On the Synthesis of Piecewise Affine Control Laws

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***Abstract*— Piecewise affine (PWA) control laws offer an attractive solution to real-time control of linear, nonlinear and hybrid systems. In this paper we provide a compact exposition of the existing state-of-the-art methods for the synthesis of PWA control laws using optimization-based methods.**

I. INTRODUCTION

Piecewise affine (PWA) functions $f_{\mathrm{PWA}} : \mathbb{R}^n \to \mathbb{R}^m$ of the form [1], [2]

$$f_{\mathrm{PWA}}(x) := F_i x + f_i \quad \text{if} \quad x \in \Omega_i, \tag{1}$$

where $\mathcal{I} = \{1, \ldots, s\}$ is a finite set of indices, $\{\Omega_i\}_{i \in \mathcal{I}}$ defines a partition of $\mathbb{R}^n$ with each Ω_i a polyhedron (not necessarily closed), $F_i \in \mathbb{R}^{m \times n}$ are matrices and $f_i \in \mathbb{R}^m$ are vectors for each $i \in \mathcal{I}$, have been largely used in modeling and control of dynamical systems. Applications of PWA functions include switched power converters [3], direct torque control of three-phase induction motors [4], automotive systems [5], paper flow in printers [6] and systems biology [7]. For discrete-time systems

$$x(t+1) = f(x(t), u(t)), \tag{2}$$

where $x(t) \in \mathbb{R}^n$ is the state vector and $u(t) \in \mathbb{R}^m$ is the control input vector, $t \in \mathbb{Z}_+$, the attractiveness of PWA control $u(t) = f_{\mathrm{PWA}}(x(t))$ is explained by various reasons. Firstly, PWA functions can be efficiently implemented on inexpensive hardware. Secondly, PWA functions are versatile and can approximate any nonlinear control law arbitrarily close. Thirdly, PWA control laws have a simple form that is close to affine/linear laws which (industrial) control engineers are familiar with. As a consequence of these interesting properties, there is a strong interest in synthesizing PWA controllers with desirable properties.

Several Lyapunov-based methods for synthesizing PWA laws exist in the literature [2], [6], [8]–[12]. However, one of the most appealing ways to construct PWA state feedback controllers for (constrained) linear and PWA system models is model predictive control (MPC). Indeed, recently it was shown that MPC, which requires on-line optimization, is in fact equivalent to an explicit PWA control law [13]–[16]. The goal of this paper is to provide a compact overview of the state-of-the-art MPC-based methods for the synthesis of PWA control laws.

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II. EXPLICIT MPC

In this section we briefly recall the basic steps and ingredients to compute explicit solutions of MPC and their relations to PWA functions. The interested reader is referred to [16] for a more extensive survey on explicit MPC.

A. Model predictive control

In MPC the control action is obtained by solving a finite horizon open-loop optimal control problem at each sampling instant. Based on a discrete-time dynamic model one can obtain a prediction of the future states and outputs of the process given a control input sequence and initial conditions, which consist of the current (measured) state of the proces. Based on these predictions the best control sequence is chosen that satisfies input, state and output constraints and optimizes a performance index. This optimization yields a sequence of optimal control moves, but only the first move is applied to the process. At the next time step, the computation is repeated over a shifted time-horizon by taking the most recently available (measured) state information as the new initial condition of the optimal control problem. For this reason, MPC is also called *receding horizon control*.

