e, u_e) and let (δ_e, z_e) be a corresponding equilibrium pair of auxiliary variables. Let t be the current time, and $x(t)$ the current state. Consider the following optimal control problem

$$\min_{\{v_0^{T-1}\}} J(v_0^{T-1}, x(t)) \triangleq \sum_{k=0}^{T-1} \|Q_1(v(k)-u_e)\|_\infty + \|Q_2(\delta(k|t)-\delta_e)\|_\infty + \|Q_3(z(k|t)-z_e)\|_\infty + \|Q_4(x(k|t)-x_e)\|_\infty + \|Q_5(y(k|t)-y_e)\|_\infty \quad (2)$$

$$\text{subj. to} \begin{cases} x(T|t) &= x_e \\ x(k+1|t) &= Ax(k|t) + B_1v(k) + B_2\delta(k|t) + B_3z(k|t) \\ y(k|t) &= Cx(k|t) + D_1v(k) + D_2\delta(k|t) + D_3z(k|t) \\ E_2\delta(k|t) &+ E_3z(k|t) \leq E_1v(k) + E_4x(k|t) + E_5 \\ u_{\min} &\leq v(t+k) \leq u_{\max}, \\ & k = 0,1,\ldots,T-1 \\ x_{\min} &\leq x(t+k|t) \leq x_{\max},\ k = 1,\ldots,N_c \end{cases} \quad (3)$$

where T and $N_c \leq T$ are the prediction and state constraint horizons, respectively, Q_1, ... ,Q_5, are nonsingular weighting matrices, $x(k|t)$ is the state predicted at time $t+k$ resulting from the input $u(t+k) = v(k)$ to (1) starting from $x(0|t) = x(t)$, $u_{\min}$, $u_{\max}$ and $x_{\min}$, $x_{\max}$ are hard bounds on the inputs and on the states, respectively. Assume for the moment that the optimal solution $\{v_t^*(k)\}_{k=0,\ldots,T-1}$ exists. According to the *receding horizon* philosophy mentioned above, we set

$$u(t) = v_t^*(0), \quad (4)$$

disregard the subsequent optimal inputs $v_t^*(1), \ldots, v_t^*(T-1)$, and repeat the whole optimization procedure at time $t+1$. The control law (2)–(4) will be referred to as the Hybrid MPC law, and is analogous to the Hybrid MPC law proposed in [6] but based on infinity norm. Note that once x_e, u_e have been fixed, consistent steady-state vectors δ_e, z_e can be obtained by choosing feasible points in the domain described by (1c), for instance by solving an MILP.

In the next section we will show how to formulate the problem (2)-(3) as a mixed integer linear program (MILP).

Several formulations of predictive controllers for MLD systems can be proposed. For instance, the number of control degrees of freedom can be reduced to $N_u < T$, by setting $u(k) \equiv u(N_u - 1)$, $\forall k = N_u, \ldots, T$. However, while in other contexts this amounts to dramatically down-sizing the optimization problem at the price of performance, here the computational gain is only partial, since all the T $\delta(k|t)$ and $z(k|t)$ variables remain in the optimization. Infinite horizon formulations are inappropriate for both practical and theoretical reasons. In fact, approximating the infinite horizon with a large T is computationally prohibitive, as the number of combination of 0-1 variables involved in the MILP as will be shown later, increases exponentially with T. The following theorem shows that the control law (2)–(4) stabilizes system (1) asymptotically

Theorem 1 *Let (x_e, u_e) be an equilibrium pair and (δ_e, z_e) definitely admissible. Assume that the initial state $x(0)$ is such that a feasible solution of problem (2) exists at time $t = 0$. Then for all non singular Q_1 and Q_4, the MIPC law (2)-(4) stabilizes the system in that*

$$\begin{aligned} \lim_{t\to\infty} x(t) &= x_e \\ \lim_{t\to\infty} u(t) &= u_e \\ \lim_{t\to\infty} \|Q_2(\delta(t)-\delta_e)\|_\infty &= 0 \\ \lim_{t\to\infty} \|Q_3(z(t)-z_e)\|_\infty &= 0 \\ \lim_{t\to\infty} \|Q_5(y(t)-y_e)\|_\infty &= 0 \end{aligned}$$

while fulfilling the dynamic/relational constraints (1c) and the input and state constraints $u_{\min} \leq u(t) \leq u_{\max}$, $x_{\min} \leq x(t) \leq \max$.

Note that if Q_2 (or Q_3, Q_5) is non singular, convergence of $\delta(t)$ (or $z(t)$, $y(t)$) follows as well.

Proof: The proof follows easily from standard Lyapunov arguments. Let $\mathcal{U}_t^*$ denote the optimal control sequence $\{v_t^*(0),\ldots,v_t^*(T-1)\}$, let

$$V(t) \triangleq J(\mathcal{U}_t^*, x(t))$$

denote the corresponding value attained by the performance index, and let $\mathcal{U}_1$ be the sequence $\{v_t^*(1),\ldots,v_t^*(T-2),u_e\}$. Then, $\mathcal{U}_1$ is feasible at time $t+1$, along with the vectors $\delta(k|t+1)=\delta(k+1|t)$, $z(k|t+1)=z(k+1|t)$, $k=0,\ldots,T-2$, $\delta(T-1|t+1)=\delta_e$, $z(T-1|t+1)=z_e$, because $x(T-1|t+1)=x(T|t)=x_e$ and (δ_e,z_e) are definitely admissible. Hence,

$$\begin{aligned} V(t+1)\le J(\mathcal{U}_1,x(t+1)) &= \\ V(t)-\|Q_4(x(t)-x_e)\|_\infty &+ \\ -\|Q_1(u(t)-u_e)\|_\infty-\|Q_2(\delta(t)-\delta_e)\|_\infty &+ \\ -\|Q_3(z(t)-z_e)\|_\infty-\|Q_5(y(t)-y_e)\|_\infty & \end{aligned} \quad (5)$$

and $V(t)$ is decreasing. Since $V(t)$ is lower-bounded by 0, there exists $V_\infty=\lim_{t\to\infty}V(t)$, which implies $V(t+1)-V(t)\to 0$. Therefore, each term of the sum

$$\begin{aligned} \|Q_4(x(t)-x_e)\|_\infty+\|Q_1(u(t)-u_e)\|_\infty &+ \\ \|Q_2(\delta(t)-\delta_e)\|_\infty+\|Q_3(z(t)-z_e)\|_\infty &+ \\ \|Q_5(y(t)-y_e)\|_\infty &\le \\ V(t)-V(t+1) & \end{aligned} \quad (6)$$

converges to zero as well, which proves the theorem as $Q_1,\ldots,Q_5$ are nonsingular. □

The end point constraint $x(T|t)=x_e$ can be relaxed by weighting the final state. However from a theoretical point of view, it is not clear how to reformulate an infinite horizon problem for an MLD system, as can be done for linear systems through weightings computed from Lyapunov or Riccati algebraic equations.

3 Piecewise Linear Solution of MPC

In the previous section we have defined an optimal receding horizon control law for MLD systems.

The MPC formulation (2)-(3) can be rewritten as a mixed-integer linear program, by using the following standard approach. The sum of the components of any vector $\{\varepsilon_0^u,\ldots,\varepsilon_{T-1}^u,\varepsilon_0^\delta,\ldots,\varepsilon_{T-1}^\delta,\varepsilon_0^z,\ldots,\varepsilon_{T-1}^z,\varepsilon_0^x,\ldots,\varepsilon_{T-1}^x,\varepsilon_0^y,\ldots,\varepsilon_{T-1}^y\}$ that satisfies