The process to be controlled is usually modeled by the system of difference equations (2). We assume for simplicity that $f(0,0) = 0$. The control and state sequences are requested to satisfy the constraints

$$x(t) \in \mathcal{X}, u(t) \in \mathcal{U} \tag{3}$$

$\forall t \in \mathbb{Z}_+$, where $\mathcal{U} \subseteq \mathbb{R}^m$ and $\mathcal{X} \subseteq \mathbb{R}^n$ are closed sets containing the origin in their interior[1]. Assuming that the control objective is to steer the state to the origin, MPC solves the following constrained regulation problem given a full measurement of the state $x(t)$ at the current time t:

$$\mathcal{P}_N(x(t)) : \min_z \quad \sum_{k=0}^{N-1} l(x_k, u_k) + F(x_N) \tag{4a}$$

$$\text{s.t.} \quad x_{k+1} = f(x_k, u_k), \quad k = 0, \ldots, N-1 \tag{4b}$$

$$x_0 = x(t), \tag{4c}$$

$$u_k \in \mathcal{U}, \quad k = 0, \ldots, N-1 \tag{4d}$$

$$x_k \in \mathcal{X}, \quad k = 1, \ldots, N-1 \tag{4e}$$

$$x_N \in \mathcal{X}_N, \tag{4f}$$

$$u_k = \kappa(x_k), \quad k = N_u, \ldots, N-1 \tag{4g}$$

[1]Mixed constraints on (x, u) can be treated as well, for instance to handle constraints on outputs with direct feedthrough $y(t) = f_y(x(t), u(t))$.

where $z \in \mathbb{R}^\ell$, is the vector of optimization variables, $z = [u_0' \ \dots \ u_{N_u-1}']'$ ($'$ denotes the transpose), $\ell = mN_u$, l is the stage cost, and the closed terminal set $\mathcal{X}_N \subseteq \mathcal{X}$, terminal cost F, and terminal gain κ are chosen to ensure closed-loop stability of the MPC scheme [17], [18]. At each time-step t, x_k denotes the predicted state vector at time $t+k$, obtained by applying the input sequence $u_0, .., u_{k-1}$ to model (2), starting from $x_0 = x(t)$. The number $N \geq 1$ is the prediction horizon and N_u is the control (input) horizon ($1 \leq N_u \leq N$). In what follows we will also use "$\leq$" for vector variables, to denote component-wise inequalities. Because N is finite, if f, l and F are continuous and $\mathcal{U}$ is also compact the minimum in (4a) exists. At each time-step t a solution to problem $\mathcal{P}_N(x(t))$ is found by solving the mathematical program

$$\begin{array}{ll} \min\limits_z & h(z, x(t)) \\ \text{s.t.} & g(z, x(t)) \leq 0, \ g \in \mathbb{R}^q \end{array} \tag{5}$$

obtained from (4), yielding the optimal control sequence $z^*(x(t))$ (assuming for simplicity the uniqueness of the optimal sequence). Only the first input is applied to system (2), i.e.

$$u(t) = z_0^*(x(t)), \tag{6}$$

and the optimization problem (4) is repeated at time $t+1$, based on the new state $x(t+1)$ obtained from the process.

The basic MPC setup (4) can be specialized to different cases, depending on the prediction model, performance index, and terminal conditions used.

1) Linear model and quadratic cost: An optimal control problem (4) with quadratic costs and linear model is obtained when

$$l(x_k, u_k) = x_k' Q x_k + u_k' R u_k, \ F(x_N) = x_N' P x_N \tag{7}$$

in (4a), where $Q = Q' \geq 0$, $R = R' > 0$, and $P = P' \geq 0$ are weight matrices and (4b) is given by

$$f(x_k, u_k) = Ax_k + Bu_k \tag{8}$$

$\kappa(x) = Kx$ in (4g), $\mathcal{U}$, $\mathcal{X}$ and $\mathcal{X}_N$ are polyhedral sets, e.g., $\mathcal{U} = \{u \in \mathbb{R}^m : \ u_{\min} \leq u \leq u_{\max}\}$. Then, by substituting $x_k = A^k x(t) + \sum_{j=0}^{k-1} A^j B u_{k-1-j}$, problem (5) becomes a quadratic program:

$$h(z, x(t)) = \frac{1}{2} z' H z + x'(t) C' z + \frac{1}{2} x'(t) Y x(t) \tag{9a}$$

$$g(z, x(t)) = Gz - W - Sx(t), \tag{9b}$$

where $H = H' > 0$ and C, Y, G, W, S are matrices of appropriate dimensions [13].