$$\begin{aligned}
-\mathbf{1_m}\varepsilon_k^u &\le Q_1(u(k|t)-u_e)\ k=0,1,\ldots,T-1\\
-\mathbf{1_m}\varepsilon_k^u &\le -Q_1(u(k|t)-u_e)\ k=0,1,\ldots,T-1\\
-\mathbf{1_{r_\ell}}\varepsilon_k^\delta &\le Q_2(\delta(k|t)-\delta_e)\ k=0,1,\ldots,T-1\\
-\mathbf{1_{r_\ell}}\varepsilon_k^\delta &\le -Q_2(\delta(k|t)-\delta_e)\ k=0,1,\ldots,T-1\\
-\mathbf{1_{r_c}}\varepsilon_k^z &\le Q_3(z(k|t)-z_e)\ k=0,1,\ldots,T-1\\
-\mathbf{1_{r_c}}\varepsilon_k^z &\le -Q_3(z(k|t)-z_e)\ k=0,1,\ldots,T-1\\
-\mathbf{1_n}\varepsilon_k^x &\le Q_4(x(k|t)-x_e)\ k=0,1,\ldots,T-1\\
-\mathbf{1_n}\varepsilon_k^x &\le -Q_4(x(k|t)-x_e)\ k=0,1,\ldots,T-1\\
-\mathbf{1_p}\varepsilon_k^y &\le Q_5(y(k|t)-y_e)\ k=0,1,\ldots,T-1\\
-\mathbf{1_p}\varepsilon_k^y &\le -Q_5(y(k|t)-y_e)\ k=0,1,\ldots,T-1
\end{aligned} \quad (7)$$

represents an upper bound on $J(v_0^{T-1},x(t))$, where 1_k is a column vector of ones of length k, and where

$$\begin{aligned} x(k|t) &= A^k x(t)+\sum_{j=0}^{k-1}A^j(B_1u(k-1-j|t)+ \\ & B_2\delta(k-1-j|t)+B_3z(k-1-j|t)) \end{aligned} \quad (8)$$

Similarly to what was shown in [10], it is easy to prove that the vector

$$\begin{aligned} p \triangleq \{&\varepsilon_0^u,\ldots,\varepsilon_{T-1}^u,\varepsilon_0^\delta,\ldots,\varepsilon_{T-1}^\delta,\varepsilon_0^z,\ldots,\varepsilon_{T-1}^z, \\ &\varepsilon_0^x,\ldots,\varepsilon_{T-1}^x,\varepsilon_0^y,\ldots,\varepsilon_{T-1}^y\} \end{aligned}$$

that satisfies equations (7) and simultaneously minimizes

$$\begin{aligned} J(p) = {} & \varepsilon_0^u+\ldots+\varepsilon_{T-1}^u+\varepsilon_0^\delta+\ldots+\varepsilon_{T-1}^\delta+ \\ & \varepsilon_0^z+\ldots+\varepsilon_{T-1}^z+\varepsilon_o^x+\ldots+\varepsilon_{T-1}^x+ \\ & \varepsilon_0^y+\ldots+\varepsilon_{T-1}^y \end{aligned} \quad (9)$$

also solves the original problem, i.e. the same optimum $J^*(v_0^{T-1},x(t))$ is achieved. Therefore, problem (2)-(3) can be reformulated as the following MILP problem

$$\begin{aligned}
\min_p \quad & J(p) \\
\text{s.t.} \quad -\mathbf{1_m}\varepsilon_k^u &\le \pm Q_1(v_e+v(k|t))\ k=0,1,\ldots,T \\
-\mathbf{1_m}\varepsilon_k^\delta &\le \pm Q_2(\delta_e+\delta(k|t))\ k=0,1,\ldots,T \\
-\mathbf{1_m}\varepsilon_k^z &\le \pm Q_3(z_e+z(k|t))\ k=0,1,\ldots,T \\
-\mathbf{1_n}\varepsilon_k^x &\le \pm Q_4(x_e+A^kx(0|t)+ \\
&\quad \textstyle\sum_{j=0}^{k-1}A^j(B_1v(k-1-j|t)+ \\
&\quad B_2\delta(k-1-j|t)+B_3v(k-1-j|t))) \\
&\quad k=1,\ldots,T \\
-\mathbf{1_n}\varepsilon_k^y &\le \pm Q_5(y_e+CA^kx(0|t)+ \\
&\quad \textstyle C\sum_{j=0}^{k-1}A^j(B_1v(k-1-j|t)+ \\
&\quad B_2\delta(k-1-j|t)+B_3v(k-1-j|t))) \\
&\quad +D_1v(k)+D_2\delta(k|t)+D_3z(k|t) \\
&\quad k=1,\ldots,T \\
x_{\min} &\le A^kx(0|t)+ \\
&\quad \textstyle\sum_{j=0}^{k-1}A^j(B_1v(k-1-j|t)+B_2\delta(k-1-j|t) \\
&\quad +B_3v(k-1-j|t))\le x_{\max}, \\
&\quad k=1,\ldots,N_c \\
u_{\min} &\le v_{k|t}\le u_{\max},\ k=0,1,\ldots,T \\
x(T|t) &= x_e \\
x(k+1|t) &= Ax(k|t)+B_1v(k)+ \\
&\quad B_2\delta(k|t)+B_3z(k|t),\ k\ge 0 \\
y(k|t) &= Cx(k|t)+D_1v(k)+ \\
&\quad D_2\delta(k|t)+D_3z(k|t) \\
E_2\delta(k|t) &+ E_3z(k|t)\le E_1v(k)+ \\
&\quad E_4x(k|t)+E_5,\ k\ge 0
\end{aligned} \quad (10)$$

where the variable $x(0|t)$ appears only in the constraints in (10) as a vector parameter. Problem (10) can be rewritten in the more compact form

$$p_t^* \triangleq \arg\min_p \quad \{l(p_c, p_d, \xi(t)) = f_c^T p_c + f_d^T p_d\}$$

$$\text{subj. to} \quad G_c p_c + G_c p_d \leq S + F\xi(t) \tag{11}$$

where $\xi(t) = x(t)$, the matrices G, S, F can be straightforwardly defined from (10), and p_c, p_d represent the continuous and discrete components, respectively, of the optimization vector p.
The MILP problem (11) depends on the current value of $\xi(t)$, and needs to be solved in order to compute the command input. Rather than solving the MILP on line, we follow the ideas of [8, 4], and propose an approach where all computations are moved off line. In fact, by treating $\xi(t)$ as a vector of parameters, the MILP becomes a *multiparametric* MILP (mp-MILP), and its solution for all admissible initial states $\xi(t)$ will be the explicit MPC controller law for MLD systems. We will also show that such a control law is piecewise affine with respect to the state vector.
As we will describe in the next section, we use the algorithm developed in [11] for solving the mp-MILP formulated above. Once the multi-parametric problem (10) has been solved off line, i.e. the solution $p_t^* = f(\xi(t))$ of (11) has been found, the model predictive controller (2)-(3) is available explicitly, as the optimal input $u(t)$ consists simply of m components of p_t^*

$$u(t) = [0 \ \ldots 0 \ I \ 0 \ \ldots \ 0] f(\xi(t)). \tag{12}$$

As the solution p^* of the mp-MILP problem is piecewise affine with respect to the state $x(t)$, the same property is inherited by the controller because of (12).

4 Multiparametric-MILP Solvers

Two main approaches have been proposed for solving mp-MILP problems. In [1], the authors develop an algorithm based on branch and bound (B&B) methods. At each node of the B&B tree an mp-LP is solved where a certain number of integer variables is relaxed to continuous values in $[0, 1]$. The solution at the root node ,where all the integer variables are relaxed, represents a valid lower bound, while the solution at a leaf node where all the integer variables have been fixed to 0 or 1 represents a valid upper bound. As in standard B&B methods, the complete enumeration of combinations of 0-1 integer variables is avoided by comparing the multiparametric solutions, and by fathoming the nodes where there is no improvement of the value function. In [11] an alternative algorithm was proposed, which only solves mp-LPs where the integer variables are fixed to the optimal value determined by an MILP,