2) Linear model and linear cost: Let ∞- or 1-norms be used to measure performance

$$l(x_k, u_k) = \|Qx_k\|_p + \|Ru_k\|_p, \ F(x_N) = \|Px_N\|_p, \tag{10}$$

where $p = 1, \infty$, $R \in \mathbb{R}^{n_R \times m}$, $Q \in \mathbb{R}^{n_Q \times n}$, $P \in \mathbb{R}^{n_P \times n}$, and use the same setup for (4b)-(4g) as in Section II-A.1. In case of ∞-norms, by introducing auxiliary variables $\epsilon_0^u, \ \dots, \ \epsilon_{N-1}^u, \ \epsilon_1^x, \ \dots, \ \epsilon_N^x$ satisfying $\epsilon_k^u \geq \|Ru_k\|_\infty$ ($k = 0, \dots, N-1$), $\epsilon_k^x \geq \|Qx_k\|_\infty$ ($k = 1, \dots, N-1$), $\epsilon_N^x \geq \|Px_N\|_\infty$, or, equivalently,

$$\begin{array}{l} \epsilon_k^u \geq \pm R^i u_k, \ i = 1, \dots, n_R, \ k = 0, \dots, N-1 \\ \epsilon_k^x \geq \pm Q^i x_k, \ i = 1, \dots, n_Q, \ k = 1, \dots, N-1 \\ \epsilon_N^x \geq \pm P^i x_N, \ i = 1, \dots, n_P, \end{array} \tag{11}$$

where the superscript i in (11) denotes the ith row, problem (4) can be mapped into the linear program [19]

$$h(z, x(t)) = [\overbrace{1 \ \dots \ 1}^{N} \ \overbrace{1 \ \dots \ 1}^{N} \ \overbrace{0 \ \dots \ 0}^{mN_u}] z \tag{12a}$$

$$g(z, x(t)) = Gz - W - Sx(t), \tag{12b}$$

where $z \triangleq [\epsilon_0^u \ \dots \ \epsilon_{N-1}^u \ \epsilon_1^x \ \dots \ \epsilon_N^x \ u_0' \ \dots \ u_{N_u-1}']'$ is the optimization vector, and G, W, S are obtained from weights Q, R, P, model matrices A, B, (11), constraint sets $\mathcal{U}$, $\mathcal{X}$, $\mathcal{X}_N$, and gain K. The case of 1-norms can be treated similarly by introducing slack variables $\epsilon_{ik}^u \geq \pm R^i u_k$, $\epsilon_{ik}^x \geq \pm Q^i x_k$, $\epsilon_{iN}^x \geq \pm P^i x_N$. This reformulation extends beyond $1/\infty$-norms to *any* convex piecewise affine cost l, F, that, thanks to the result of [20], can be rewritten as the max of a finite set of affine functions.

The use of linear programming in optimization-based control dates back to the early sixties [21].

3) Hybrid model and linear or quadratic costs: The MPC setup also extends to the case in which (4b) (and thus also (2)) is a hybrid dynamical model [22] of the form

$$x_{k+1} = f(x_k, u_k, \delta_k, \zeta_k) = Ax_k + B_1 u_k + B_2 \delta_k + B_3 \zeta_k \tag{13a}$$

$$E_2 \delta_k + E_3 \zeta_k \leq E_1 u_k + E_4 x_k + E_5, \tag{13b}$$

involving both real and binary variables, denoted as the Mixed Logical Dynamical (MLD) model [23], where $x_k \in \mathbb{R}^{n_c} \times \{0,1\}^{n_b}$ is the state vector, $u_k \in \mathbb{R}^{m_c} \times \{0,1\}^{m_b}$ is the input vector, and $\zeta_k \in \mathbb{R}^{r_c}$, $\delta_k \in \{0,1\}^{r_b}$ are auxiliary variables implicitly defined by (13b) for any given pair (x_k, u_k). Matrices A, B_i, ($i = 1,2,3$), and E_i ($i = 1, \dots, 5$) denote real constant matrices, and the inequalities (13b) must be interpreted component-wise. In [22] it was shown that this MLD model has strong equivalence relations to PWA models, linear complementarity models [24] and other hybrid models.