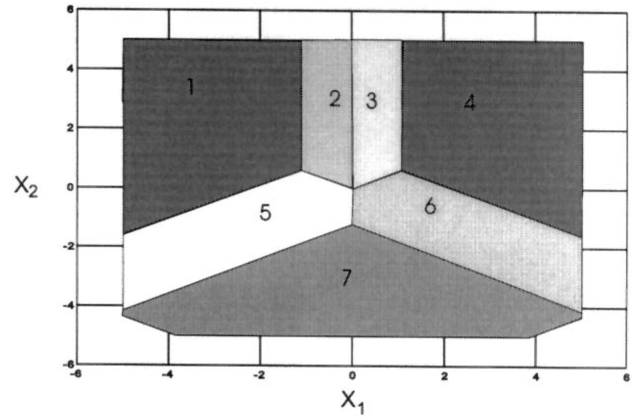

Figure 1: Polyhedral partition of the state-space

instead of solving mp-LP problems with relaxed integer variables. More in detail, problem (11) is alternatively decomposed into an mp-LP and an MILP subproblem. First an MILP problem is solved by considering also parameters as variables. Then an mp-LP is solved where the binary variables are fixed to the optimal values determined by the previous MILP. The solution of the mp-LP provides a parametric upper bound. A new integer vector is determined by solving an MILP that includes an additional constraint imposing a decrease of the value function with respect to the previous mp-LP (see [11] for more details). The algorithmic implementation of the mp-MILP [11] algorithm adopted in this paper relies on [9] for solving mp-LP problems, and on [12] for solving MILP's.

5 An Example

Consider the following simple system [6]

$$\left\{ \begin{array}{rcl} x(t+1) & = & 0.8 \begin{bmatrix} \cos\alpha(t) & -\sin\alpha(t) \\ \sin\alpha(t) & \cos\alpha(t) \end{bmatrix} x(t) + \\ & & \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \\ y(t) & = & [1 \ 0] x(t) \\ \alpha(t) & = & \left\{ \begin{array}{ll} \frac{\pi}{3} & \text{if} \quad [1 \ 0] x(t) \geq 0 \\ -\frac{\pi}{3} & \text{if} \quad [1 \ 0] x(t) < 0 \end{array} \right. \\ x(t) & \in & [-5, 5] \times [-5, 5] \\ u(t) & \in & [-1, 1] \end{array} \right. \tag{13}$$

By using auxiliary variables $z(t) \in \mathbb{R}^4$ and $\delta(t) \in \{0, 1\}$ such that $[\delta(t) = 1] \leftrightarrow [[1 \ 0] x(t) \geq 0]$, Eq. (13) can be rewritten in the form (1) as in [6].
In order to optimally transfer the state from $x_0 = [-1 \ 1]'$ to $x_f = [0 \ 0]'$, the performance index (2) is minimized subject to (3) and the MLD system dynamics (13), along with the weights $Q_1 = 0.001$, $Q_2 = 0.04$, $Q_3 = 30 I_4$, $Q_4 = 10 I_2$, $Q_5 = 700$. By solving the mp-MILP associated with this MPC problem we obtain the

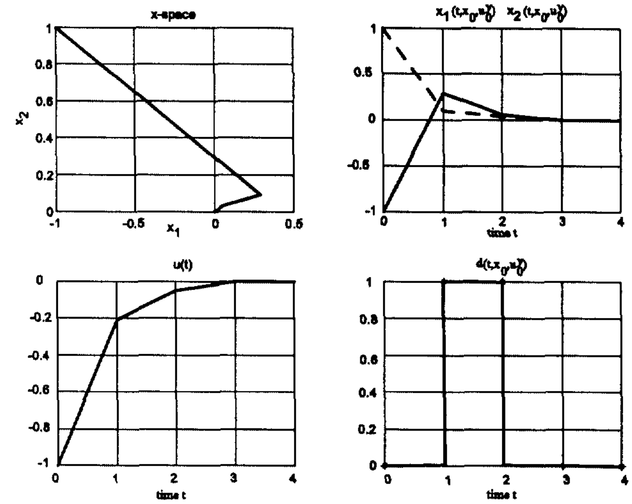

Figure 2: Closed-loop MPC control of system (13)