The associated finite-horizon optimal control problem based on quadratic costs takes the form (9) with $z = [u_0' \cdots u_{N-1}' \ \delta_0' \cdots \delta_{N-1}' \ \zeta_0' \cdots \zeta_{N-1}']'$ and subject to the further restriction that some of the components of z must be binary. The hybrid MPC problem maps into a Mixed-Integer Quadratic Programming (MIQP) problem when the quadratic costs (7) are used in (4a) [25], or a Mixed-Integer Linear Programming (MILP) problem when ∞- or 1-norms are used as in (10) [26].

B. Multiparametric programming

Consider the following mathematical program

$$\mathcal{MP}(x): \min_z \quad f(z,x) \tag{14a}$$
$$\text{s.t.} \quad g(z,x) \leq 0 \tag{14b}$$
$$Az + Bx + d = 0, \tag{14c}$$

where $z \in \mathbb{R}^\ell$ collects the decision variables, $x \in \mathbb{R}^n$ is a vector of parameters, $f : \mathbb{R}^\ell \times \mathbb{R}^n \to \mathbb{R}$ is the objective function, $g : \mathbb{R}^\ell \times \mathbb{R}^n \to \mathbb{R}^q$, A is a $q_e \times \ell$ real matrix, B is a $q_e \times n$ real matrix, and $d \in \mathbb{R}^{q_e}$. Problem (14) is referred to as a *multiparametric programming problem*. We are interested in characterizing the (optimal) solution of problem (14) a function of x for a given polytopic set X of parameters. The solution of a multiparametric problem is a quadruple (V^*, Z^*, z^*, X_f), where (i) the *set of feasible parameters* X_f is the set of all $x \in X$ for which problem (14) admits a solution, $X_f = \{x \in X : \exists z \in \mathbb{R}^\ell,\ g(z,x) \leq 0,\ Az+Bx+d=0\}$; (ii) The *value function* $V^* : X_f \to \mathbb{R}$ associates with every $x \in X_f$ the corresponding optimal value of (14); (iii) The *optimal set* $Z^* : X_f \to 2^{\mathbb{R}^\ell}$ associates to each parameter $x \in X_f$ the corresponding set of optimizers $Z^*(x) = \{z \in \mathbb{R}^\ell : f(z,x) = V^*(x),\ g(z,x) \leq 0,\ Az + Bx + d = 0\}$ of problem (14); (iv) An *optimizer function* $z^* : X_f \to \mathbb{R}^\ell$ associates to each parameter $x \in X_f$ an optimizer $z \in Z^*(x)$ ($Z^*(x)$ is just a singleton if $\mathcal{MP}(x)$ is strictly convex).

By treating $x(t)$ as the vector of parameters, the QP problem arising from the linear MPC formulation of Section II-A.1 can be treated as the multiparametric QP (mpQP)

$$\mathcal{QP}(x): V^*(x) = \frac{1}{2}x'Yx + \min_z \quad \frac{1}{2}z'Hz + x'F'z \tag{15a}$$
$$\text{s.t.} \quad Gz \leq W + Sx. \tag{15b}$$

In [13], the authors investigated the analytical properties of the mpQP solution, that are summarized by the following theorem.

Theorem II.1 *Consider a multiparametric quadratic program with $H > 0$, $\left[\begin{smallmatrix} H & F \\ F' & Y \end{smallmatrix}\right] \geq 0$. The set X_f of parameters x for which the problem is feasible is a polyhedral set, the value function $V^* : X_f \mapsto \mathbb{R}$ is continuous, convex, and piecewise quadratic, and the optimizer $z^* : X_f \mapsto \mathbb{R}^\ell$ is piecewise affine and continuous.*

The immediate corollary is that the quadratic MPC approach based on linear costs described in Section II-A.2 admits a continuous piecewise-affine explicit solution of the form (1).

By treating $x(t)$ as the vector of parameters, the linear MPC formulation of Section II-A.2 can be treated as the multiparametric LP (mpLP)

$$\mathcal{LP}(x): \min_z \quad c'z \tag{16a}$$
$$\text{s.t.} \quad Gz \leq W + Sx, \tag{16b}$$

where $z \in \mathbb{R}^\ell$ is the optimization vector, $x \in X \subset \mathbb{R}^n$ is the vector of parameters, c, G, W, S are suitable constant matrices and X is the set of parameters of interest.

Theorem II.2 [27] *Consider the mpLP problem (16). Then, the set X_f is a convex polyhedral set, there exists an optimizer function $z^* : \mathbb{R}^n \to \mathbb{R}^l$, which is a continuous and piecewise affine function of x, and the value function $V^* : \mathbb{R}^n \to \mathbb{R}$ is a continuous, convex, and piecewise affine function of x.*

The first methods for solving parametric linear programs appeared in 1952 in the master thesis published in [28], and independently in [29]. Since then, extensive research has been devoted to (multi)parametric analysis and programming, see the references in [20], [27], [27], [30]–[32]. Recently, there is a renewed interest in this field, mainly pushed by the application of mpLP in explicit MPC, see e.g. the recent survey [33].

As detailed in [26], the MPC formulation based on ∞-,1-norms or quadratic costs subject to the MLD dynamics (13) can be solved explicitly by treating the optimization problem associated with MPC as a multiparametric mixed integer linear programming (mpMILP) problem or a multiparametric mixed integer quadratic programming (mpMIQP) problem, respectively. For further details regarding explicit PWA solutions to these types of mixed integer problems the interested reader is referred to [16].

III. Implementation of PWA control laws

A PWA control (1) is essentially a lookup table of linear feedback gains. The right gain is selected on-line by finding the region Ω_i of the polyhedral partition where the current state $x(t)$ lies. This latter problem has been referred to as *point-location problem*. Note that if function f_{PWA} is continuous, then one can always define the regions of the partitions as closed polyhedra $\Omega_i = \{x \in \mathbb{R}^n : H_i x \leq k_i\}$, as no ambiguity in the definition of f_{PWA} would arise on possibly overlapping boundaries of different sets Ω_i, Ω_j. The most straightforward solution is to store all the M polyhedra of the partition and carry out an on-line search through them until the right one is found. While this procedure is extremely easy to implement in a computer code, more efficient ways have been proposed for evaluating explicit MPC controllers. For instance, by exploiting the properties of multiparametric solutions, several solutions are proposed in the last few years, see e.g. [34]–[39] and the references therein.

From the hardware synthesis viewpoint, [40] showed that explicit MPC solutions can be implemented in an application specific integrated circuit (ASIC) with about $20,000$ gates, leading to computation times in order of 1 μs. In the end, whether the explicit PWA form is preferable to the one based on on-line optimization depends on available CPU time, data memory, and program memory (see e.g. [41, Table II] for a comparison in the linear quadratic case).

IV. Conclusions

In this short paper we gave a compact overview of recent MPC-based techniques for the synthesis of PWA control laws, together with pointers to the relevant literature for more details on this appealing topic. The potential of PWA controllers is enormous and we envision a tremendous growth of PWA control in the years to come in various application fields. Especially, if the implementation of PWA control laws on real-time hardware can be performed more efficiently and systematically, we foresee that PWA and MPC control will also be applied to faster systems that require sample frequencies in the order of 1kHz and above.

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