explicit controller

$$u = \begin{cases} -1.0000 & \text{if } \begin{bmatrix} 1.0000 & -1.7296 \\ -1.0000 & 0.0000 \\ 0.0000 & 1.0000 \\ 94.5067 & -1.0000 \end{bmatrix} x \leq \begin{bmatrix} -2.1680 \\ 5.0000 \\ 5.0000 \\ -105.1385 \end{bmatrix} \text{ (Region \#1)} \\ \begin{bmatrix} 0.9238 & 0.0000 \end{bmatrix} x & \text{if } \begin{bmatrix} -188.1504 & 1.0000 \\ -20.3158 & -34.1997 \\ 1.0000 & 0.0000 \\ 0.0000 & 1.0000 \end{bmatrix} x \leq \begin{bmatrix} 204.3621 \\ 1.0000 \\ 0.0000 \\ 5.0000 \end{bmatrix} \text{ (Region \#2)} \\ \begin{bmatrix} -0.9238 & -0.0000 \end{bmatrix} x & \text{if } \begin{bmatrix} -217.00 & 1.0000 \\ 38.1919 & -64.29 \\ 1.0000 & 0.0000 \\ 0.0000 & 1.0000 \end{bmatrix} x \leq \begin{bmatrix} 4.9708 \\ 1.0000 \\ 1.0000 \\ 5.0000 \end{bmatrix} \text{ (Region \#3)} \\ -1.0000 & \text{if } \begin{bmatrix} -188.15 & -1.0000 \\ -1.0000 & -1.6858 \\ 1.0000 & 0.0000 \\ 1.0000 & 133.68 \end{bmatrix} x \leq \begin{bmatrix} -204.39 \\ -2.1673 \\ 5.0000 \\ 665.60 \end{bmatrix} \text{ (Region \#4)} \\ \begin{bmatrix} 0.4619 & -0.8000 \end{bmatrix} x & \text{if } \begin{bmatrix} 1.0000 & -1.7193 \\ -1.0000 & 1.7295 \\ -1.0000 & 0.0000 \\ 13.4029 & 23.6383 \\ 43.4009 & 0.0000 \end{bmatrix} x \leq \begin{bmatrix} 2.1387 \\ 2.1174 \\ 5.0000 \\ -1.0000 \\ -1.0000 \end{bmatrix} \text{ (Region \#5)} \\ \begin{bmatrix} -0.4619 & -0.8000 \end{bmatrix} x & \text{if } \begin{bmatrix} 1.0000 & 1.7175 \\ -20.2527 & 34.1997 \\ -1.0000 & 0.0000 \\ -1.0000 & -1.7369 \\ 106.3247 & 1.0000 \end{bmatrix} x \leq \begin{bmatrix} 2.0413 \\ -1.0000 \\ 0.0000 \\ 2.2345 \\ 525.0259 \end{bmatrix} \text{ (Region \#6)} \\ 1.0000 & \text{if } \begin{bmatrix} 1.0000 & -1.6446 \\ -1.0000 & 1.7449 \\ -1.0000 & 0.0000 \\ -1.0000 & -1.6788 \\ 1.0000 & -265.137 \\ 1.0000 & 1.7369 \\ 1.0000 & 0.0000 \end{bmatrix} x \leq \begin{bmatrix} 12.0940 \\ -2.2448 \\ 5.0000 \\ 12.3141 \\ 1329.55 \\ -2.2345 \\ 5.0000 \end{bmatrix} \text{ (Region \#7)} \end{cases}$$

In Fig. 1 the state space regions are depicted. The resulting optimal trajectories are shown in Fig. 2.

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